Functional Differential Equations with Causal Operators

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Abstract: The aim of this paper is to establish the existence of solutions and some properties of set solutions for a class of functional differential equation with causal operator under assumption that the equation satisfies the Carathéodory type condition.

Keywords: functional differential equation; causal operator; existence results

1 Introduction

The study of differential equations with causal operators has a rapid development in the last years and some results are assembled in a recent monograph [1]. The term of causal operators is adopted from engineering literature and the theory these operators has the powerful quality of unifying ordinary differential equations, integro-differential equations, differential equations with finite or infinite delay, Volterra integral equations, and neutral functional equations, to name a few.

In the paper [6] is considered the class $\mathcal{S}$ of all infinite-dimensional nonlinear $M$-input $u$, $M$-output $y$ systems $(\rho, f, g, Q)$ given by the following controlled nonlinear functional equation

$$\begin{align*}
y'(t) &= f(p(t), (\hat{Q}y)(t)) + g(p(t), (\hat{Q}y)(t), u(t)), \\
y|_{[-\sigma,0]} &= y^0 \in C([-\sigma,0], \mathbb{R}^M),
\end{align*}$$

where $\sigma > 0$ quantifies the memory of the system, $p$ is a perturbation term, and $\hat{Q}$ is a nonlinear causal operator. The aim of the control objective is the development of an adaptive servomechanism which ensures practical tracking, by the system output, of an arbitrary reference signal assumed to be in the class $\mathcal{R}$ of all locally absolutely continuous and bounded with essentially bounded derivative.

More exactly, the control objective is to determine an $(\mathcal{R}, \mathcal{S})$-servomechanism, that is, to determine the continuous functions $\Phi : \mathbb{R}^M \to \mathbb{R}^M$ and $\psi_{\lambda} : \mathbb{R}_+ \to \mathbb{R}_+$ (parametrized by $\lambda > 0$) such that, for every system of class $\mathcal{S}$ and every reference signal $r \in \mathcal{R}$, the control

$$u(t) = -k(t)\Phi(y(t) - r(t)), \quad k(t) = \psi_{\lambda}(|y(t) - r(t)|), \quad k_{[-\sigma,0]} = k^0,$$

applied to (1) ensures convergence of controller gain, and tracking of $r(\cdot)$ with asymptotic accuracy quantified by $\lambda > 0$, in the sense that $\max\{||y(t) - r(t)|| - \lambda, 0\} \to 0$ as $t \to \infty$. For more details (see [6, 10]).

Using (2), we can write (1) as

$$x'(t) = F(t, (Qx)(t)), \quad x|_{[-\sigma,0]} = x^0 \in C([-\sigma,0], \mathbb{R}^N),$$

where $N = M + 1$, $x(t) := (y(t), k(t))$, $x^0 = (y^0, k^0)$, and $Q$ is an operator defined on $C([-\sigma,0], \mathbb{R}^N)$ by

$$(Qx)(t) = (Q(y, k))(t) := ((\hat{Q}y)(t), y(t), k(t)).$$

The purpose of this article is to study the topological properties of the initial value problem (3). For this we will use ideas from papers [4], [5]. Also, we give an existence result for this problem, assuming only the continuity of the operator $Q$. In the paper [6] is also obtained an existence result assuming that the operator $Q$ is a locally Lipschitz operator.

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2 Preliminaries

Let \( \mathbb{R}^n \) be the \( n \)-dimensional Euclidean space with norm \( |·| \). For \( x \in \mathbb{R}^n \) and \( r > 0 \) let \( B_r(x) := \{ y \in \mathbb{R}^n; |y - x| < r \} \) be the open ball centered at \( x \) with radius \( r \), and let \( B_r[x] \) be its closure. If \( I = [0, b) \subset \mathbb{R} \), \( b \in (0, \infty) \), then we denote by \( C(I, \mathbb{R}^n) \) the Banach space of continuous functions from \( I \) into \( \mathbb{R}^n \). If \( \sigma \) is a positive number then we let \( C_{\sigma} := C([−\sigma, 0], \mathbb{R}^n) \) and let \( L^\infty_{\text{loc}}(I, \mathbb{R}^n) \) denote the space of measurable locally essentially bounded functions \( x(·) : I \rightarrow \mathbb{R}^n \).

