Active Anti–Synchronization of two Identical and Different Fractional–Order Chaotic Systems

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Abstract: Fractional order dynamical systems have attracted increasing interest in recent years. In this particular paper, the anti-synchronization behavior of the fractional–order chaotic systems is intricately explored by using an active control method. The sufficient conditions for achieving the anti-synchronization of two fractional–order chaotic systems are derived based on the stability theory of fractional–order systems. Theoretical analysis and numerical simulations are shown to verify the results.

Keywords: Active control; Anti–synchronization; Fractional Lorenz system; Fractional Liu system.

1 Introduction

Chaotic nonlinear systems appear ubiquitously in nature and can occur in man-made systems. These systems are recognized by their great sensitivity to initial conditions. In recent studies, chaos research has expanded to the applications of dynamical systems in fractional–order. The theory of fractional order derivative was developed mainly in the 19th century and fractional calculus has enabled the operations of integration and differentiation to any fractional order.

Many physical systems are known to display fractional order dynamics. Some examples are viscoelastic systems, colored noise, electrode-electrolyte polarization and electromagnetic waves. Due to this fact and the numerous potential applications, many scientists have worked towards achieving the synchronization of different fractional–order chaotic systems. So far, a wide variety of approaches have been proposed for the synchronization of the fractional–order chaotic. Some of these proposed methods are, the unidirectional linear error feedback coupling [1], the one-way coupling scheme [2, 3], the Pecora-Carroll (PC) method [4], the active sliding mode controller [5], a scalar transmitted signal [6, 7] and the active control method [8]. The aim of this work is to further develop the state observer method for constructing anti-synchronized slave system for fractional–order chaotic systems by using the active control method.

The outline of the rest of the paper is organized as follows. Firstly, Section 2 provides a brief review of the anti-synchronization theory for the fractional–order systems and also the systems’ description. Next, in Section 3, the active control method is applied to different fractional–order chaotic systems by using the active control method. The outline of the rest of the paper is organized as follows. Firstly, Section 2 provides a brief review of the anti-synchronization theory for the fractional–order systems and also the systems’ description. Next, in Section 3, the active control method is applied to different fractional–order chaotic systems. In Section 4, the active control method is then applied to anti-synchronize two different fractional–order chaotic systems, namely the fractional–order chaotic Lorenz and Liu systems. Finally, Section 5 provides a summary of our results.

2 Preliminaries and system’s description

2.1 Active fractional anti–synchronization

The idea of fractional integrals and derivatives has been known since the development of the regular calculus, with the first reference probably being associated with Leibniz in 1695. There are several definitions of fractional derivatives. The Caputo derivative [9] is a time domain computation method. In real applications, the Caputo derivative is more popular.
since the un-homogenous initial conditions are permitted if such conditions are necessary. Furthermore, these initial values are prone to measure since they all have idiographic meanings. The Caputo derivative definition is given by

$$\frac{d^n f(t)}{dt^n} = J^{n-\alpha} \frac{d^\alpha f(t)}{dt^\alpha}$$

(1)

where \( n \) is the first integer which is not less than \( \alpha \) and \( J^\alpha \) is the \( \alpha \)-order Riemann-Liouville integral operator which is described as follows:

$$J^\alpha \varphi(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\varphi(\tau)}{(t-\tau)^{\alpha-1}} \, d\tau$$

(2)

where \( \Gamma(\cdot) \) is the gamma function, \( 0 < \theta \leq 1 \). Consider the drive fractional–order chaotic system in the form of

$$\frac{d^n x}{dt^n} = Ax + Bf(Cx) + G$$

(3)

where \( x \in \mathbb{R}^{n \times 1}, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{1 \times n} \) and \( G \in \mathbb{R}^{n \times 1} \) are continuous matrixes, let \( f : \mathbb{R}^1 \rightarrow \mathbb{R}^m \) is a nonlinear function, \( 0 < \alpha \leq 1 \). On the other hand, the response system is assumed by

$$\frac{d^n y}{dt^n} = Ay + Bf(Cy) + G + U(t)$$

(4)

