.
ACADIMIC
World Academic Union

ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.4 (2007) No.1,pp.3-9

Optimal Control of the b-family Equation

Lixin Tian *, Chunyu Shen
Nonlinear Scientific Research Center, Jiangsu University
Zhenjiang, Jiangsu, 212013, P.R. China
(Received 19 April 2007, accepted 18 June 2007)

Abstract: This paper studies the problem of optimal control of the b-family equation. Result existence and uniqueness of weak solution in the interval to the b-family equation. According to variational inequality, optimal control theories and distributed parameter system control theories, it is proved that in the special Banach space, the norm of solution is related to the control item and initial value. The optimal control of the b-family equation under boundary condition is given in L^2 space, and the existence of optimal solution is proved.

Keywords: b-family equation; optimal control; optimal solution; distributed optimal control

1 Introduction

In [1], D.D.Holm and M.F.Staley introduced the b-family PDEs that described the balance between convection and stretching for small viscosity in the dynamics of 1D nonlinear wave in fluids:

$$m_t + \underbrace{um_x}_{convection} + \underbrace{bu_x m}_{stretching} = 0 \quad with \quad u = g * m \ , \tag{1.1}$$

in independent variables time t and one spatial coordinate x. Here $u = g * m$ denotes the convolution (or filtering),

$$u(x) = \int_{-\infty}^{\infty} g(x-y)m(y)dy, \tag{1.2}$$

which relates velocity u to momentum density m by integration against kernel $g(x)$ over the real line. We shall choose $g(x)$ to be an even function, so that u and m have the same parity. The family of Eq. (1.1) is characterized by the kernel $g(x)$ and the real dimensionless constant b, which is the ratio of stretching to convective transport. As we see, b is also the number of covariant dimensions associated with the momentum density m. The function $g(x)$ will determine the traveling wave shape and length scale for Eq. (1.1), while the constant b will provide a balance or bifurcation parameter for the nonlinear solution behavior. Special values of b will include the first few positive and negative integers. The quadratic terms in Eq. (1.1) represent the competition, or balance, in fluid convection between nonlinear transport and amplification due to b dimensional stretching. For example, if m is fluid momentum (a one form density in one dimension) then $b = 2$. Eq. (1.1) with $b = 2$ arises in the nonlinear dynamics of shallow water waves, as shown in ([2]) and ([5]). Eq. (1.1) with $b = 2$ and $b = 3$ appears in the theory of integrable partial differential equations ([2, 4, 5]). The three-dimensional analog of Eq. (1.1) with $b = 2$ was introduced in ([6, 7]). Applying the proper viscosity to this three-dimensional analog with $b = 2$ produces the Navier-Stokes-alpha model of turbulence ([3]). The 1D version of this turbulence model is

$$m_t + \underbrace{um_x}_{convection} + \underbrace{bu_x m}_{stretching} = \underbrace{\varepsilon m_{xx}}_{vis\cos ity} , \quad with \quad u = g * m \ . \tag{1.3}$$

***Corresponding author**. *E-mail address*: tianlx@ujs.edu.cn

IJNS.2007.08.15/087

In addition, as an important component of modern control theories, the optimal control has a wider application in modern engineering. Modern optimal control theories and applied models are both represented by ODE, which have developed perfectly. With the development and application of technology, it is necessary to solve the problem of the optimal control theories with PDE. Recently, S.Volkwein has studied the optimal control of Burgers equation using lagrangian-SQP method ([8]). Volkweins discussed the instantaneous control for the Burgers equation ([9]). The distributed control problem for Burgers equation was studied in [10]. Zhu, Tian, Zhao studied the optimal control of KdV-Burgers equation and sufficient nonlinear KdV-Burgers equation ([11, 12]). A. Armaou and P. D. Christofides studied the feedback control of Kuramto-Sivashing equation ([13]). Zhifeng Zhao studied the optimal control of Kuramto-sivashing ([14]). Bernt Oksendal proved a sufficient maximum principle for the optimal control systems which are described by a quasi-linear stochastic heat equation ([15]). [16] considered the problem of boundary optimal control of a wave equation with boundary dissipation in the way of a time-domain decomposition of the corresponding optimality system. Omar Ghattas and Jai-Hyeong Bark studied the optimal control of two and three dimensional in compressible Navier-Stokes flows ([17]). Guangcao Ji and Clyde Martin studied the optimal boundary control of the heat equation with target function at terminal time ([18]).