**Definition 1** Let \( \sigma \geq 0 \). An operator \( Q : C([−\sigma, b], \mathbb{R}^n) \rightarrow L^\infty_{\text{loc}}([0, b), \mathbb{R}^n) \) is a causal operator if, for each \( \tau \in [0, b) \) and for all \( x(·), y(·) \in C([−\sigma, b], \mathbb{R}^n) \), with \( x(t) = y(t) \) for every \( t \in [−\sigma, \tau] \), we have \( (Qx)(t) = (Qy)(t) \) for a.e. \( t \in [0, \tau] \).

Two significant examples of causal operators are: the Niemytzki operator

\[
(Qx)(t) = f(t, x(t))
\]

and the Volterra integral operator

\[
(Qx)(t) = g(t) + \int_0^t k(t, s)f(s, x(s))ds.
\]

For \( i = 0, 1, \ldots, p \), we consider the functions \( g_i : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n \), \((t, x) \rightarrow g_i(t, x)\), that are measurable in \( t \) and continuous in \( x \). Set \( \sigma := \max_{i=1}^p \sigma_i \), where \( \sigma_i \geq 0 \), and let

\[
(Qx)(t) = \int_0^t g_0(s, x(t+s))ds + \sum_{i=1}^p g_i(t, x(t-\sigma_i)), \quad t \geq 0.
\]

Then, the operator \( Q \), so defined, is a causal operator (for details, see [6], [10]). For another concrete examples which serve to illustrate that the class of causal operators is very large, we refer to the monograph [1].

We consider the initial-valued problem with causal operator

\[
x'(t) = F(t, x(t), (Qx)(t)), \quad x[−\sigma, 0] = \varphi \in \mathcal{C}_\sigma,
\]

under the following assumptions:

\( (h_1) \) \( Q \) is continuous;

\( (h_2) \) for each \( r > 0 \) and each \( \tau \in (0, b) \), there exists \( M > 0 \) such that, for all \( x(·) \in C([−\sigma, b], \mathbb{R}^n) \) with \( \sup_{−\sigma \leq t \leq \tau} ||x(t)|| \leq r \), we have \( ||(Qx)(t)|| \leq M \) for a.e. \( t \in [0, \tau] \);

\( (h_3) \) \( F : [−\sigma, b) \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a Carathéodory function, that is:

(a) for a.e. \( t \in [−\sigma, b) \), \( F(t, ·, ·) \) is continuous,

(b) for each fixed \( (x, v) \in \mathbb{R}^n \times \mathbb{R}^n \), \( F(·, x, v) \) is measurable,

(c) for every bounded \( B \subset \mathbb{R}^n \times \mathbb{R}^n \), there exists \( \mu(·) \in L^1_{\text{loc}}([−\sigma, b), \mathbb{R}_+) \) such that

\[
||F(t, x)|| \leq \mu(t) \quad \text{for a.e. } t \in [−\sigma, b) \text{ and all } (x, v) \in B.
\]

By a solution of (4) on \([−\sigma, T] \), we mean a function \( x(·) \in C([−\sigma, T], \mathbb{R}^n) \), with \( T \in (0, b] \) and \( x[−\sigma, 0] = \varphi \), such that \( x|_{[0, T]} \) is absolutely continuous and satisfies (4) for a.e. \( t \in [0, T] \).