where \( y \in \mathbb{R}^{m \times 1}, \) and \( U(t) \) is the input active controller to be determined for the purpose of anti–synchronizing two identical fractional–order chaotic systems. In order to choose certain controller, we add (4) to (3). It is convenient to define the state errors between the two fractional–order chaotic systems by using \( e(t) = y(t) - x(t) \). Using this notation, we obtain

$$\frac{d^n e}{dt^n} = Ae(t) + BE(y, x) + U(t)$$

(5)

where \( E(y, x) \) can directly be determined from Eq. (4) and (5). Then let the control function to be

$$U(t) = H(y, x) + Me(t)$$

(6)

where \( H(y, x) \) is determined by \( BE(y, x) \), and \( M \) is the control parameter matrix. For this end, we propose the following Theorem.

**Theorem 1** In \( m \)-dimensional fractional system, if all the eigenvalues \((\lambda_1, \lambda_2, \ldots, \lambda_m)\) of the Jacobian matrix of some equilibrium point satisfy

$$|\text{arg}(\lambda_i)| > \frac{\beta \pi}{2}, \quad \beta = \max(\alpha_1, \alpha_2, \ldots, \alpha_m), \quad (i = 1, 2, \ldots, m)$$

(7)

then the fractional–order system is asymptotically steady at the equilibrium.

The stability of fractional differential equations with application to control processing was undertaken by Denis Matignon [10], where he proposed that the stabilities are guaranteed if the roots of some polynomial (the eigenvalues of the matrix of dynamics or the poles of the transfer matrix) lie outside the closed angular sector \( |\text{arg}(\lambda_i)| > \beta \pi/2 \), which implies that if \( |\text{arg}(\lambda_i)| > \beta \pi/2, \beta = \max(\alpha_1, \alpha_2, \ldots, \alpha_m)\), \((i = 1, 2, \ldots, m)\) is satisfied, then the eigenvalues of the matrix must lie outside the closed angular sector \( |\text{arg}(\lambda_i)| > \beta \pi/2 \), then the stabilities are guaranteed. Figure (1) illustrates Theorem 1. If one equilibrium is stable, then the fractional–order system will be steady at one point. When no equilibrium is stable, then the fractional–order system is considered to be in chaos. It is obvious that zero point is an equilibrium of Eq. (5). Here, the fractional order is always less than 1. If the value of \( M \) can satisfy all latent roots \((\lambda_1, \lambda_2, \ldots, \lambda_m)\) of the Jacobian matrix of Eq. (5) at zero with the value of \(-1\), the condition in (7) can be satisfied. So Eq. (5) is asymptotic stable according to Theorem 1. The error system in (5) can finally be steady, i.e., the drive system in (3) and the response system in (4) can achieve asymptotic anti–synchronization.
2.2 System’s description

The Chen system [11], introduced by Chen and Ueta in 1999, is a chaotic system with a double scroll attractor. The fractional–order Chen system is described by

\[
\begin{align*}
\frac{d^\alpha x}{dt^\alpha} &= a(y - x), \\
\frac{d^\alpha y}{dt^\alpha} &= (c - a)x - xz + cy, \\
\frac{d^\alpha z}{dt^\alpha} &= xy - bz.
\end{align*}
\]

where \(0 < \alpha \leq 1\). When \(\alpha = 1\), system (8) is the original integer order of the Chen system, which is chaotic when \(a = 35, b = 3\) and \(c = 28\). When \(\alpha\) is selected as \(\alpha = 0.9\), then system (8) will display a chaotic attractor [12], the chaotic attractor is shown in Fig. (2)–(a).

