Existence and Uniqueness Results for Impulsive Fractional Integrodifferential Equations in Banach Spaces

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Abstract: In this paper, we prove the existence and uniqueness of mild solutions for a class of impulsive fractional integrodifferential equations of the form

\[
D^{\alpha}_t x(t) = A x(t) + f(t, x(t), \int_0^t a(t, s, x(s))ds, \int_0^t b(t, s, x(s))ds), \quad t \in I = [0, T], t \neq t_k, k = 1, 2, ..., m, \\
x(0) = x_0 \in \mathcal{X},
\]

where \(T > 0\), with \(0 < \alpha < 1\). To prove the existence (and uniqueness) of solutions, assuming that \(A\) is a sectorial operator on a Banach space \(\mathcal{X}\) by means of Banach Contraction Principle and Leray-Schauder’s Alternative fixed point theorem.

Keywords: existence and uniqueness; fractional differential equation; integrodifferential equation; impulsive condition.

1 Introduction

Fractional differential equations is a generalization of ordinary differential equations and integration to arbitrary non-integer orders. It is widely and efficiently used to describe many phenomena arising in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see[19, 22, 23, 28, 39, 42, 48]). In fact, fractional differential equations are considered as models alternative to nonlinear differential equations [4, 6, 14, 20] and other kind of equations [29–31]. For more details on this theory and applications, we refer the monographs of Lakshmikantham et al. [37], Miller and Ross [43], Samko et al. [51], Kilbas et al. [35], Podlubny [49] and the papers of [5, 32, 33, 38, 40, 44, 45].

Most of the practical systems are integrodifferential equations in nature and hence the study of integrodifferential equations is very important. The fractional integrodifferential equations have been studied by many authors [7–9, 21, 54, 57] and references therein. Recently Mophou et al. [44, 45, 47] studied some semilinear fractional differential equations with nonlocal conditions and neutral fractional functional evolution equations with infinite delay in Banach spaces, where as Matar [40] studied the existence and uniqueness for fractional semilinear mixed Volterra-Fredholm integrodifferential equations with nonlocal conditions in Banach spaces.

On the other hand, impulsive differential equations have become important in recent years as mathematical models of phenomena in both the physical and social sciences. There has a significant development in impulsive theory especially in the area of impulsive differential equations with fixed moments, see for instance the monographs by Lakshmikantham et al. [36], Bainov et al. [11], Samoilenko et al. [52] and Benchahra et al. [12] and the papers [11–3, 15–17, 25–27, 34]. However impulsive fractional differential equations have been studied by few authors, see for instance [18, 46, 53, 55, 56].

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In [46] Mophou et al. studied the existence and uniqueness of mild solution to impulsive fractional differential equations of the form

\[ D_0^\alpha x(t) = Ax(t) + f(t, x(t)), \quad t \in I = [0, T], \quad t \neq t_k, \quad k = 1, 2, ..., m, \]  \hfill (1.1)
\[ x(0) = x_0 \in \mathbb{X}, \]  \hfill (1.2)
\[ \Delta x|_{t=t_k} = I_k(x(t_k^+)), \quad k = 1, 2, ..., m, \]  \hfill (1.3)

where \( 0 < \alpha < 1 \), the operator \( A \) is the infinitesimal generator of a strongly continuous semigroup \( T(t)_{t \geq 0} \) on a Banach space \( \mathbb{X} \). \( D_0^\alpha \) is the Caputo fractional derivative.

Very Recently, Xiao-Bao Shu et al. [55] studied the existence of mild solutions for impulsive fractional differential equations of the form (1.1)-(1.3) with \( A \) is a sectorial operator in Banach space \( \mathbb{X} \). Motivated by the above mentioned works [40, 46, 55] the purpose of this paper is to discuss the following impulsive fractional integrodifferential equations

\[ D_0^\alpha x(t) = Ax(t) + f \left( t, x(t), \int_0^t a(t, s, x(s))ds, \int_0^t b(t, s, x(s))ds \right), \quad t \in I = [0, T], \quad t \neq t_k, \quad k = 1, 2, ..., m, \]  \hfill (1.4)
\[ x(0) = x_0 \in \mathbb{X}, \]  \hfill (1.5)
\[ \Delta x|_{t=t_k} = I_k(x(t_k^+)), \quad k = 1, 2, ..., m, \]  \hfill (1.6)