We remark that, \( x(·) \in C([−\sigma, T], \mathbb{R}^n) \) is a solution for (4) on \([−\sigma, T] \), if and only if \( x[−\sigma, 0] = \varphi \) and

\[
x(t) = \varphi(0) + \int_0^t F(s, x(s), (Qx)(s))ds \quad \text{for } t \in (0, T).
\]

The existence of solutions for this kind of Cauchy problem has been studied in [6], when \( Q : C([−\sigma, b], \mathbb{R}^n) \rightarrow L^\infty_{\text{loc}}([0, b), \mathbb{R}^n) \) is a locally Lipschitz operator. The existence of solutions for this kind of Cauchy problem has been studied by [2], in the case that \( Q : C([0, b), E) \rightarrow C([0, b), E) \) is a local Lipschitz operator. Also, for other results (see [3, 7–9]).
3 Existence of solutions

In the first half of this section, we present an existence result of the solutions for Cauchy problem (4), under conditions (h₁) – (h₃).

**Theorem 1** Assume that the conditions (h₁) – (h₃) hold. Then, for every \( \varphi \in C_\sigma \), there exists a solution \( x(\cdot) : [-\sigma, T] \to \mathbb{R}^n \) for Cauchy problem (4) on some interval \([-\sigma, T]\) with \( T \in (0, b) \).

**Proof.** Let \( \delta > 0 \) be any number and let \( r := \|\varphi\|_\sigma + \delta \). If \( x^0(\cdot) \in C([-\sigma, b], \mathbb{R}^n) \) denotes the function defined by

\[
x^0(t) = \begin{cases} 
\varphi(t), & \text{for } t \in [-\sigma, 0) \\
\varphi(0), & \text{for } t \in [0, b]
\end{cases},
\]

then \( \sup_{0 \leq t < b} |x^0(t)| \leq r \). Therefore, by (h₂), we have \( ||(Qx^0)(t)|| \leq M \) for a.e. \( t \in [0, b) \). Since \( F \) is a Carathéodory function, there exists \( \mu(\cdot) \in L^1_{loc}([0, b], \mathbb{R}^+) \) such that

\[
|F(t, x, v)| \leq \mu(t) \text{ for a.e. } t \in [0, b) \text{ and } (x, v) \in B_r(0) \times B_M(0).
\]

We choose \( T \in (0, b) \) such that \( \int_0^T \mu(t)dt < \delta \) and we consider the set \( B \) defined as follows

\[
B = \{ x \in C([-\sigma, T], \mathbb{R}^n); x|_{[-\sigma, 0]} = \varphi, \ sup_{0 \leq t \leq T} |x(t) - x^0(t)| \leq \delta \}.
\]

Further on, we consider the integral operator \( P : B \to C([-\sigma, T], \mathbb{R}^n) \) given by

\[
(Pu)(t) = \begin{cases}
\varphi(t), & \text{for } t \in [-\sigma, 0) \\
\varphi(0) + \int_0^t F(s, x(s), (Qx)(s))ds, & \text{for } t \in [0, b]
\end{cases},
\]

and we prove that this is a continuous operator from \( B \) into \( B \). First, we observe that if \( x(\cdot) \in B \), then \( \sup_{0 \leq t \leq T} |x(t)| < r \) and so, by (h₂), \( ||(Qx)(t)|| \leq M \) for a.e. \( t \in [0, b) \). Hence, for each \( x(\cdot) \in B \), we have

\[
\sup_{0 \leq t \leq T} ||(P \varphi(t) - x^0(t)|| = \sup_{0 \leq t \leq T} \int_0^t |F(s, x(s), (Qx)(s))|ds
\]

\[
\leq \int_0^T |F(s, x(s), (Qx)(s))|ds
\]

\[
\leq \int_0^T \mu(t)dt < \delta
\]

and thus, \( P(B) \subset B \). Moreover, it follows that \( P(B) \) is uniformly bounded. Further, let \( x_m \to x \) in \( B \). We have

\[
\sup_{0 \leq t \leq T} ||(P \varphi_m(t) - (P \varphi)(t)||
\]