In this research, we are concerned with distributed control applied to the b-family equation. As a model we take the distributed optimal control problem

$$(P)\begin{cases}\min J(m,\varpi)=\frac{1}{2}\|Cm-z\|_S^2+\frac{\delta}{2}\|\varpi\|_{L^2(Q_o)}^2\\ \quad s.t.\quad m_t-\varepsilon m_{xx}+bu_xm+um_x=f+B^*\varpi \qquad in\,(0,T)\times(0,1)\\ u(0,t)=u(1,t)=u_x(0,t)=u_x(1,t)=u_{xx}(0,t)=u_{xx}(1,t)=0\\ u(0)=u_0(x)\end{cases},$$

where $m=u-\alpha^2u_{xx}$. Here the control target is to match the given desired state z in L^2-sense by adjusting the body force, ϖ in a control volume $Q_o\subseteq Q=(0,T)\times\Omega$ in the L^2-sense, i.e. with minimal energy and work, the first term in the cost functional measures the physical objective, and the second one is the size of the control, where the parameter $\delta>0$ plays the role of a weight.

This paper is organized as follows. Section 2 gives some notations. Section 3 is devoted to discuss the existence of the weak solution to the b-family equation. Section 4 is devoted to the study of (P). We discuss the optimal control of the b-family equation and prove the existence of optimal solution.

2 Notations

For fixed $T>0$, we set $\Omega=(0,1)$ and $Q=(0,T)\times\Omega$. Let $Q_o\subseteq Q$ be an open set with positive measure. Let $V=H_0^1(0,1)$, $H=L^2(0,1)$, $f\in L^2(V^*)$ and $\phi(x)\in H$. We supply V with the inner product $\langle\varphi,\psi\rangle_V=\langle\varphi_x,\psi_x\rangle_H,\forall\varphi,\psi\in V$. $V^*=H^{-1}(0,1)$ is dual space. Further, the extension operator $B^*\in L\left(L^2(Q_o),L^2(V^*)\right)$ is given by $B^*q=\begin{cases}q, inQ_o\\ 0, inQ\backslash Q_o\end{cases}$.

We denote $u(t)$ and $f(t)$ the functions $u(t,\cdot)$ and $f(t,\cdot)$ respectively, considered as functions of x only when t is fixed. Define $\|u\|_{H^m(\Omega)}=\|D^mu\|_H$, where $D^m=\frac{\partial^m}{\partial x^m},m=0,1,\cdots$. For $T>0$ the space $L^2(0,T;V)$ and $C(0,T;H)$ denote the space of square integrable and continuous functions, respectively, in the sense of Bochner from $[0,T]$ to V and $[0,T]$ to H. The space $W(0,T;V)$ is defined by $W(0,T;V)=\left\{\varphi;\varphi\in L^2(V),\varphi_t\in L^2(V^*)\right\}$, which is a Hilbert space endowed with common inner product. For brevity we write $L^2(V)$, $C(H)$ and $W(V)$ in place of $L^2(0,T;V)$, $C(0,T;H)$ and $W(0,T;V)$ respectively.

3 The existence of weak solution to the b-family equation

For a control $\varpi\in L^2(Q_o)$, the state u is given by the weak solution of the b-family equation

$$\begin{cases}u_t-\alpha^2u_{xxt}-\varepsilon\left(u_{xx}-\alpha^2u_{xxxx}\right)+(b+1)uu_x-b\alpha^2u_xu_{xx}-\alpha^2uu_{xxx}\\ \qquad\qquad =f+B^*\varpi,\qquad f+B^*\varpi\in L^2(V^*)\\ u(x,0)=u_0(x)\\ u(0,t)=u(1,t)=u_x(0,t)=u_x(1,t)=u_{xx}(0,t)=u_{xx}(1,t)=0\end{cases}.$$