where \( 0 < \alpha < 1 \), \( A \) is a sectorial operator on a Banach space \( \mathbb{X} \). \( D_0^\alpha \) is the Caputo fractional derivative, \( a : D = \{(t, s) \in I \times I : s \leq t \}, b : D = \{(t, s) \in I \times I : s \leq t \}, f : I \times \mathbb{X} \times \mathbb{X} \times \mathbb{X} \to \mathbb{X} \) is a given continuous function. \( I_k : \mathbb{X} \to \mathbb{X}, 0 = t_0 < t_1 < ... < t_k < ... < t_m < t_{m+1} = T, \Delta x|_{t=t_k} = I_k(x(t_k^+)), x(t_k^+) = \lim_{h \to 0^+} x(t_k + h) \) and \( x(t_k^-) = \lim_{h \to 0^-} x(t_k + h) \) represent the right and left limits of \( x(t) \) at \( t = t_k \) respectively.

2 Preliminaries

In this section, we recall some notions about sectorial operators, solution operators and analytic solution operators, and then present the definition of a mild solution of (1.4)-(1.6) by investigating the classical solutions of systems (1.4)-(1.6).

An operator \( A \) is said to be sectorial if there are constants \( \omega \in \mathbb{R}, \theta \in [\frac{\pi}{2}, \pi] \), \( M > 0 \) such that the following two conditions are satisfied:

\[
\begin{cases}
(1) \rho(A) \subset \sum_{\theta, \omega} = \{ \lambda \in C : \lambda \neq \omega, |\arg(\lambda - \omega)| < \theta \}, \\
(2) \| R(\lambda, A) \|_{L(\mathbb{X})} \leq \frac{M}{|\lambda - \omega|}, \quad \lambda \in \sum_{\theta, \omega}.
\end{cases}
\]

Consider the following Cauchy problem for the Caputo fractional derivative evolution equation of order \( \alpha \) \((m - 1 < \alpha < m, m > 0 \) is an integer):

\[ D_0^\alpha u(t) = Au(t), \quad u(0) = x, \quad u^{(k)}(0) = 0, k = 1, 2, ..., m - 1, \]  \hfill (2.1)

where \( A \) is a sectorial operator. The solution operator \( S_\alpha(t) \) of (2.1) is defined by (see [10])

\[ S_\alpha(t) = \frac{1}{2\pi i} \int_{\gamma} e^{\lambda t} \lambda^{\alpha - 1} R(\lambda^\alpha, A)d\lambda, \]

where \( \gamma \) is a suitable path lying in \( \sum_{\theta, \omega} \).

An operator \( A \) is said to belong to \( \wp^\alpha(\mathbb{X}; M, \omega) \) or \( \wp^\alpha(M, \omega) \), if problem (2.1) has a solution operator \( S_\alpha(t) \) satisfying \( \| S_\alpha(t) \| \leq Me^{\omega t}, t \geq 0 \). Denote \( \wp^\alpha(\omega) = \bigcup \{ \wp^\alpha(M, \omega) : M \geq 1 \} \) and \( \wp^\alpha = \bigcup \{ \wp^\alpha(\omega) : \omega \geq 0 \} \).

**Definition 1** ([10, 55]). A solution operator \( S_\alpha(t) \) of (2.1) is called analytic if \( S_\alpha(t) \) admits an analytic extension to a sector \( \sum_{\theta_0} := \{ \lambda \in C \setminus \{0\} : |\arg \lambda| < \theta_0 \} \) for some \( \theta_0 \in (0, \frac{\pi}{2}) \). An analytic solution operator is said to be of analyticity type \((\theta_0, \omega_0)\) if for each \( \theta < \theta_0 \) and \( \omega > \omega_0 \) there is an \( M = M(\theta, \omega) \) such that \( \| S_\alpha(t) \| \leq Me^{\omega t}, t \in \sum_{\theta} := \{ t \in C \setminus \{0\} : |\arg t| < \theta \} \). Denote \( A(\theta_0, \omega_0) := \{ A \in \wp^\alpha : A \) generates analytic solution operators \( S_\alpha(t) \) of type \((\theta_0, \omega_0)\).
Theorem 4