In 2003, I. Grigorenko and E. Grigorenko [13] introduced a generalization of the Lorenz dynamical system using fractional derivatives. This system can have an effective non–integer dimension of \(\sum\), which is defined as a sum of the orders of all involved derivatives. They found that the system with \(\sum < 3\) can exhibit chaotic behaviour. A striking finding is that there is a critical value of the effective dimension \(\sum_{cr}\), under which the system undergoes a transition from chaotic dynamics to a regular one. This system can be described by

\[
\begin{align*}
\frac{d^\alpha x}{dt^\alpha} &= \sigma(y - x), \\
\frac{d^\alpha y}{dt^\alpha} &= \rho x - xz - y, \\
\frac{d^\alpha z}{dt^\alpha} &= xy - bz.
\end{align*}
\]

where \(0 < \alpha \leq 1\). When \(\alpha = 1\), system (9) is the original integer order Lorenz system, which is chaotic when \(\sigma = 10, b = \frac{8}{3}\) and \(\rho = 28\). When \(\alpha\) selected as \(\alpha = 0.99\), then system (9) displays a chaotic attractor [13], the chaotic attractor is shown in Fig.(2)–(b).

The Liu system is described by

\[
\begin{align*}
\frac{d^\alpha x}{dt^\alpha} &= -ax - dy^2, \\
\frac{d^\alpha y}{dt^\alpha} &= \mu y - kxz, \\
\frac{d^\alpha z}{dt^\alpha} &= -cz + mxy,
\end{align*}
\]

where \(a = 1, d = 1, \mu = 2.5, k = 4, c = 5\) and \(m = 4\). The lowest value of \(\alpha\) for which the system exhibits chaos is given by 0.92 [14], the chaotic attractor is shown in Fig.(2)–(c).
3 Active anti-synchronization of the fractional–order chaotic Chen system

In order to observe the anti–synchronization behaviour in two fractional–order Chen systems (8), we assume that we have two identical fractional–order Chen systems. Therefore, we define the drive and response systems as follows

\[
\begin{align*}
\frac{d^\alpha x_1}{dt^\alpha} &= a(y_1 - x_1), \\
\frac{d^\alpha y_1}{dt^\alpha} &= (c - a)x_1 - x_1z_1 + cy_1, \\
\frac{d^\alpha z_1}{dt^\alpha} &= x_1y_1 - bz_1,
\end{align*}
\]

and

\[
\begin{align*}
\frac{d^\alpha x_2}{dt^\alpha} &= a(y_2 - x_2) + u_1(t), \\
\frac{d^\alpha y_2}{dt^\alpha} &= (c - a)x_2 - x_2z_2 + cy_2 + u_2(t), \\
\frac{d^\alpha z_2}{dt^\alpha} &= x_2y_2 - bz_2 + u_3(t).
\end{align*}
\]

where \(u_1(t), u_2(t)\) and \(u_3(t)\) are three control functions being introduced. In order to estimate the control functions, we add Eq. (12) to Eq. (11). We define the error system as the sum of the response system (12) and the drive system (11). Let us define the state errors between the response system (12) and the drive system (11) as

\[
e_1 = x_2 + x_1, \quad e_2 = y_2 + y_1, \quad e_3 = z_2 + z_1
\]

Add Eq. (12) to Eq. (11) and by using the notation in (13) yields

\[
\begin{align*}
\frac{d^\alpha e_1}{dt^\alpha} &= a(e_2 - e_1) + u_1(t), \\
\frac{d^\alpha e_2}{dt^\alpha} &= (c - a)e_1 - x_2z_2 - x_1z_1 + ce_2 + u_2(t), \\
\frac{d^\alpha e_3}{dt^\alpha} &= x_2y_2 + x_1y_1 - be_3 + u_3(t).
\end{align*}
\]

We define the active control functions \(u_1(t), u_2(t)\) and \(u_3(t)\) as follows:

\[
\begin{align*}
u_1(t) &= V_1(t), \\
u_2(t) &= x_2z_2 + x_1z_1 + V_2(t), \\
u_3(t) &= -x_2y_2 - x_1y_1 + V_3(t).
\end{align*}
\]
Hence the error system in (14) becomes

\[
\begin{align*}
\frac{d^\alpha e_1}{dt^\alpha} &= a(e_2 - e_1) + V_1(t), \\
\frac{d^\alpha e_2}{dt^\alpha} &= (c - a)e_1 + ce_2 + V_2(t), \\
\frac{d^\alpha e_3}{dt^\alpha} &= -be_3 + V_3(t).
\end{align*}
\]