For the sake of convenience, we denote $m = u - \alpha^2 u_{xx}$. Rewrite above equations as

$$\begin{cases} m_t - \varepsilon m_{xx} + bmu_x + um_x = f + B^*\varpi, f + B^*\varpi \in L^2(V^*) \\ m(x,0) = \phi(x), \phi(x) \in H \\ u(0,t) = u(1,t) = u_x(0,t) = u_x(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0 \end{cases} \tag{3.1}$$

Definition 3.1 *A function $u(x,t)$ is called a weak solution to b-family equation, if*

$$\begin{array}{l} \frac{d}{dt}\left(u - \alpha^2 u_{xx}, \varphi\right)_H - \varepsilon\left(u_{xx} - \alpha^2 u_{xxxx}, \varphi\right)_H + \left((b+1)uu_x, \varphi\right)_H - \left(b\alpha^2 u_x u_{xx}, \varphi\right)_H \\ -\left(\alpha^2 u u_{xxx}, \varphi\right)_H = \langle f + B^*\varpi, \varphi\rangle_{V^*,V}, \forall \varphi \in V, t \in [0,T] \end{array}.$$

Theorem 3.1 *With $\phi \in H, f + B^*\varpi \in L^2(V^*)$ holding, the equation (3.1) admits a unique weak solution $m(x,t) \in W(0,T;V)$ in the interval $[0,T]$.*

Proof: 1. The existence follows from the standard application of the Galerkin method [see reference 21-23].
2. Uniqueness of the solution is an immediate consequence of 1 [see reference 9]. □

Lemma 3.1 *With $f \in L^2(V^*)$ and $\phi \in H$ holding, there exists a constant $C_1 > 0$, such that $\|m\|^2_{W(V)} \le 2C_1\left[\left(\|\phi\|_H + \|f\|_{L^2(V^*)}\right)^2 + \|\varpi\|^2_{L^2(Q_o)}\right]$.*

Proof: Multiplying m on both sides of equation (3.1).Then,

$$mm_t - \varepsilon mm_{xx} + bm^2u_x + umm_x = (f + B^*\varpi)m.$$

Integrating the result equation over the interval $(0,1)$ with respect to x. Then,

$$\frac{1}{2}\frac{d}{dt}\|m\|^2_H + \varepsilon\int_0^1 m_x^2 dx + (1-2b)\int_0^1 umm_x dx = \langle f + B^*\varpi, m\rangle_{V^*,V}. \tag{3.2}$$

Since $\left|(1-2b)\int_0^1 umm_x dx\right| \le \frac{|1-2b|}{2}K_1\|u\|_V\left(\lambda_1\|m_x\|^2_H + \|m_x\|^2_H\right)$, where K_1 is embedding constant and λ_1 is poincare coefficient.

Taking inner product in (3.1) to be u on Ω, it follows that

$$\begin{array}{r} \frac{1}{2}\frac{d}{dt}\left(\|u\|^2_H + \alpha^2\|u_x\|^2_H\right) + \varepsilon\left(\|u_x\|^2_H + \alpha^2\|u_{xx}\|^2_H\right) + (2-b)\alpha^2\int_0^1 uu_xu_{xx}dx \\ = \langle f + B^*\varpi, u\rangle_{V^*,V} \end{array}.$$

Because $f + B^*\varpi \in L^2(V^*)$ is the control item, we assume $\|f + B^*\varpi\|_{V^*} \le M_1$, where M_1 is positive constant.

Since $\left|(2-b)\alpha^2\int_0^1 uu_xu_{xx}dx\right| \le K_2|2-b|\alpha^2\|u\|_{H^2}\|u\|^2_V$, where K_2 is nonnegative embedding constant.

