ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.4(2007) No.1,pp.10-16

On the Extended Tanh Method Applications of Nonlinear Equations

E. Yusufoğlu, A. Bekir *
Dumlupinar University
Science Faculty Department of Mathematics
Kütahya-TURKEY

(Received 18 December 2006, accepted 6 April 2007)

Abstract: The extended tanh method is used to derive abundant solitary wave solutions of nonlinear wave equations. The obtained solutions include solitons and kinks solutions. The extended tanh method presents a wider applicability for handling nonlinear wave equations.

Key words: Extended tanh method, solitons, kinks, Benjamin-Bona-Mahony (BBM) equation, modified Benjamin-Bona-Mahony (MBBM) equation, KdV-Burger equation.

1 Introduction

Nonlinear evolution equations have a major role in various scientific and engineering fields, such as fluid mechanics, plasma physics, optical fibers, solid state physics, chemical kinematics, chemical physics and geochemistry. Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in nonlinear wave equations. In recent years, quite a few methods for obtaining explicit traveling and solitary wave solutions of nonlinear evolution equations have been proposed. A variety of powerful methods, such as inverse scattering method [1, 16], bilinear tranformation [9, 11, 15], the tanh-sech method [12, 14, 21], extended tanh method [4, 6, 23], sine-cosine method [24–26, 30], F-expansion method [3], and homogeneous balance method [5, 17, 29] were used to investigate nonlinear dispersive and dissipative problems.

The pioneer work Malfiet in [12, 13] introduced the powerful tanh method for a reliable treatment of the nonlinear wave equations. The useful tanh method is widely used by many work such as in [18–20, 23] and by the references therein. Later, the extended tanh method, developed by Wazwaz [27, 28], is a direct and effective algebraic method for handling nonlinear equations. Various extensions of the method were developed as well.

Our first interest in present work being in implementing the extended tanh method to stress its power in handling nonlinear equations so that one can apply it to models of various types of nonlinearity. The next interest is in the determination of exact travelling wave solutions for Benjamin-Bona-Mahony equation, modified Benjamin-Bona-Mahony equation and KdV-Burger equation. Searching for exact solutions of nonlinear problems has attracted a considerable amount of research work where computer symbolic systems facilitate the computational work.

2 The extended tanh method

Wazwaz summarized the main steps introduced in this method, as follows [27, 28].

A PDE

$$P(u, u_t, u_x, u_{xx}, ...) = 0. \tag{2.1}$$

*Corresponding author. *E-mail address*: abekir@dumlupinar.edu.tr

IJNS.2007.08.15/088

can be converted to an ODE

$$Q(U, U', U'', U''', ...) = 0. \tag{2.2}$$

upon using a wave variable $\xi = x - \beta t$. Equation (2.2) is then integrated as long as all terms contain derivatives where integration constants are considered zeros, Introducing a new independent variable

$$Y = \tanh(\xi) \tag{2.3}$$

leads to change of derivatives:

$$\begin{aligned}
&\tfrac{d}{d\xi} = (1 - Y^2)\tfrac{d}{dY} \\
&\tfrac{d^2}{d\xi^2} = (1 - Y^2)\left(-2Y\tfrac{d}{dY} + \left(1 - Y^2\right)\tfrac{d^2}{dY^2}\right) \\
&\tfrac{d^3}{d\xi^3} = (1 - Y^2)\left(\left(6Y^2 - 2\right)\tfrac{d}{dY} - 6Y\left(1 - Y^2\right)\tfrac{d^2}{dY^2} + \left(1 - Y^2\right)^2\tfrac{d^3}{dY^3}\right)
\end{aligned} \tag{2.4}$$

The extended tanh method admits the use of the finite expansion

$$U(\xi) = S(Y) = \sum_{k=0}^{m} a_k Y^k + \sum_{k=1}^{m} b_k Y^{-k}, \tag{2.5}$$

where m is a positive integer, for this method, that will be determined. Expansion (2.5) reduces to the standard tanh method [12] for $b_k = 0$, $1 \leq k \leq m$. The parameter m is usually obtained, as stated before, by balancing the linear terms of the highest order in the resulting equation with the highest order nonlinear terms. If m is not an integer, then a transformation formula should be used to overcome this difficulty. Substituting (2.5) into the ODE results an algebraic system of equations in powers of Y that will lead to the determination of the parameters a_k $(k = 0, ..., m)$, b_k $(k = 1, ..., m)$ and β.