\[
= \sup_{0 \leq t \leq T} \int_0^t |F(s, x(s), (Qx_m)(s)) - F(s, x(s), (Qx)(s))|ds
\]

\[
\leq \sup_{0 \leq t \leq T} \int_0^T |F(s, x(s), (Qx_m)(s)) - F(s, x(s), (Qx)(s))|ds
\]

\[
\leq \int_0^T |F(s, x(s), (Qx_m)(s)) - F(s, x(s), (Qx)(s))|ds
\]

and so, by (h₃), it follows that \( \sup_{0 \leq t \leq T} ||(P \varphi_m(t) - (P \varphi)(t)|| \to 0 \) as \( m \to \infty \). Since \( x_m|_{[-\sigma, 0]} = \varphi \) for every \( m \in \mathbb{N} \), we deduce that \( P : B \to B \) is a continuous operator. Further, we show that \( P(B) \) is uniformly equicontinuous on \([-\sigma, T]\). Let \( \varepsilon > 0 \). On the closed set \([0, T]\), the function \( t \to \int_0^t \mu(s)ds \) is uniformly continuous, and so there exists \( \eta' > 0 \) such that

\[
\left| \int_s^t \mu(\tau)d\tau \right| \leq \varepsilon/2, \text{ for every } t, s \in [0, T] \text{ with } |t-s| < \eta'.
\]
On the other hand, since $\varphi \in C_r$ is a continuous function on $[-\sigma, 0]$, then there exists $\eta'' > 0$ such that

$$||\varphi(t) - \varphi(s)|| \leq \varepsilon/2, \text{ for every } t, s \in [0, T] \text{ with } |t - s| < \eta''.$$  

Let $t, s \in [-\sigma, T]$ are such that $|t - s| \leq \eta$, where $\eta = \min\{\eta', \eta''\}$. If $-\sigma \leq s \leq t \leq 0$ then, for each $x(\cdot) \in B$, we have

$$||(Pu)(t) - (Pu)(s)|| = ||\varphi(0) + \int_0^t F(\tau, x(\tau), (Qx)(\tau))d\tau - \varphi(s)||$$

$$\leq ||\varphi(0) - \varphi(s)|| + \int_0^s ||F(\tau, x(\tau), (Qx)(\tau))||d\tau \leq ||\varphi(0) - \varphi(s)|| +$$

$$\left|\int_0^t \mu(\tau)d\tau\right| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq \varepsilon.$$

Finally, if $0 \leq s \leq t \leq T$ then, for each $x(\cdot) \in B$, we have

$$||(Pu)(t) - (Pu)(s)|| = \int_0^s ||F(\tau, x(\tau), (Qx)(\tau))||d\tau \leq \int_0^s \mu(\tau)d\tau \leq \varepsilon.$$  

Therefore, we conclude that $P(B)$ is uniformly equicontinuous on $[-\sigma, T]$. Therefore, by Arzela-Ascoli theorem we deduce that $P(B)$ is a compact operator. Since $B$ is nonempty, convex closed subset of $C([-\sigma, T], \mathbb{R}^n)$, then the Schauder fixed point theorem implies that $P$ has a fixed point in $B$. ■

**Theorem 2** Assume that the conditions $(h_1) - (h_3)$ hold. Then, there exist $T \in (0, b)$ and a sequence of absolutely continuous functions $x_m(\cdot) : [-\sigma, T] \to \mathbb{R}^n$ such that, extracting a subsequence if necessary, $x_m(t) \to x(t)$ uniformly on $[-\sigma, T]$ and $x(\cdot) : [-\sigma, T] \to \mathbb{R}^n$ is a solution for the Cauchy problem (4).  