By Young's inequality, it follows that

$$\frac{d}{dt}\left(\|u\|^2_H + \alpha^2\|u\|^2_V\right) \le \frac{(2-b)^2K_2^2}{2\varepsilon}\left(\|u\|^2_H + \alpha^2\|u\|^2_V\right)^2 + \frac{M_1^2}{2\varepsilon}.$$

Then,

$$\begin{array}{r} \|u\|^2_H + \alpha^2\|u\|^2_V \le \frac{M_1}{|2-b|K_2}\tan\left\{\frac{|2-b|K_2M_1}{2\varepsilon}t + \arctan\left[\frac{|2-b|K_2\left(\|u_0\|^2_H + \alpha^2\|u_0\|^2_V\right)}{M_1}\right]\right\} \\ \triangleq M_2^2 \quad \forall t \in [0,T],\ T < \frac{\pi\varepsilon}{|2-b|K_2M_1} \end{array}.$$

From above analysis, we know $\|u\|_H \le M_2$, $\|u\|_V \le \frac{M_2}{\alpha}\triangleq M_3$, where M_2 and M_3 are positive constant. Then we get

$$\left|(1-2b)\int_0^1 umm_x dx\right| \le M_4\|m\|^2_V,$$

where $M_4 = \frac{|1-2b|}{2} K_1 M_3 (\lambda_1 + 1)$.

Hence (3.2) change into

$$\frac{1}{2}\frac{d}{dt}\|m\|_H^2 + \varepsilon\|m\|_V^2 \le M_4\|m\|_V^2 + \langle f + B^*\varpi, m\rangle_{V^*,V}. \tag{3.3}$$

Integrating (3.3) over the interval $(0,T)$ with respect to t, thus we derive

$$\frac{1}{2}\|m(T)\|_H^2 - \frac{1}{2}\|\phi\|_H^2 + (\varepsilon - M_4)\|m\|_{L^2(V)}^2 \le \int_0^T \langle f + B^*\varpi, m\rangle_{V^*,V} dt,$$

where $\varepsilon > M_4$.

From Holder's inequality, gives

$$\int_0^T \langle f + B^*\varpi, m\rangle_{V^*,V} dt \le \|f + B^*\varpi\|_{L^2(V^*)}\|m\|_{L^2(V)}.$$

Then,

$$\|m\|_H^2 - \|\phi\|_H^2 + 2(\varepsilon - M_4)\|m\|_{L^2(V)}^2 \le 2\|f + B^*\varpi\|_{L^2(V^*)}\|m\|_{L^2(V)}.$$

From Young's inequality, we can derive.

$$\|m\|_{L^2(V)}^2 \le C_0\left(\|\phi\|_H + \|f + B^*\varpi\|_{L^2(V^*)}\right)^2, \tag{3.4}$$

where $C_0 = \max\left\{\frac{1}{\varepsilon - M_4}, \frac{1}{(\varepsilon - M_4)^2}\right\}$ and $\varepsilon > M_4$.

Due to (3.3), it follows that

$$\frac{1}{2}\frac{d}{dt}\|m\|_H^2 \le M_4\|m\|_V^2 + \langle f + B^*\varpi, m\rangle_{V^*,V}.$$

Integrating above inequality over the interval $(0,T)$ with respect to t, thus we derive

$$\begin{aligned} \|m\|_H^2 &\le \|\phi\|_H^2 + 2\|f + B^*\varpi\|_{L^2(V^*)}\|m\|_{L^2(V)} + 2M_4\|m\|_{L^2(V)}^2 \\ &\le 2\max\left(1, 2\sqrt{C_0}, 2C_0M_4\right)\left(\|\phi\|_H + \|f + B^*\varpi\|_{L^2(V^*)}\right)^2 \end{aligned}.$$

By (3.1) we have

$$\|m_t\|_{V^*} \le \|f + B^*\varpi\|_{V^*} + \varepsilon\|m\|_V + b\|m\|_V\|u\|_H + \|u\|_V\|m\|_H.$$

Since $\|u\|_H \le M_2, \|u\|_V \le M_3$. We can infer.

$$\|m_t\|_{V^*}^2 \le 3\|f + B^*\varpi\|_{V^*}^2 + 3(\varepsilon + bM_2)^2\|m\|_V^2 + 3M_3^2\|m\|_H^2 \quad . \tag{3.5}$$