To show the efficiency of the method described in the previous part, we present some examples.

3 The BBM equation

We consider the Benjamin-Bona-Mahony equation

$$u_t + u_x + auu_x - bu_{xxt} = 0, \quad a, b > 0. \tag{3.1}$$

Benjamin-Bona-Mahony were derived for the description of the unidirectional propagation of small-amplitude long waves on the surface of water in a channel [2]. Using the wave variable $\xi = x - \beta t$ carries Eq. (3.1) into the ODE

$$(1 - \beta)U + \frac{a}{2}U^2 - \beta b U'' = 0 \tag{3.2}$$

obtained after integrating the ODE once and setting the constant of integration equal to zero. Balancing U'' with U^2 in (3.2) gives

$$m + 2 = 2m, \tag{3.3}$$

so that

$$m = 2. \tag{3.4}$$

The extended tanh method (2.5) admits the use of the finite expansion

$$U(\xi) = S(Y) = a_0 + a_1 Y + a_2 Y^2 + \frac{b_1}{Y} + \frac{b_2}{Y^2} \tag{3.5}$$

Substituting Eq. (3.5) into Eq. (3.2), and equating the coefficients of the powers Y then we obtain a system of algebraic equations for a_0, a_1, a_2, b_1, b_2 and β, by solving we obtained the six sets of solutions

i) *The first set:*

$$a_0 = \frac{4b}{(4b+1)a}, \qquad a_1 = b_1 = a_2 = 0, \qquad b_2 = -\frac{12b}{(4b+1)a} \qquad \beta = \frac{1}{4b+1}. \tag{3.6}$$

ii) *The second set:*

$$a_0 = -\frac{12b}{(4b-1)a}, \qquad a_1 = b_1 = a_2 = 0, \qquad b_2 = \frac{12b}{(4b-1)a}, \qquad \beta = -\frac{1}{4b-1}. \tag{3.7}$$

iii) *The third set:*

$$a_0 = \frac{4b}{(4b+1)a}, \qquad a_1 = b_1 = b_2 = 0, \qquad a_2 = -\frac{12b}{(4b+1)a}, \qquad \beta = \frac{1}{4b+1}. \tag{3.8}$$

iv) *The fourth set:*

$$a_0 = -\frac{12b}{(4b-1)a}, \qquad a_1 = b_1 = b_2 = 0, \qquad a_2 = \frac{12b}{(4b-1)a}, \qquad \beta = -\frac{1}{4b-1}. \tag{3.9}$$

v) *The fifth set:*

$$\begin{aligned} &a_0 = -\frac{8b}{(16b+1)a}, \qquad a_1 = b_1 = 0, \qquad a_2 = -\frac{12b}{(16b+1)a}, \\ &b_2 = -\frac{12b}{(16b+1)a}, \qquad \beta = \frac{1}{16b+1}. \end{aligned} \tag{3.10}$$

vi) *The sixth set:*

$$\begin{aligned} &a_0 = -\frac{24b}{(16b-1)a}, \qquad a_1 = b_1 = 0, \qquad a_2 = \frac{12b}{(16b-1)a}, \\ &b_2 = \frac{12b}{(16b-1)a}, \qquad \beta = -\frac{1}{16b-1}. \end{aligned} \tag{3.11}$$

In view of this we obtain the following solutions

Family 1: *Soliton solutions*

$$u_1 = \frac{4b}{(4b+1)a} - \frac{12b}{(4b+1)a}\coth^2(x - \frac{t}{4b+1}), \tag{3.12}$$

$$u_2 = -\frac{12b}{(4b-1)a} + \frac{12b}{(4b-1)a}\coth^2(x + \frac{t}{4b-1}), \tag{3.13}$$

$$u_3 = -\frac{8b}{(16b+1)a} - \frac{12b}{(16b+1)a}\tanh^2(x - \frac{t}{16b+1}) - \frac{12b}{(16b+1)a}\coth^2(x - \frac{t}{16b+1}), \tag{3.14}$$

$$u_4 = -\frac{24b}{(16b-1)a} + \frac{12b}{(16b-1)a}\tanh^2(x + \frac{t}{16b-1}) + \frac{12b}{(16b-1)a}\coth^2(x + \frac{t}{16b-1}). \tag{3.15}$$