**Proof.** Let $\delta > 0$ and $T \in (0, b)$ are that in the proof of the Theorem 1. For each $m \geq 1$ we consider the sequence $(x_m(\cdot))_{m \geq 1}$ given by

$$x_m(t) = \begin{cases} 
  x^0(t), & \text{for } -\sigma \leq t \leq T/m, \\
  \varphi(0) + \int_0^{t-T/m} F(s, x_m(s), (Qx_m)(s))ds, & \text{for } T/m \leq t \leq T.
\end{cases}$$

Then, for all $m \geq 1$ we have $x_m(\cdot) \in B$. Moreover, for $0 \leq t \leq T/m$, we have

$$||(Pu_m)(t) - x_m(t)|| \leq \int_0^{T/m} ||F(s, x_m(s), (Qx_m)(s))||ds \leq \int_0^{T/m} \mu(s)ds$$

and for $T/m \leq t \leq T$, we have

$$||(Pu_m)(t) - x_m(t)|| = ||(Pu_m)(t) - (Pu_m)(t - T/m)|| = \int_0^{t-T/m} F(s, x_m(s), (Qx_m)(s))ds - \int_0^t F(s, x_m(s), (Qx_m)(s))ds$$

$$= \int_{t-T/m}^{t} F(s, x_m(s), (Qx_m)(s))ds$$

$$\leq \int_{t-T/m}^{t} |F(s, x_m(s), (Qx_m)(s))|ds \leq \int_{t-T/m}^{t} \mu(s)ds.$$

Therefore, it follows that $\sup_{0 \leq t \leq T} ||(Pu_m)(t) - x_m(t)|| \to 0$ as $m \to \infty$. On the other hand, the set $A = \{x_m; m \geq 1\}$ is relatively compact in $C([0, T], \mathbb{R}^n)$. Hence, extracting a subsequence if necessary, we can assume that $x_m(\cdot) \to x(\cdot)$ in $C([0, T], \mathbb{R}^n)$. Therefore, since

$$\sup_{0 \leq t \leq T} ||(Pu)(t) - x(t)|| \leq \sup_{0 \leq t \leq T} ||(Pu)(t) - (Pu_m)(t)|| + \sup_{0 \leq t \leq T} ||(Pu_m)(t) - x_m(t)|| + \sup_{0 \leq t \leq T} ||x_m(t) - x(t)||$$

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then, by the fact that $P$ is a continuous operator, we obtain that $\sup_{0 \leq t \leq T} \|(Pu)(t) - x(t)\| = 0$. It follows that $x(t) = (Pu)(t) = \varphi(0) + \int_0^t F(s,x(s),(Qx)(s))ds$ for every $t \in [0,T]$. Now, if we define $\tilde{x}(\cdot) : [-\sigma,T] \rightarrow \mathbb{R}^n$ by
\[
\tilde{x}(t) = \begin{cases} 
\varphi(t), & \text{for } t \in [-\sigma,0), \\
\varphi(0) + \int_0^t F(s,x(s),(Qx)(s))ds, & \text{for } t \in [0,b),
\end{cases}
\]
then $\tilde{x}(\cdot)$ solve the Cauchy problem (4). □

### 4 Dependence

In the following, for a fixed $\varphi \in \mathcal{C}_\sigma$ and a bounded set $K \subset \mathbb{R}^n$, by $\mathcal{S}_T(\varphi, K)$ we denote the set of solutions $x(\cdot)$ of Cauchy problem (4) on $[-\sigma,T]$ with $T \in (0,b)$ and such that $x(t) \in K$ for all $t \in [-\sigma,T]$. By $\mathcal{A}_T(\varphi, K)$ we denote the attainable set; that is, $\mathcal{A}_T(\varphi, K) = \{x(T); x(\cdot) \in \mathcal{S}_T(\varphi, K)\}$.

**Proposition 3** Assume that the conditions $(h_1)$ - $(h_3)$ hold. Then, for every $\varphi \in \mathcal{C}_\sigma$, $\mathcal{S}_T(\varphi, K)$ is compact set in $\mathcal{C}([-\sigma,T], \mathbb{R}^n)$.

