Abstract: A direct Ritz method for solving variational problems with fixed and free boundary conditions is demonstrated. Walsh-hybrid method bases are used as the basis functions. It is shown how the form of the operational matrix of integration affects accuracy of the obtained approximate solutions. The properties of the Walsh-hybrid functions with the operational matrix of integration and the cross product of two Walsh-hybrid function vectors are utilized to reduce a variational problem to the solution of algebraic equations. The method is computationally attractive and applications are demonstrated through illustrative examples.

Keywords: Walsh-hybrid; variational problems; direct method; orthogonal functions; algebraic equations

1 Introduction

The orthogonal functions are used to approximate given functions are well known in engineering mathematics. This technique has been set up to provide solutions to many problems in engineering fields such as heat transfer and control systems. The most attractive characteristic of this technique is that it can reduce the equations of a problem to algebraic equations, thus greatly simplifying the analysis. The available sets of orthogonal functions can be divided into three classes. The first is the set of piecewise constant basis functions (e.g., Haar, block-pulse, Walsh, etc.). The second consists of the set of orthogonal polynomials (e.g., Laguerre, Legendre, Chebyshev, Hermite, etc.). The third is the widely used set of sine-cosine functions in Fourier series. In these methods, a truncated orthogonal series is used for variational problem, with the goal of obtaining efficient computational solutions. Typical examples are the rationalized Haar functions [1, 2], block-pulse functions [3], Walsh functions [4], Laguerre polynomials [5], Legendre polynomials [6], Chebyshev polynomials [7], Fourier series [8] and Hartley series [9].

In this paper we use Walsh-hybrid functions [10] for solving variational problems, which are the combination of block-pulse functions and Walsh functions. The Walsh function was initiated by Rademacher [11] and independently developed by Walsh [12] in the early nineteen twenties. In recent decade, the Walsh theory has been innovated and applied to various problems in engineering and science. In the present work we apply Walsh-hybrid functions to finding the extremum of the functional

$$J(x) = \int_0^1 F[t, x(t), \dot{x}(t)] dt.$$ (1)

The method consist of expanding the $\dot{x}(t)$ by Walsh-hybrid functions with unknown coefficient. The properties of the Walsh-hybrid functions are then utilized to reduce the variational problems to solving algebraic equations.

The paper is organized as follows: Section 2, is devoted to the basic formulation of the Walsh functions required for our subsequent development. In section 3 the proposed numerical method is applied to obtain the numerical solution of variational problems. In section 4 the numerical findings are reported and the accuracy of the proposed method is demonstrated.

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2 Properties of Walsh-hybrid functions

2.1 Walsh-hybrid functions

Walsh-hybrid functions \( h_{nm}(t), n = 1, 2, \ldots, N, \ m = 0, 1, \ldots, M - 1, \) have three arguments, \( n, m \) are the order for block-pulse and Walsh functions respectively and \( t \) is the normalized time (Also note that \( N \) is fixed). They are defined on the interval \([0, 1]\) as [10]

\[
h_{nm}(t) = \begin{cases} 
\omega_m(Nt + 1 - n), & t \in \left[\frac{n-1}{N}, \frac{n}{N}\right), \\
0, & \text{otherwise}, 
\end{cases}
\]

(2)

Here, \( \omega_m(t) \) are the Walsh functions of order \( m \) which are orthogonal in the interval \([0, 1]\) and satisfy [13]

\[
\omega_m(t) = \prod_{i=1}^{r} r_{m_i}(t), \quad \omega_0(t) = 1,
\]

(3)

where

\[
m = \sum_{i=1}^{r} 2^{m_i}, \quad m = 1, 2, \ldots, M - 1, \quad M = 2^\alpha, \quad \alpha = 1, 2, \ldots.
\]

Here, \( r_i(t) \) are the Rademacher functions of order \( i \), which are orthogonal in the interval \([0, 1]\), but are not a complete orthogonal set in \( L^2[0, 1] \) space and satisfy the conditions given by [13]

\[
r_i(t) = \text{sgn}(\sin((2^{i+1}+1)\pi t)), \quad i = 0, 1, 2, \ldots,
\]