where \( V_1(t), V_2(t) \) and \( V_3(t) \) are controls input. According to Theorem 1 and the stability theory of fractional–order systems, there are many possible choices for the control functions \( V_1(t), V_2(t) \) and \( V_3(t) \). We choose

\[
\begin{bmatrix}
V_1 \\
V_2 \\
V_3
\end{bmatrix} = M
\begin{bmatrix}
e_1 \\
e_2 \\
e_3
\end{bmatrix}
\]

where \( M \) is a \( 3 \times 3 \) constant matrix. In order to make the closed loop of the fractional–order system stable, the proper choice of the elements of the matrix \( M \) is such that the fractional–order feedback system must have all eigenvalues with negative real parts. Let the matrix \( M \) is chosen in the following form:

\[
\begin{bmatrix}
-1 + a & -a & 0 \\
(a - c) & -(1 + c) & 0 \\
0 & 0 & -1 + b
\end{bmatrix}
\]

In this particular choice, the closed loop fractional system (16) has the eigenvalues \((-1, -1, -1)\). This choice will lead the error states \( e_1, e_2 \) and \( e_3 \) to converge to zero as time \( t \) tends to infinity and this implies that the anti-synchronization of two identical fractional–order Chen systems is achieved.

3.1 Simulation results

To verify and demonstrate the effectiveness of the proposed method, we discuss the simulation result for the anti-synchronization behavior of two identical fractional–order Chen systems. In the numerical simulations, the generalization of Adams-Bashforth-Moulton method is used to solve the systems with time step size 0.005, where the fractional–order \( \alpha \) is chosen as \( \alpha = 0.9 \). The initial states of the drive system (11) and the response system (12) are taken as \( x_1(0) = -10, y_1(0) = 0, z_1(0) = 37 \) and \( x_2(0) = -5, y_2(0) = 5, z_2(0) = 32 \), respectively. Hence the error system has the initial values \( e_1(0) = -15, e_2(0) = 5 \) and \( e_3(0) = 69 \). The three unknown parameters are chosen as \( a = 35, b = 3 \) and \( c = 28 \) in all simulations, so that the fractional–order Chen system exhibits a chaotic behavior. Anti-synchronization behaviour of the systems in (11) and (12) via active control is shown in Figures (3)–(4). Figure (3) display state trajectories of the drive system in (11) and the response system in (12). Figure (4) display the anti-synchronization errors between systems (11) and (12).

Figure 3: State trajectories of the drive system (11) and the response system (12).
4 Chaos anti-synchronization between fractional-order Lorenz and Liu system

In this section, we study the anti–synchronization between the Lorenz and the Liu systems. Assuming that the Lorenz system is the drive system and the Liu system is the response system, we have:

\[
\begin{align*}
\frac{d^\alpha x_1}{dt^\alpha} &= \sigma(y_1 - x_1), \\
\frac{d^\alpha y_1}{dt^\alpha} &= \rho x_1 - x_1 z_1 - y_1, \\
\frac{d^\alpha z_1}{dt^\alpha} &= x_1 y_1 - b z_1,
\end{align*}
\]

and

\[
\begin{align*}
\frac{d^\alpha x_2}{dt^\alpha} &= -a x_2 - dy_2^2 + u_1(t), \\
\frac{d^\alpha y_2}{dt^\alpha} &= \mu y_2 - k x_2 z_2 + u_2(t), \\
\frac{d^\alpha z_2}{dt^\alpha} &= c z_2 + m x_2 y_2 + u_3(t),
\end{align*}
\]

where \( U = [u_1, u_2, u_3]^T \) is the active controlling function to be determined. The error function is defined as

\[
e_1 = x_1 + x_2, \quad e_2 = y_1 + y_2, \quad e_3 = z_1 + z_2.
\]  (21)