Family 2: *Bell shaped solutions*

$$u_5 = \frac{4b}{(4b+1)a} - \frac{12b}{(4b+1)a}\tanh^2(x - \frac{t}{4b+1}), \tag{3.16}$$

$$u_6 = -\frac{12b}{(4b-1)a} + \frac{12b}{(4b-1)a}\tanh^2(x + \frac{t}{4b-1}), \tag{3.17}$$

4 The MBBM equation

We consider the modified Benjamin-Bona-Mahony equation

$$u_t + u_x + au^2 u_x + bu_{xxt} = 0. \tag{4.1}$$

Using the wave variable $\xi = x - \beta t$ carries Eq. (4.1) into the ODE

$$(1-\beta)U + \frac{a}{3}U^3 - \beta b U'' = 0 \tag{4.2}$$

obtained after integrating the ODE once and setting the constant of integration equal to zero. Balancing U'' with U^3 in (4.2) gives

$$m + 2 = 3m, \tag{4.3}$$

so that

$$m = 1. \tag{4.4}$$

The extended tanh method (2.5) admits the use of the finite expansion

$$U(\xi) = S(Y) = a_0 + a_1 Y + \frac{b_1}{Y} \tag{4.5}$$

Substituting Eq. (4.5) into Eq. (4.2), and equating the coefficients of the powers Y then we obtain a system of algebraic equations for a_0, a_1, b_1 and β, and by solving we obtain the four sets of solutions

i) *The first set:*

$$a_0 = 0, \quad a_1 = 0, \quad b_1 = \sqrt{\tfrac{-6b}{a(2b-1)}}, \quad \beta = -\tfrac{1}{2b-1}. \tag{4.6}$$

ii) *The second set:*

$$a_0 = 0, \quad a_1 = \sqrt{\tfrac{-6b}{a(2b-1)}}, \quad b_1 = 0, \quad \beta = -\tfrac{1}{2b-1}. \tag{4.7}$$

iii) *The third set:*

$$a_0 = 0, \quad a_1 = \sqrt{\tfrac{-6b}{a(8b-1)}}, \quad b_1 = -\sqrt{\tfrac{-6b}{a(8b-1)}}, \quad \beta = -\tfrac{1}{8b-1}. \tag{4.8}$$

iv) *The fourth set:*

$$a_0 = 0, \quad a_1 = \sqrt{\tfrac{6b}{a(4b+1)}}, \quad b_1 = -\sqrt{\tfrac{6b}{a(4b+1)}}, \quad \beta = \tfrac{1}{4b+1}. \tag{4.9}$$

In view of this we obtain the following kink shaped solitary wave solutions

$$u_1 = \sqrt{\tfrac{-6b}{a(2b-1)}} \coth(x + \tfrac{t}{2b-1}), \tag{4.10}$$

$$u_2 = \sqrt{\tfrac{-6b}{a(2b-1)}} \tanh(x + \tfrac{t}{2b-1}), \tag{4.11}$$

$$u_3 = \sqrt{\tfrac{-6b}{a(8b-1)}} \tanh(x + \tfrac{t}{8b-1}) - \sqrt{\tfrac{-6b}{a(8b-1)}} \coth(x + \tfrac{t}{8b-1}), \tag{4.12}$$

$$u_4 = \sqrt{\tfrac{6b}{a(4b+1)}} \tanh(x - \tfrac{t}{4b+1}) - \sqrt{\tfrac{6b}{a(4b+1)}} \coth(x - \tfrac{t}{4b+1}), \tag{4.13}$$