(4)

where \( \text{sgn} \) denotes the sign function, \( \text{sgn}(t) \) gives \(-1, 0, \) or \(1\) depending on whether \( t \) is negative, zero, or positive. Since \( h_{nm}(t) \) is the combination of Walsh functions and block-pulse functions which are both complete and orthogonal, thus the set of hybrid functions are complete orthogonal set. The orthogonality property is given by

\[
\int_0^1 h_{nm}(t) h_{n'm'}(t) dt = \begin{cases} 
\frac{\pi}{N}, & n = n', m = m' \\
0, & \text{otherwise}
\end{cases}
\]

(5)

where \( n' = 1, 2, \ldots, N \) and \( m' = 0, 1, 2, \ldots, M - 1, \)

2.2 Function approximation

A function \( f \) in \( L^2[0, 1] \) space may be expanded in terms of hybrid functions as

\[
f(t) = \sum_{m=0}^{+\infty} \sum_{n=1}^{N} c_{nm} h_{nm}(t),
\]

(6)

where \( c_{nm} \) are given by

\[
c_{nm} = N \int_0^1 f(t) h_{nm}(t) dt.
\]

(7)

The series in Eq.(6) contains an infinite number of terms. If we let \( m = 0, 1, 2, \ldots, M - 1 \) then the infinite series in Eq.(6) is truncated up to its first \( NM \) terms as

\[
f(t) \approx \sum_{m=0}^{M-1} \sum_{n=1}^{N} c_{nm} h_{nm}(t) = C^T H(t),
\]

(8)

where

\[
C = [c_{10}, c_{11}, \ldots, c_{1M-1}|c_{20}, c_{21}, \ldots, c_{2M-1}|\cdots|c_{N0}, c_{N1}, \ldots, c_{NM-1}]^T,
\]

(9)

and

\[
H(t) = [h_{10}(t), h_{11}(t), \ldots, h_{1M-1}(t)|h_{20}(t), h_{21}(t), \ldots, h_{2M-1}(t)|\cdots|h_{N0}(t), h_{N1}(t), \ldots, h_{NM-1}(t)]^T.
\]

(10)

Also, the integration of the cross product of two vectors \( H(t) \) in Eq.(10) is

\[
D = \int_0^1 H(t) H^T(t) dt = \frac{1}{N} \text{diag}(I, I, \ldots, I),
\]

(11)

where \( D \) is an \( NM \times NM \) diagonal matrix and \( I \) is \( M \times M \) identity matrix.
2.3 Operational matrix of integration

The integration of the vector $H(t)$ defined in Eq.(10) is given by [10]

$$
\int_0^t H(t')dt' \simeq PH(t),
$$

(12)

where $P$ is the $NM \times NM$ operational matrix for integration given by

$$
P = \frac{1}{N} \begin{bmatrix}
P_w & E & E & \cdots & E & E \\
O & P_w & E & \cdots & E & E \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
O & \cdots & \cdots & \cdots & P_w & E \\
O & \cdots & \cdots & \cdots & O & P_w
\end{bmatrix}.
$$

Also $E$ is the $M \times M$ matrix represented by

$$
E = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & 0
\end{bmatrix},
$$

and $P_w$ is the $M \times M$ operational matrix for Walsh functions given in [14] as

$$
P_w = (P_w)_{M \times M} = \begin{bmatrix}
(P_w)_{M \times M} & -\frac{1}{2M}I_{M \times M} \\
-\frac{1}{2M}I_{M \times M} & O
\end{bmatrix},
$$

where $(P_w)_{1 \times 1} = [\frac{1}{2}]$ and $I_{M \times M}$ is the $M \times M$ identity matrix.