Therefore, by the inverse Laplace transform, we have

\[
\|\lambda^{\alpha-1}R(\lambda^\alpha, A)\| \leq \frac{C}{|\lambda - \omega|}, \quad \text{for } \lambda \in \sum_{\theta+\pi/2} (\omega).
\]

Next, we consider the definition of the mild solution of (1.4)-(1.6). Consider the following Cauchy problem

\[
\begin{cases}
(D_\alpha^t u) = Au(t) + f(t), & 0 < \alpha < 1, \\
u(0) = x_0 \in \mathcal{X},
\end{cases}
\]

where \( f \) is an abstract function defined on \([0, \infty)\) and with values in \( \mathcal{X} \), \( A \) is a sectorial operator.

Theorem 2 [55]. If \( f \) satisfies the uniform Holder condition with exponent \( \beta \in (0, 1] \) and \( A \) is a sectorial operator, then the unique solution of the Cauchy problem (2.2) is given by

\[
u(t) = S_\alpha(t)x_0 + \int_0^t T_\alpha(t-s)f(s)ds,
\]

where

\[
S_\alpha(t) := \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} \lambda^{\alpha-1} R(\lambda^\alpha, A) d\lambda, \\
T_\alpha(t) := \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} R(\lambda^\alpha, A) d\lambda
\]

and \( \Gamma \) is a suitable path lying on \( \sum_{\theta, \omega} \).

**Proof.** Taking the Laplace transform of the equation (2.2), we have

\[
\lambda^\alpha (Lu)(\lambda) - \lambda^{\alpha-1}x_0 = A(Lu)(\lambda) + (Lf)(\lambda).
\]

Since \( (\lambda^\alpha I - A)^{-1} \) exists, i.e., \( \lambda^\alpha \in \rho(A) \), from the above equation, we obtain

\[
(Lu)(\lambda) = \lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1}x_0 + (\lambda^\alpha I - A)^{-1}(Lf)(\lambda)
\]

Therefore, by the inverse Laplace transform, we have

\[
u(t) = S_\alpha(t)x_0 + \int_0^t T_\alpha(t-s)f(s)ds.
\]

Theorem 3 [10, 55]. If \( \alpha \in (0, 1) \) and \( A \in \mathcal{A}^\alpha(\theta_0, \omega_0) \), then for any \( x \in \mathcal{X} \) and \( t > 0 \), we have

\[
\|T_\alpha(t)\| \leq C e^{\omega t} (1 + t^{\alpha-1}), \quad t > 0, \quad \omega > \omega_0.
\]

**Theorem 4** If \( f \) satisfies a uniform Holder condition with exponent \( \beta \in (0, 1] \) and \( A \) is a sectorial operator, then any solution of the Cauchy problem (1.4)-(1.6) is a fixed point of the operator given below

\[
\Gamma x(t) = \begin{cases}
S_\alpha(t)x_0 + \\
\int_0^t T_\alpha(t-s)f \left( s, x(t), \int_0^s a(s, \tau, x(\tau))d\tau, \int_0^T b(s, \tau, x(\tau))d\tau \right) ds, & t \in (0, t_1]; \\
S_\alpha(t-t_1)(x(t_1^-)) + \\
\int_{t_1}^t T_\alpha(t-s)f \left( s, x(t), \int_0^s a(s, \tau, x(\tau))d\tau, \int_0^T b(s, \tau, x(\tau))d\tau \right) ds, & t \in (t_1, t_2]; \\
\cdot \cdot \cdot \\
S_\alpha(t-t_m)(x(t_m^-)) + \\
\int_{t_m}^t T_\alpha(t-s)f \left( s, x(t), \int_0^s a(s, \tau, x(\tau))d\tau, \int_0^T b(s, \tau, x(\tau))d\tau \right) ds, & t \in (t_m, T].
\end{cases}
\]

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In fact, from (2.3) it is easy to see that Theorem 2.3 holds.