5 The KdV-Burger equation

The general form of the KdV-Burger equation is given by

$$u_t + auu_x + bu_{xx} + cu_{xxx} = 0, \tag{5.1}$$

where a, b and c are some constant coefficients [8]. The KdV-Burger equation involves both dispersion and dissipation problems [10]. Using the wave variable $\xi = x - \beta t$ carries Eq. (5.1) into the ODE

$$-\beta U + \frac{a}{2}U^2 + bU' + cU'' = 0. \tag{5.2}$$

obtained after integrating the ODE once and setting the constant of integration equal to zero. Balancing U'' with U^2 in (5.2) gives

$$m + 2 = 2m, \tag{5.3}$$

so that

$$m = 2. \tag{5.4}$$

The extended tanh method (2.5) admits the use of the finite expansion

$$U(\xi) = S(Y) = a_0 + a_1 Y + a_2 Y^2 + \frac{b_1}{Y} + \frac{b_2}{Y^2} \tag{5.5}$$

Substituting Eq. (5.5) into Eq. (5.2), and proceeding as before we obtain the three sets of solutions
i) *The first set:*

$$a_0 = \tfrac{3\beta}{a}, \qquad a_1 = a_2 = b_1 = 0, \qquad b_2 = -\tfrac{3\beta}{a}, \qquad \beta = 4c. \tag{5.6}$$

ii) *The second set:*

$$a_0 = -\tfrac{\beta}{a}, \qquad a_1 = a_2 = b_1 = 0, \qquad b_2 = \tfrac{3\beta}{a}, \qquad \beta = -4c. \tag{5.7}$$

iii) *The third set:*

$$a_0 = \tfrac{\beta}{a}, \qquad a_1 = a_2 = b_2 = 0, \qquad b_1 = \tfrac{\beta}{a}, \qquad \beta = 2b. \tag{5.8}$$

In view of this we obtain the following soliton solutions

$$u_1 = \tfrac{3\beta}{a} - \tfrac{3\beta}{a}\coth^2(x - 4ct), \tag{5.9}$$

$$u_2 = -\tfrac{\beta}{a} + \tfrac{3\beta}{a}\coth^2(x + 4ct), \tag{5.10}$$

$$u_3 = \tfrac{\beta}{a} + \tfrac{\beta}{a}\coth(x - 2bt). \tag{5.11}$$

6 Conclusion

The main goal of this work, travelling wave solutions were formally derived to BBM, MBBM and KdV-Burger equations. The transformation formulae were used for every type of nonlinearity to show that our analysis is applicable to a variety of nonlinear problems. We have emphasized in this work that this relevant transformation is powerful and can be effectively used to discuss nonlinear evolution equations and related models in scientific fields. The availability of computer systems like *Mathematica* or *Maple* facilitates the tedious algebraic calculations.

References

[1] Ablowitz, M.J., Segur, H.: Solitons and inverse scattering transform.*SIAM*.Philadelphia (1981)

[2] Benjamin, T. B., Bona, J. L., and Mahony, J. J.:Model Equations for Long Waves in Nonlinear Dispersive Systems. *Philos. Trans. Roy. Soc. London Ser. A* 272,47-48 (1972)

[3] Cai, G., Wang, Q., Huang, J.:A Modified F-expansion Method for Solving Breaking Soliton Equation.*International Journal of Nonlinear Science* 2,122-128(2006)

[4] S.A. El-Wakil and M.A. Abdou:New exact travelling wave solutions using modified extended tanh-function method. *Chaos, Solitons and Fractals*. 31(4),840-852 (2007)

[5] Fan, E., Zhang H.:A note on the homogeneous balance method. *Phys. Lett. A*.246,403-406 (1998)

[6] Fan, E.: Extended tanh-function method and its applications to nonlinear equations.*Phys. Lett. A*277,212 (2000)

[7] Fan, E., Hon, Y.C.:Generalized tanh method extended to special types of nonlinear equations.*Z. Naturforsch*57a,692-700 (2002)

[8] Demiray, H.: A travelling wave solution to the KdV-Burgers equation.*Applied Mathematics and Computation* 154, 665-670(2004)

[9] Hirota R.: Direct method of finding exact solutions of nonlinear evolution equations. *R. Bullough, P. Caudrey (Eds.), Backlund transformations, Springer, Berlin* 1157-1175(1980)