**Proof** We consider a sequence $\{x_m(\cdot)\}_{m \geq 1} \in \mathcal{S}_T(\varphi, K)$ and we shall show that this sequence contains a subsequence which converges, uniformly on $[-\sigma,T]$, to a solution $x(\cdot) \in \mathcal{S}_T(\varphi, K)$. Since $K$ is a bounded set, then there exists $M > 0$ such that $K \subset B_r(0)$. By $(h_2)$, there exists $M > 0$ such that $|||Qx(t)||| \leq M$ for every $x(\cdot) \in \mathcal{C}([-\sigma,T], \mathbb{R}^n)$ with $\sup_{-\sigma \leq t \leq T} ||x(t)|| < r$. Since $F$ is a Carathéodory function, there exists $\mu(\cdot) \in L^1([0,T], \mathbb{R}_+) such that $||F(t,x,v)|| \leq \mu(t)$ for a.e. $t \in [0,T]$ and $(x,v) \in B_r(0) \times B_M(0)$.

Since $x_m|_{[-\sigma,0]} = \varphi$, we have that $x_m(\cdot) \rightarrow \varphi(\cdot)$ uniformly on $[-\sigma,0]$. On the other hand, since $x_n(t) = \varphi(0) + \int_0^t F(s,x_n(s),(Qx_n)(s))ds$ for all $t \in [0,T]$, we have that
\[
x_n(t) - x_n(s) = \int_s^t F(x_n(t),(Qx_n)(\tau))d\tau \text{ for } s,t \in (0,T]
\]
and hence
\[
||x_n(t) - x_n(s)|| \leq \int_s^t ||F(x_n(t),(Qx_n)(\tau))||d\tau \leq \int_s^t \mu(\tau)d\tau \leq r|t-s| \text{ for } s,t \in (0,T].
\]
Therefore, $\{x_n(\cdot)\}_{n \geq 1}$ is equicontinuous on $(0,T]$. Since, for every $n \geq 1$, $x_n(\cdot)$ satisfies the same initial condition, $x_n(0) = \varphi(0)$, then we deduce that $\{x_n(\cdot)\}_{n \geq 1}$ is equibounded on $(0,T]$. Further, by the Ascoli-Arzela theorem and extracting a subsequence if necessary, we may assume that the sequence $\{x_n(\cdot)\}_{n \geq 1}$ converges uniformly on $(0,T]$ to a continuous function $x(\cdot)$. If we extend $x(\cdot)$ to $[-\sigma,T]$ such that $x|_{[-\sigma,0]} = \varphi$ then is clearly $x_n(\cdot) \rightarrow x(\cdot)$ uniformly on $[-\sigma,T]$. Now, by $(h_1)$, we have that $\lim_{n \rightarrow \infty} Qx_n = Qx$ in $L^\infty([0,T], \mathbb{R}^n)$. Therefore, $\lim_{n \rightarrow \infty} (Qx_n)(t) = (Qx)(t)$ for a.e. $t \in (0,T]$ and so, by the continuity of the function $F(t,\cdot)$, we have
\[
\lim_{n \rightarrow \infty} F(t,x_n(t),(Qx_n)(t)) = F(t,(Qx)(t)) \text{ for a.e. } t \in [0,T].
\]
Since $\{x_n(\cdot)\}_{n \geq 1}$ is equibounded on $[-\sigma,T]$, then there exists $r > 0$ such that $\sup_{-\sigma \leq t \leq T} ||x_n(t)|| < r$ for all $n \geq 1$. It follows that there exists $M > 0$ such that $||(Qx_n)(t)|| < M$ for a.e. $t \in [0,T]$ and for all $n \geq 1$, and so, we have that $||F(t,x_n(t),(Qx_n)(t))|| \leq \mu(t)$ for a.e. $t \in [0,T]$ and for all $n \geq 1$. Therefore, by the Lebesgue dominated convergence theorem, we have
\[
\lim_{n \rightarrow \infty} \int_0^t F(s,x_n(t),(Qx_n)(s))ds = \int_0^t F(s,x(t),(Qx)(s))ds, \text{ for all } t \in [0,T].
\]
It follows that $x(t) = \lim_{n \rightarrow \infty} x_n(t) = \varphi(0) + \int_0^t F(s,x(t),(Qx)(s))ds,$ for all $t \in [0,T]$ and so $x(\cdot) \in \mathcal{S}_T(\varphi, K)$. □
Proposition 4 Assume that the conditions \((h_1)-(h_3)\) hold. Then, the multifunction \(S_T : C_\sigma \rightarrow C([-\sigma,T],\mathbb{R}^n)\) is upper semicontinuous.