In order to define the concept of mild solution, we need to define the following spaces.

\[ PC(I, \mathbb{X}) = \{ x : I \to \mathbb{X} : x \in C((t_k, t_{k+1}), \mathbb{X}), k = 0, 1, 2, \ldots, m, \text{ and there exist } x(t^-_k) \text{ and } x(t^+_k), \, k = 1, 2, \ldots, m \] with \( x(t^-_k) = x(t_k) \). \]

Endowed with the norm \( \| x \|_{PC} = \sup_{t \in I} \| x(t) \| \), \( (PC(I, \mathbb{X}), \| \cdot \|) \) is a Banach space.

From Theorem 2.3, we can define the mild solution of system (1.4)-(1.6) as follows:

**Definition 2** A function \( x : I \to \mathbb{X} \) is called a mild solution of a system (1.4)-(1.6) if \( x \in PC(I, \mathbb{X}) \) and satisfies the following equation:

\[
\begin{align*}
 x(t) = & \quad \left\{ \begin{array}{ll}
 S_0(t)x_0 + \\
 & \int_0^t T_0(t-s)f(s, x(s), \int_0^s a(s, \tau, x(\tau))d\tau, \int_0^s b(s, \tau, x(\tau))d\tau)ds, \, t \in (0, t_1]; \\
 & S_0(t-t_1)(x(t^-_1) + I_1(x(t^-_1)) + \\
 & \int_1^t T_0(t-s)f(s, x(s), \int_0^s a(s, \tau, x(\tau))d\tau, \int_0^s b(s, \tau, x(\tau))d\tau)ds, \, t \in (t_1, t_2]; \\
 & \ldots \\
 & S_0(t-m_1)(x(t^-_{m_1}) + I_1(x(t^-_{m_1})) + \\
 & \int_{m_1}^t T_0(t-s)f(s, x(s), \int_0^s a(s, \tau, x(\tau))d\tau, \int_0^s b(s, \tau, x(\tau))d\tau)ds, \, t \in (t_{m_1}, T]. \\
\end{array} \right.
\end{align*}
\]

## 3 Existence results

In this section, we give the main results on the existence of mild solutions of the system (1.4)-(1.6).

If \( A \in \mathcal{A}^\alpha(\theta_0, \omega_0) \), then \( \| S_0(t) \| \leq Me^{\omega t} \) and \( \| T_0(t) \| \leq Ce^{\omega t}(1 + t^{1-\alpha}) \).

Let \( \tilde{M}_S := \sup_{0 \leq t \leq T} \| S_0(t) \|_{L(x)}, \tilde{M}_T := \sup_{0 \leq t \leq T} Ce^{\omega t}(1 + t^{1-\alpha}) \), where \( L(x) \) is the Banach space of bounded linear operators from \( X \) into \( X \) equipped with its natural topology. So, we have

\[
\| S_0(t) \|_{L(x)} \leq \tilde{M}_S, \quad \| T_0(t) \|_{L(x)} \leq t^{1-\alpha}\tilde{M}_T.
\]

To establish our results, we introduce the following hypotheses:

\((H_1)\) \( f : I \times X \times X \times X \to X \) is continuous and there exist functions \( L_1, L_2, L_3 \in L^1(I, R^+) \) such that

\[
\| f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2) \| \leq L_1(t)\| x_1 - x_2 \| + L_2(t)\| y_1 - y_2 \| + L_3(t)\| z_1 - z_2 \|
\]

for all \( t \in [0, T], x_1, y_1, z_1 \in X, i = 1, 2.\)

\((H_2)\) \( a : D := \{(t, s) \in I \times I : s \leq t \} \times X \to X \) and \( b : D := \{(t, s) \in I \times I : s \leq t \} \times X \to X \) and there exist constants \( M_a > 0, M_b > 0 \), such that

(a) \( \| \int_0^t a(t, s, x) - a(t, s, y)ds \| \leq M_a\| x - y \|, \) for \( x, y \in X, \)

(b) \( \| \int_0^t b(t, s, x) - b(t, s, y)ds \| \leq M_b\| x - y \|, \) for \( x, y \in X. \)