Integrating (3.5) over the interval $(0,T)$, then

$$\|m_t\|_{L^2(V^*)}^2 \le 3\|f + B^*\varpi\|_{L^2(V^*)}^2 + 3(\varepsilon + bM_2)^2\|m\|_{L^2(V^*)}^2 + 3M_3^2T\|m\|_H^2. \tag{3.6}$$

From (3.4) and (3.6), we can get

$$\begin{aligned} \|m\|_{W(V)}^2 &= \|m\|_{L^2(V)}^2 + \|m_t\|_{L^2(V^*)}^2 \\ &\le 2C_1\left[\left(\|\phi\|_H + \|f\|_{L^2(V^*)}\right)^2 + \|B^*\varpi\|_{L^2(V^*)}^2\right] \\ &\le 2C_1\left[\left(\|\phi\|_H + \|f\|_{L^2(V^*)}\right)^2 + \|\varpi\|_{L^2(Q_0)}^2\right] \end{aligned},$$

where $C_1 = \left[C_0 + 3 + 3(\varepsilon + bM_2)^2 C_0 + 6M_3^2T \cdot \max\left(1, 2\sqrt{C_0}, 2C_0M_4\right)\right]$. $\square$

4 The distributed optimal control of the b-family equation

Let a control $\varpi \in L^2(Q_o), m \in W(V)$ be a weak solution to (3.1).

There exists a weak solution m to (3.1) from Theorem 1. Then we get weak solution u, where $m = u - \alpha^2 u_{xx}$.

Given an observation operator $C \in L(W(V), S)$,in which S is a real Hilbert space and C is continuous. Choosing performance index of tracking type, we obtain

$$J(m, \varpi) = \frac{1}{2}\|Cm - z\|_S^2 + \frac{\delta}{2}\|\varpi\|_{L^2(Q_o)}^2,$$

where $z \in S$ is a desired state and $\delta > 0$ is fixed.

Control problem about (3.1) is

$$\min J(m, \varpi), where\, (m, \varpi)\, satisfies (3.1). \tag{4.1}$$

Set $X = W(V) \times L^2(Q_o)$ and $Y = L^2(V) \times H$.

We define an operator

$$e = e(e_1, e_2) : X \to Y.\, e(m, \varpi) = \begin{bmatrix} G \\ m(x, 0) - \phi(x) \end{bmatrix},$$

where $G = (-\Delta)^{-1}(m_t - \varepsilon m_{xx} + bu_x m + um_x - f - B^*\varpi)$ and Δ is an operator from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$.

Then we write (4.1) in following form

$$\min J(m, \varpi)\, subject\ to\, e(m, \varpi) = 0.$$

Theorem 4.1 *There exists an optimal control solution of problem (P).*

Proof: Let $(m, \varpi) \in X$ satisfy the equation $e(m, \varpi) = 0$. We have

$$J(m, \varpi) \geq \frac{\delta}{2}\|\varpi\|_{L^2(Q_o)}^2.$$

From Lemma 3.1, we can conclude that $\|m\|_{W(V)} \to \infty$yields $\|\varpi\|_{L^2(Q_o)} \to \infty$.

Hence,

$$J(m, \varpi) \to +\infty, when\, \|(m, \varpi)\|_X \to \infty. \tag{4.2}$$

As the norm is weakly lowered semi-continuous, we achieve that J is weakly lowered semi-continuous. Since $J(m, \varpi) \geq 0$, for all $(m, \varpi) \in X$ holds, there exists $\zeta \geq 0$ with

$$\zeta = \inf\left\{J(m, \varpi)\,\middle|\,(m, \varpi) \in X \quad with\ \ e(m, \varpi) = 0\right\}.$$

This implies the existence of a minimizing sequence $\{(m^n, \varpi^n)\}_{n\in N}$ in X such that

$$\zeta = \lim_{n\to\infty} J(m^n, \varpi^n)\, and\, e(m^n, \varpi^n) = 0, for\ all\ n \in \mathrm{N}.$$

Due to (4.2), there exists an element $(m^*, \varpi^*) \in X$ with

$$m^n \overset{weak}{\longrightarrow} m^*, n \to \infty, m \in W(V) \quad , \tag{4.3}$$

$$\varpi^n \overset{weak}{\longrightarrow} \varpi^*, n \to \infty, \varpi \in L^2(Q_o). \tag{4.4}$$