Thus, by adding Eq. (20) to Eq. (19) yields the error dynamical system between Eq. (19) and Eq. (20):

\[
\begin{align*}
\frac{d^\alpha e_1}{dt^\alpha} &= \sigma(e_2 - e_1) - \sigma(y_2 - x_2) - ax_2 - dy_2^2 + u_1(t), \\
\frac{d^\alpha e_2}{dt^\alpha} &= \rho e_1 - e_2 + (\mu + 1)y_2 - k x_2 z_2 - x_1 z_1 + u_2(t), \\
\frac{d^\alpha e_3}{dt^\alpha} &= (b - c)z_2 - b c_3 + x_1 y_1 + m x_2 y_2 + u_3(t),
\end{align*}
\]  (22)

We define the active control functions \( U \) as follows:

\[
\begin{align*}
u_1 &= V_1 + \sigma(y_2 - x_2) + ax_1 + dy_2^2, \\
u_2 &= V_2 - (\mu + 1)y_2 + k x_2 z_2 + x_1 z_1, \\
u_3 &= V_3 - (b - c)z_2 - x_1 y_1 - m x_2 y_2,
\end{align*}
\]  (23)
Hence the error system (22) becomes:

\[
\frac{d^\alpha e_1}{dt^\alpha} = \sigma (e_2 - e_1) + V_1, \\
\frac{d^\alpha e_2}{dt^\alpha} = \rho e_1 - e_2 + V_2, \\
\frac{d^\alpha e_3}{dt^\alpha} = -be_3 + V_2.
\]

(24)

where \( V_1(t) \), \( V_2(t) \) and \( V_3(t) \) are controls input. According to Theorem 1 and the stability theory of fractional–order systems, there are many possible choices for the control functions \( V_1(t) \), \( V_2(t) \) and \( V_3(t) \). We choose:

\[
\begin{bmatrix}
V_1 \\
V_2 \\
V_3
\end{bmatrix} = A \begin{bmatrix}
e_1 \\
e_2 \\
e_3
\end{bmatrix}
\]

(25)

where \( A \) is a \( 3 \times 3 \) constant matrix. In order to make the closed loop of the fractional–order system stable, the proper choice of the elements of the matrix \( A \) is such that the fractional–order feedback system must have all eigenvalues with negative real parts. Let the matrix \( A \) is chosen in the following form:

\[
\begin{bmatrix}
\sigma - 1 & -\sigma & 0 \\
-\rho & 0 & 0 \\
0 & 0 & b - 1
\end{bmatrix}
\]

(26)

In this particular choice, the closed loop fractional system (24) has the eigenvalues \((-1, -1, -1)\). This choice will lead the error states \( e_1, e_2 \) and \( e_3 \) to converge to zero as time \( t \) tends to infinity and this implies that the anti-synchronization of two different fractional–order chaotic systems is achieved.

4.1 Simulation results

To verify and demonstrate the effectiveness of the proposed method, we discuss the simulation result for the anti-synchronization behaviour of two different fractional–order systems. In the numerical simulations, the generalization of Adams-Bashforth-Moulton method is used to solve the systems by using time step size 0.005, where the fractional–order \( \alpha \) is chosen as \( \alpha = 0.99 \) for both systems. The initial states of the drive system in (19) and the response system in (20) are taken as \( x_1(0) = 10, y_1(0) = 5, z_1(0) = 10 \) and \( x_2(0) = 0.2, y_2(0) = 0, z_2(0) = 0.5 \), respectively. Hence the error system has the initial values \( e_1(0) = 10.2, e_2(0) = 10 \) and \( e_3(0) = 10.5 \). The other parameters were selected to be \( \sigma = 10, r = 28, b = 8/3, a = 1, d = 1, \mu = 2.5, k = 4, e = 5 \) and \( m = 4 \) in all simulations, such that the fractional–order systems exhibit chaotic behavior. Anti-synchronization behavior of the systems in (19) and (20) via active control is shown in Figures (5)–(6). Figure (5) displays state trajectories of the drive system in (19) and the response system in (20). Figure (6) displays the anti-synchronization errors between systems (19) and (20).

5 Conclusion

In this present work, we have demonstrated that two identical and different fractional–order chaotic systems can be anti-synchronized using active control and fractional–order stability theory. Finally, a numerical simulation is provided to show the effectiveness of our method.
Figure 6: Anti-synchronization errors between systems (19) and (20).

References