Proof. Let \(K\) be a closed set in \(C([-\sigma,T],\mathbb{R}^n)\) and \(B = \{ \varphi \in C_\sigma ; S_T(\varphi, K) \cap K \neq \emptyset \}\). We must show that \(B\) is closed in \(C_\sigma\). For this, let \(\{ \varphi_n \}_{n \geq 1} \) a sequence in \(B\) such that \(\varphi_n \rightarrow \varphi\) on \([-\sigma,0]\). Further, for any \(n \geq 1\), let \(x_n(.) \in S_T(\varphi_n, K) \cap K\). Then, \(x_n = \varphi_n\) on \([-\sigma,0]\) for all \(n \geq 1\), and

\[
x_n(t) = \varphi_n(0) + \int_0^t F(s, x_n(s), (Q x_n)(s))ds,
\]

for all \(t \in (0, T]\). As in proof of Proposition 3 we can show that \(\{x_n(.)\}_{n \geq 1}\) is equicontinuous and equibounded on \([0, T]\).

Therefore, by the Ascoli-Arzela theorem and extracting a subsequence if necessary, we may assume that the sequence \(\{x_n(.)\}_{n \geq 1}\) converges uniformly on \([-\sigma, T]\) to a continuous function \(x(.) \in K\). Since

\[
x(t) = \lim_{n \to \infty} x_n(t) = \varphi(0) + \int_0^t F(s, x(s), (Q x)(s))ds
\]

for all \(t \in [0,T]\), we deduce that \(x(.) \in S_T(\varphi, K) \cap K\). This prove that \(B\) is closed and so \(\varphi \rightarrow S_T(\varphi)\) is upper semicontinuous. 

Corollary 5 Assume that the conditions \((h_1)-(h_3)\) hold. Then, for any \(\varphi \in C_\sigma\) and any \(t \in [0,T]\) the attainable set \(A_t(\varphi, K)\) is compact in \(C([-\sigma,t],\mathbb{R}^n)\) and the multifunction \((t,\varphi) \rightarrow A_t(\varphi, K)\) is jointly upper semicontinuous.

In the following, we consider the following control problem:

\[
\begin{cases}
x'(t) = F(t, x(t), (Qx)(t)) \quad \text{for a.e. } t \in [0,T] \\
x|_{[-\sigma,0]} = \varphi \\
\text{minimize } g(x(T))
\end{cases}
\]

where \(g : \mathbb{R}^n \rightarrow \mathbb{R}\) is a given function.

Theorem 6 Let \(K_0\) be a compact set in \(C_\sigma\) and let \(g : \mathbb{R}^n \rightarrow \mathbb{R}\) be a lower semicontinuous function. If the conditions \((h_1)-(h_3)\) hold, then the control problem (5) has an optimal solution; that is, there exists \(\varphi_0 \in K_0\) and \(x_0(.) \in S_T(\varphi_0, K)\) such that

\[
g(x_0(T)) = \inf \{g(x(T)); x(.) \in S_T(\varphi, K), \varphi \in K_0\}.
\]

Proof. From Corollary 5 we deduce that the attainable set \(A_T(\varphi)\) is upper semicontinuous. Then the set

\[
A_T(K_0) = \{ x(T) ; x(.) \in S_T(\varphi, K), \varphi \in K_0 \} = \cup_{\varphi \in K_0} A_T(\varphi, K)
\]

is compact in \(\mathbb{R}^n\) and so, since \(g\) is lower semicontinuous, there exists \(\varphi_0 \in K_0\) such that \(g(x_0(T)) = \inf \{g(x(T)); x(.) \in S_T(\varphi, K), \varphi \in K_0\}\).