3 Walsh-hybrid direct method

Consider the variational problem given in Eq. (1). The necessary condition for $x(t)$ to extremize $J(x)$ is that it should satisfy the Euler-Lagrange equation [15]:

$$
\frac{\partial F}{\partial x} - \frac{d}{dt} \left( \frac{\partial F}{\partial \dot{x}} \right) = 0
$$

with appropriate boundary conditions. However, the above differential equation can be integrated easily only for simple cases. Thus numerical and direct methods such as the well-known Ritz and Galerkin methods have been developed to solve variational problems. Here we consider a Ritz direct method for solving Eq. (1) using Walsh-hybrid functions.

Suppose, the rate variable $\dot{x}(t)$ can be expressed approximately as

$$
\dot{x}(t) \simeq \sum_{m=0}^{M-1} \sum_{n=1}^{N} c_{nm} \dot{h}_{nm}(t) = C^T H(t),
$$

(13)

using Eq. (12), $x(t)$ can be represented as

$$
x(t) = \int_0^t \dot{x}(t')dt' + x(0) \simeq C^T PH(t) + X_0 H(t) = (C^T P + X_0) H(t),
$$

(14)

where

$$
X_0 = [x(0), 0, \cdots, 0|x(0), 0, \cdots, 0|0, \cdots, 0|0, \cdots, 0]_T.
$$

we can also express $t$ in terms of $H(t)$ as

$$
t \simeq d^T H(t),
$$

(15)
where
\[ d^T = \frac{1}{N} (d_{1,M}, d_{2,M}, \cdots, d_{N,M}), \]
and
\[ d_{n,M} = \begin{pmatrix} d_{n,\frac{M}{4}}, -\frac{1}{2M}, 0, \cdots, 0 \\ 0 \end{pmatrix}, \quad d_{n,2} = \left( \frac{2n-1}{2}, -\frac{1}{4} \right), \quad n = 1, 2, \cdots, N. \]

For \( N = 4, M = 8, d^T \) is
\[ d^T = \frac{1}{4} \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & \frac{1}{2} & 0 \\ -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} & 0 \\ \frac{3}{2} & -\frac{1}{4} & \frac{3}{2} & 0 \\ -\frac{1}{4} & \frac{3}{2} & -\frac{1}{4} & 0 \end{bmatrix}, \]
Substituting Eqs. (13)-(15) in Eq. (1) the functional \( J(x) \) becomes a function of \( c_{nm}, n = 1, 2, \cdots, N, m = 0, 1, \cdots, M-1. \)

Here, to find the extremum of \( J(x) \) we find
\[ \frac{\partial J}{\partial c_{nm}} = 0, \quad n = 1, 2, \cdots, N, \quad m = 0, 1, \cdots, M-1. \] (16)

The above procedure is now used to solve the following variational problems. It is noted from Eq. (14) is gotten
\[ x(t) = C^T \int_0^t H(t')dt' + x(0), \]
since \( \int_0^t H(t')dt' \) is a continuous vector hence we get a continuous solution for \( x(t) \).

## 4 Illustrative examples

### 4.1 Example 1

Consider the problem of finding the minimum of the functional \([2,5,6]\]
\[ J(x) = \int_0^1 (\dot{x}^2 + t\ddot{x} + x^2)dt, \] (17)
with boundary conditions
\[ x(0) = 0, \quad x(1) = \frac{1}{4}. \] (18)

The equation (17) together with boundary conditions (18) has the exact solution like this:
\[ x(t) = \frac{(e^{-t} - 1)(e - 2e^2 - 2t + e^t + 1)}{4(e^2 - 1)}. \]

Using Eqs. (13)-(15) in Eq. (17) and using Eq. (11) give
\[ J(x) = \int_0^1 \left[ C^T H(t)H^T(t)C + C^T H(t)H^T(t)d + C^T PH(t)H^T(t)P^T C \right]dt \]
\[ \simeq C^T DC + C^T Dd + C^T PD^T P^T C. \] (19)

Also, using Eq. (14) and the boundary conditions in Eq. (18), we obtain
\[ C^T d_0 - \frac{1}{4} = 0, \] (20)
where
\[ d_0 = \int_0^1 H(t) dt = [1, 0, \cdots, 0] [1, 0, \cdots, 0].^T \]