We can infer from (4.3) that

$$\lim_{n\to\infty}\int_0^T (m_t^n(t) - m_t^*, \varphi(t))_{V^*,V}\, dt = 0, \quad \forall\varphi \in L^2(V) \quad .$$

Since $W(V)$ is compactly embedded into $L^2(L^\infty)$, we have $m^n \to m^*$ strongly in $L^2(L^\infty)$, as $n \to \infty$. Since $W(V)$ is continuously embedded into $C(H)$, we can derive that $m^n \to m^*$ strongly in $C(H)$, as $n \to \infty$. We can infer $u^n \to u^*$ strongly in $C(H)$ also, as $n \to \infty$.

As the sequence $\{m^n\}_{n\in N}$ converges weakly, $\|m^n\|_{W(V)}$ is bounded. $\|m^n\|_{L^2(L^\infty)}$ is also bounded. Since $m^n \to m^*$ strongly in $L^2(L^\infty)$, we can infer that $\|m^*\|_{L^2(L^\infty)}$ is bounded.

Thus, it follows by Holder's inequality that

$$\begin{array}{l} \left|\int_0^T\int_0^1 (m^n u_x^n - m^* u_x^*)\,\varphi dxdt\right| \\ \leq \left|\int_0^T\int_0^1 (m^n - m^*)\,u_x^n \varphi dxdt\right| + \left|\int_0^T\int_0^1 m^*(u_x^n - u_x^*)\,\varphi dxdt\right| \\ \leq \|m^n - m^*\|_{L^2(L^\infty)}\|u^n\|_{C(H)}\|\varphi\|_{L^2(V)} + \|u^n - u^*\|_{C(H)}\|m^*\|_{L^2(L^\infty)}\|\varphi\|_{L^2(V)} \to 0 \end{array},$$

as $n \to \infty$, for $\forall\varphi \in L^2(V)$.

$$\begin{array}{l} \left|\int_0^T\int_0^1 (u^n m_x^n - u^* m_x^*)\,\varphi dxdt\right| \\ \leq \left|\int_0^T\int_0^1 (u_x^* m^* - u_x^n m^n)\,\varphi dxdt\right| \\ \quad + \|u^* - u^n\|_{C(H)}\|m^*\|_{L^2(L^\infty)}\|\varphi\|_{L^2(V)} + \|u^n\|_{C(H)}\|m^* - m^n\|_{L^2(L^\infty)}\|\varphi\|_{L^2(V)} \\ \to 0 \end{array},$$

as $n \to \infty$, for $\forall\varphi \in L^2(V)$.

From (4.4), we can infer $\left|\int_0^T\int_0^1 (B^*\varpi^n - B^*\varpi^*)\,\varphi dxdt\right| \stackrel{n\to\infty}{\longrightarrow} 0$,for all $\varphi \in L^2(V)$.

Based on the above discussion, we can conclude that

$$e_1(m^*, \varpi^*) = 0 \quad \forall n \in \mathrm{N}.$$

From $m^* \in W(V)$, we can derive that $m^*(0) \in H$.

Since $m^n \stackrel{weak}{\longrightarrow} m^*$ in $W(V)$, we can infer $m^n(0) \stackrel{weak}{\longrightarrow} m^*(0)$, when $n \to \infty$.

Thus we have

$$\langle m^n(0) - m^*(0), \psi\rangle_H \to 0, (n \to \infty), \forall\psi \in H.$$

Then we can get

$$e(m^*(0), \varpi^*(0)) = 0.$$

There exists an optimal solution (m^*, ϖ^*) to problem (P). In the meantime, from $m = u - \alpha^2 u_{xx}$, we can derive b-family equation have optimal solution (u^*, ϖ^*). □

Acknowledgements

Research was supported by the National Nature Science Foundation of China (No: 90610031) and Outstanding Personnel Program in Six Fields of Jiangsu(No:6-A-029) and Teaching and Research Award Program for Outstanding Young Teachers in Higher Education Institutions of POE, China(No:2002-383).