5 An example

We consider the following integro - differential equation with delay

\[
x'(t) = F(t, x(t), g(t) + \int_{t-\sigma}^t k(t,s)f(s, x(s))ds),
\]

with the initial condition

\[
x|_{[-\sigma,0]} = \varphi \in C_\sigma.
\]

Also, we assume that the following assumptions hold:

\((h'_1)\) \(f : [-\sigma,b] \times \mathbb{R}^n \rightarrow \mathbb{R}^n\) is a Carathéodory function; that is,

1. \(f(\cdot, x)\) is a measurable function for each fixed \(x \in \mathbb{R}^n\),
2. \(f(t, \cdot)\) is continuous for each fixed \(t \in [-\sigma,b]\),

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(3) For each \( r > 0 \) there exist \( \mu_r (\cdot) \in L^1([-\sigma, b], \mathbb{R}) \) such that \( \| f(t, x) \| \leq \mu_r(t) \) a.e. on \([-\sigma, b]\) for every \( x \in \mathbb{R}^n \) with \( \| x \| \leq r \).

\((h'_2)\) \( k : [0, b) \times [-\sigma, b] \to \mathbb{R}_+ \) is a continuous bounded function with

\[ a = \sup \{ k(t, s); t \in [0, b], s \in [-\sigma, b] \}; \]

\((h'_3)\) \( g(\cdot) \in L^\infty([0, b], \mathbb{R}^n) \).

If we define the operator \( Q : C([-\sigma, b], \mathbb{R}^n) \to L^\infty([0, b], \mathbb{R}^n) \) by

\[ (Q x)(t) = g(t) + \int_{-\sigma}^{t} k(t, s) f(s, x(s)) ds, \ t \geq 0, \]

then \( Q \) is a causal operator.

In the following, we prove that the operator \( Q \) satisfies the conditions \((h_1)\) and \((h_2)\).

Let \( x_m \to x \) in \( C([-\sigma, b], \mathbb{R}^n) \). Then, for \( t \in [0, b] \), we have

\[ \|(Q x_m)(t) - (Q x)(t)\| = \| \int_0^t k(t, s) f(s, x_m(s)) ds - \int_0^t k(t, s) f(s, x(s)) ds \| \]

\[ \leq \int_0^t a \| f(s, x_m(s)) - f(s, x(s)) \| ds \]

and, by \((h'_4)\), it follows that \( \|(Q x_m)(t) - (Q x)(t)\| \to 0 \) as \( n \to \infty \). Hence, \( Q \) is a continuous operator.

Next, let \( r \) be any positive number. If \( \sup_{-\sigma \leq t \leq b} \| x(t) \| \leq r \), then \( \|(Q x)(t)\| \leq M \), for a.e. \( t \in [0, b] \), where

\[ M := \sup_{0 \leq t \leq b} \| g(t) \| + a \int_0^b \mu_r(s) ds. \]

Thus, \((h_2)\) holds.

Therefore, we obtain the following result.

**Theorem 7** Assume that the assumptions \((h'_1)\) - \((h'_3)\) hold. If \( F : [-\sigma, b] \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \) is a Carathéodory function, then the problem \((6)\)-(7) has a solution on some interval \([0, T]\), with \( T \in (0, b] \).

**References**


LINS homepage: http://www.nonlinearscience.org.uk/