[10] Jeffrey, A, Mohamad, M.N.B.:Exact solutions to the KdV-Burgers' equation. *Wave motion* 14 ,369-375(1991)

[11] Lu, D., Hong, B., Tian, L.:Backlund Transformation and N-soliton-like Solutions to the Combined KdV-Burgers Equation with Variable Coefficients.*International Journal of Nonlinear Science* 2,3-10(2006)

[12] Malfliet, W.:Solitary wave solutions of nonlinear wave equations. *Am. J. Phys.* 60, 650-654(1992)

[13] Malfliet, W., Hereman, W.: The tanh method. I: Exact solutions of nonlinear evolution and wave equations.*Phys. Scr.* 54, 563-568(1996)

[14] A. H. Khater, W. Malfliet, D. K. Callebaut and E. S. Kamel:The tanh method, a simple transformation and exact analytical solutions for nonlinear reaction–diffusion equations.*Chaos, Solitons and Fractals* 14(2), 513-522(2002)

[15] Tanoglu, G.: Hirota Method for Solving Reaction-Diffusion Equations with Generalized Nonlinearity.*International Journal of Nonlinear Science* 1, 30-36(2006)

[16] Vakhnenko, V. O., Parkes, E. J. and Morrison, A. J.: A B acklund transformation and the inverse scattering transform method for the generalised Vakhnenko equation. *Chaos, Solitons and Fractals* 17(4), 683-692(2003)

[17] Wang, L., Zhou, J., Ren, L.:The Exact Solitary Wave Solutions for a Family of BBM Equation.*International Journal of Nonlinear Science*1 ,58-64(2006)

[18] Wazwaz, A.M.:The tanh method for travelling wave solutions of nonlinear equations. *Applied Mathematics and Computation* 1543, 713-723(2004)

[19] Wazwaz, A.M.: The tanh method for generalized forms of nonlinear head conduction and Burger-Fisher equations.*Applied Mathematics and Computation* 169, 321-338(2005)

[20] Wazwaz, A.M.:Travelling wave solutions of generalized forms of Burger, Burger-KdV and Burger-Huxley equations.*Applied Mathematics and Computation* 169, 639-656(2005)

[21] Wazwaz, A.M:Two reliable methods for solving variants of the KdV equation with compact and non-compact structures. *Chaos, Solitons and Fractals* 28(2), 454-462 (2006)

[22] Wazwaz, A.M.:The tanh method: Exact solutions of the Sine-Gordon and Sinh-Gordon equations. *Applied Maths. And Computation* 167, 1196-1210(2005)

[23] Wazwaz, A.M.:The tanh method: solitons and periodic solutions for the Dodd–Bullough–Mikhailov and the Tzitzeica–Dodd–Bullough equations. *Chaos, Solitons and Fractals* 25(1),55-63 (2005)

[24] Wazwaz, A.M.:The sine-cosine method for obtaining solutions with compact and noncompact structures. *Applied Mathematics and Computation* 159(2) ,559-576(2004)

[25] Wazwaz, A.M, Helal, M.A.:Nonlinear variants of the BBM equation with compact and noncompact physical structures. *Chaos, Solitons and Fractals*26(3),767-776 (2005)

[26] Wazwaz, A.M.: A sine-cosine method for handling nonlinear wave equations.*Math. and Comput. Modelling* 40, 499-508(2004)

[27] Wazwaz, A.M.: The extended tanh method for new soliton solutions for many forms of the fifth-order KdV equations. *Applied Mathematics and Computation*. In press

[28] Wazwaz, A.M.: New solitary wave solutions to modified forms of Degasperis-Procesi and Camassa-Holm equations. *Applied Mathematics and Computation*. In press

[29] Yusufoglu, E., Bekir, A.: New Exact and Explicit TravelingWave Solutions for the CD and PKP Equations. *International Journal of Nonlinear Science* 3, 8-14(2007)

[30] Yusufoglu, E., Bekir, A.: Solitons and periodic solutions of coupled nonlinear evolution equations by using Sine-Cosine method. *International Journal of Computer Mathematics* 83 (12),915-924 (2006)