References

[1] Darryl D.Holm, Martin F.Staley: Wave structures and nonlinear balances in a family of 1+1 evolutionary PDEs. *SIAM J Applied Dynamical System*. 2(3): 323-380 (2003)

[2] R.Camassa and D.D.Holm: An integrable shallow water equation with peaked solitons. *Phys.Rev.Lett.* 71: 1661-1664 (1993)

[3] S.Chen, C.Foias, D.D.Holm, E.J.Olson, E.S.Titi and S.Wynne: The Camassa-Holm equations as a closure model for turbulent channel and pipe flows. *Phys.Rev.Lett.* 81: 5338-5341 (1998)

[4] A.Degasperis, D.D.Holm and A.N.W.Hone: A new integrable equation with peakon solutions. Submitted to *Theo.Math.Phys.* 133: 1463-1474 (2002)

[5] H.Dullin, G.Gottwald and D.D.Holm: An integrable shallow water equation with linear and nonlinear dispersion. *Phys.Rev.Lett.* 87: 194501-04 (2001)

[6] D.D.Holm, J.E.Marsden and T.S.Ratiu: The Euler–Poincar/e equations and semidirect products with applications to continuum theories. *Adv.in Math.* 137: 1-81 (1998)

[7] D.D.Holm, J.E.Marsden and T.S.Ratiu: Euler–Poincar/e models of ideal fluids with nonlinear dispersion. *Phys.Rev.Lett.* 80: 4173-4177 (1998)

[8] S.Volkwein: "Mesh-independence of an augmented lagrangian-SQP method in Hilbert spaces and control problems for burgers equation"*PhD thesis*, Department of mathematics, Technical university of Berlin .October(1997)

[9] M Hinze, S Volkwein: Analysis of instantaneous control for the Burgers equation. *Nonlinear Analysis*. 50 (1): 1-26 (2002)

[10] S.Volkwein: Distributed control problems for the Burgers equation.1999.*Computational Optimization and Applications*. 16: 57-81 (2000)

[11] Min Zhu, Lixin Tian, Zhifeng Zhao.: Optimal control of KdV-burgers equation .*Journal of Jiangsu University (Natural Science Edition)*. 25(3): 235-238 (2004)

[12] Zhifeng Zhao and Lixin Tian: Optimal control of sufficient nonlinear KdV-burgers equation.*Journal of Jiangsu University (Natural Science Edition)*. 26(2): 140-143 (2005)

[13] A Armaou, P D christofides: Feedback control of kuramto-sivashing equation, *physice D.* 137: 49-61 (2000)

[14] Zhifeng Zhao: Optimal control of Kuramto-Sivashing equation. *International Journal of Nonlinear Science.* 1 (1): 54-58 (2006)

[15] Bernt Oksendal: Optimal control of stochastic partial differential equations. *Stochastic Analysis and Applications*. 23(1): 165-179 (2005)

[16] J.E.Lagnese.G.Leugering: Time-domain decomposition of optimal control problems for the wave equation. *System &Control Letters*. 48: 229-242 (2003)

[17] Omar Ghattas and Jai-Hyeong Bark: Optimal control of two- and three-dimensional in compressible Navier-Stokes flows. *Journal of Computational Physics*. 136: 231-244 (1997)

[18] GuangCao Ji, Clyde Martin: Optimal boundary control of the heat equation with target function at terminal time. *Applied Mathematics and Computation*. 127: 335-345 (2002)

[19] A.Pazy: Semigroups of linear operators and applications to partial differential equation . NewYork:*Spring-verlag* (1983)

[20] Adams RA: Sobolev space, *Academic Press*. New York (1975)

[21] M.Marion and R.Teman: Nonlinear Galerkin methods. *SIAMJ.Numer,Anal.* 26: 1139-1157 (1989)

[22] Danping Ding, Lixin Tian, Gang Xu and Xuedi Wang: The nonlinear Galerkin approximate solution of the dissipatve Camassa-Holm equation. *Far East J.Appl.Math.* 22(3): 249-263 (2006)

[23] Gongqin Zhang: Functional Analysis. *Peking University Press*. (2005)