\((H_3)\) For each \( k = 1, 2, \ldots, m \), there exists a \( \rho_k > 0 \) such that \( \| I_k(x) - I_k(y) \| \leq \rho_k\| x - y \|, \) for all \( x, y \in X. \)

**Theorem 5** Assume that the hypotheses \((H_1)-(H_3)\) holds. If \( A \in \mathcal{A}^\alpha(\theta_0, \omega_0) \), then the system (1.4)-(1.6) has a unique mild solution \( x \in PC(I, \mathbb{X}) \) provided

\[
\Theta := \max_{1 \leq i \leq m} \left\{ \tilde{M}_S(\rho_i + 1) + \frac{1}{\alpha} \tilde{M}_T T^\alpha \left[ \| L_1 \|_{L^1(I, R^+)} + M_a\| L_2 \|_{L^1(I, R^+)} + M_b\| L_3 \|_{L^1(I, R^+)} \right] \right\} < 1.
\]

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Proof. Define the operator $\Gamma : PC(I, \mathbb{X}) \to PC(I, \mathbb{X})$ by

$$
\Gamma x(t) = \begin{cases} 
S_a(t)x_0 + 
\int_0^t T_a(t-s)f \left( x(s), \int_0^s a(s, \tau, x(\tau))d\tau, \int_0^T b(s, \tau, x(\tau))d\tau \right) ds, & t \in (0, t_1]; \\
S_a(t-t_1)(x(t_1^-) + I_1(x(t_1^-))) + 
\int_{t_1}^t T_a(t-s)f \left( x(s), \int_0^s a(s, \tau, x(\tau))d\tau, \int_0^T b(s, \tau, x(\tau))d\tau \right) ds, & t \in (t_1, t_2]; \\
S_a(t-t_m)(x(t_m^-) + I_1(x(t_m^-))) + 
\int_{t_m}^t T_a(t-s)f \left( x(s), \int_0^s a(s, \tau, x(\tau))d\tau, \int_0^T b(s, \tau, x(\tau))d\tau \right) ds, & t \in (t_m, T].
\end{cases}
$$

Note that $\Gamma$ is well defined on $PC(I, \mathbb{X})$.

Let us take $t \in [0, t_1]$ and $x, y \in PC(I, \mathbb{X})$. From the equation (3.1) and the hypotheses (H1)-(H2), we have

$$
\| (\Gamma x)(t) - (\Gamma y)(t) \| 
\leq M_T \int_0^t (t-s)^{\alpha-1} \left[ L_1(s) \| x(s) - y(s) \| + L_2(s) \| a(s, \tau, x(\tau)) - a(s, \tau, y(\tau)) \| d\tau \right] 
+ L_3(s) \int_0^T \| b(s, \tau, x(\tau)) - b(s, \tau, y(\tau)) \| d\tau \right] ds 
\leq M_T \| x - y \|_{PC} \int_0^t (t-s)^{\alpha-1} L_1(s) ds 
+ M_0 M_T \| x - y \|_{PC} \int_0^t (t-s)^{\alpha-1} L_2(s) ds 
\leq \frac{1}{\alpha} M_T \left[ L_1 \cdot L^1(I, R^+) + M_0 \cdot L_2 \cdot L^1(I, R^+) + M_0 \cdot L_3 \cdot L^1(I, R^+) \right] T^\alpha \| x - y \|_{PC}.
$$

For $t \in (t_1, t_2]$ and by using (3.1), (H1)-(H3), we have

$$
\| (\Gamma x)(t) - (\Gamma y)(t) \| 
\leq M_T \left[ \| x(t_1^-) - y(t_1^-) \| + \rho_1 \| x(t_1^-) - y(t_1^-) \| \right] + M_T \int_0^t (t-s)^{\alpha-1} \left[ L_1(s) \| x(s) - y(s) \| 
+ L_2(s) M_0 \| x(s) - y(s) \| \right] ds 
\leq \left[ M_T (\rho_1 + 1) + \frac{1}{\alpha} M_T T^\alpha \left[ L_1 \cdot L^1(I, R^+) + M_0 \cdot L_2 \cdot L^1(I, R^+) + M_0 \cdot L_3 \cdot L^1(I, R^+) \right] \right] \| x - y \|_{PC}.
$$