We now minimize Eq. (19) subject to Eq. (20) using the Lagrange multiplier technique. Suppose
\[ \bar{J}(x) = J(x) + \lambda \left( C^T d_0 - \frac{1}{4} \right), \]
where \( \lambda \) is the Lagrange multiplier. Using Eq. (16), we get
\[ \frac{\partial \bar{J}}{\partial C} = 0, \quad \frac{\partial \bar{J}}{\partial \lambda} = 0 \quad \text{or} \quad 2DC + Dd + 2PDP^T C + \lambda d_0 = 0, \quad C^T d_0 - \frac{1}{4} = 0. \quad (21) \]

We applied the method presented in this paper and solved Eq. (21). Table 1 shows the approximate solution obtained by using the rationalized Haar method [2], together with the results from the present method with \( N = 4, M = 4, 8 \) and the exact solution. This table also shows that by increasing \( M \) the accuracy of solution will increase.

**Table 1. Approximate and exact solutions for Example 1.**

<table>
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<tr>
<th>t</th>
<th>Method of [2] for ( k = 8 )</th>
<th>Present method for ( N = 4, M = 4 )</th>
<th>Present method for ( N = 4, M = 8 )</th>
<th>Exact</th>
</tr>
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<tbody>
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<td>0</td>
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</table>

### 4.2 Example 2

Let us consider the problem of searching for the minimum of the functional [16]
\[ J(x) = \int_0^1 (\dot{x}^2 + 2x\ddot{x} + 4x^2) dt, \quad (22) \]
with the following boundary conditions
\[ x(0) = 1, \quad x(1) = \text{Free}. \quad (23) \]
As it can be easily verified the exact solution of this problem is
\[ x(t) = \frac{3e^{4-2t} - e^{2t}}{3e^4 - 1}. \]
Using Eqs. (13)-(15) in Eq. (22) and using Eq. (11) we obtain
\[ J(x) = \int_0^1 \left[ C^T H(t)H^T(t)C + 2(C^T P + X_0)H(t)H^T(t)C + 4(C^T P + X_0)H(t)H^T(t)(C^T P + X_0)^T \right] dt \]
\[ \approx C^T DC + 2(C^T P + X_0)DC + 4(C^T P + X_0)D(C^T P + X_0)^T. \quad (24) \]
In this example \( X_0 = d_0^T \) and boundary conditions (23) are resulted [16]:
\[ \frac{\partial F}{\partial \dot{x}} \bigg|_{t=1} = 0 \quad \text{or} \quad \dot{x}(1) + x(1) = 0. \quad (25) \]
Using Eqs. (13)-(14) in Eq. (25) gives
\[ C^T (H(1) + d_0) + 1 = 0. \]  
(26)

We now minimize Eq. (24) subject to Eq. (26) using the Lagrange multiplier technique. Suppose
\[ J(x) = J(x) + \lambda (C^T (H(1) + d_0) + 1) , \]

where \( \lambda \) is the Lagrange multiplier. We applied the method presented in section 3 and solved this example. The computational results for \( N = 4, 8, M = 8 \) together with the exact solution \( x(t) \) are given in Table 2. Table 2 shows that by increasing \( N \) the accuracy of solution will increase.

Table 2. Approximate and exact solutions for Example 2.

<table>
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<th>Present method for ( N = 4, M = 8 )</th>
<th>Present method for ( N = 8, M = 8 )</th>
<th>Exact</th>
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</table>

5 Conclusion

The Walsh-hybrid operational matrix of integration \( P \) together with the integration of the product of two Walsh-hybrid function vectors \( D \) are applied to solve variational problems. The method is based upon reducing the system into a set of algebraic equations. The matrices \( P \) and \( D \) have many zeros; hence this method is much faster than rationalized Haar method [2], reducing the required CPU time and memory, while retaining the accuracy of the solution. The numerical examples support this claim.

Acknowledgements

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References


IJNS homepage: http://www.nonlinearscience.org.uk/