Similarly, for $t \in (t_1, t_{i+1})$

$$
\| (\Gamma x)(t) - (\Gamma y)(t) \| 
\leq \left[ M_T (\rho_{i+1} + 1) + \frac{1}{\alpha} M_T T^\alpha \left[ L_1 \cdot L^1(I, R^+) + M_0 \cdot L_2 \cdot L^1(I, R^+) + M_0 \cdot L_3 \cdot L^1(I, R^+) \right] \right] \| x - y \|_{PC}.
$$

and for $t \in (t_m, T]$,

$$
\{ (\Gamma x)(t) - (\Gamma y)(t) \| 
\leq \left[ M_T (\rho_m + 1) + \frac{1}{\alpha} M_T T^\alpha \left[ L_1 \cdot L^1(I, R^+) + M_0 \cdot L_2 \cdot L^1(I, R^+) + M_0 \cdot L_3 \cdot L^1(I, R^+) \right] \right] \| x - y \|_{PC}.
$$

Thus, for all $t \in [0, T]$, we have

$$
\| (\Gamma x)(t) - (\Gamma y)(t) \| 
\leq \max_{1 \leq i \leq m} \left\{ M_T (\rho_i + 1) + \frac{1}{\alpha} M_T T^\alpha \left[ L_1 \cdot L^1(I, R^+) + M_0 \cdot L_2 \cdot L^1(I, R^+) + M_0 \cdot L_3 \cdot L^1(I, R^+) \right] \right\} \| x - y \|_{PC} 
= \Theta \| x - y \|_{PC},
$$

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Since \( 0 \leq \Theta < 1 \), from Banach contraction principle \( \Gamma \) has a contraction mapping and there exists a unique fixed point \( x_0 \in PC(I, \mathcal{X}) \). Therefore, this \( x_0 \) is a mild solution of the system (1.4)-(1.6).

Next, we study the existence results for the system (1.4)-(1.6). In order to study of this system, we need to apply the following theorem and hypotheses.

**Theorem 6** ([24], Theorem 6.5.4(Leray - Schauder’s Alternative). Let \( D \) be the closed convex subset of a Banach space \( \mathcal{X} \) and assume that \( 0 \in D \). If \( T : D \to D \) be a completely continuous map. Then, either the set \( \{ x \in D : x = \lambda \Gamma(x), 0 < \lambda < 1 \} \) is unbounded or the map \( \Gamma \) has a fixed point in \( D \).

Now, we make the following hypotheses:

\((\text{H}_5)\) For each \((t, s) \in D\), the functions \( a(t, s, \cdot), b(t, s, \cdot) : \mathcal{X} \to \mathcal{X} \) are continuous and for each \( x \in \mathcal{X} \) the functions \( a(t, s, \cdot), b(t, s, \cdot) : D \to \mathcal{X} \) is strongly measurable.

\((\text{H}_6)\) For \( t \in I \), the function \( f(t, \cdot, \cdot, \cdot) : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to \mathcal{X} \) is continuous and for \( x, y, z \in \mathcal{X} \), the function \( f(\cdot, x, y, z) : I \to \mathcal{X} \) is strongly measurable.

\((\text{H}_7)\) There exists a continuous function \( p_1 : I \to [0, \infty] \) such that
\[
\left\| \int_0^t a(t, s, x(s))ds \right\| \leq p_1(t) \psi(\|x(s)\|), \text{ for every } t, s \in I \text{ and } x \in \mathcal{X},
\]
where \( \psi : [0, +\infty) \to (0, \infty) \) is a continuous non-decreasing function.

\((\text{H}_8)\) There exists a continuous function \( p_2 : I \to [0, \infty] \) such that
\[
\left\| \int_0^t b(t, s, x(s))ds \right\| \leq p_2(t) \psi(\|x(s)\|), \text{ for every } t, s \in I \text{ and } x \in \mathcal{X},
\]
where \( \psi : [0, +\infty) \to (0, \infty) \) is a continuous non-decreasing function.

\((\text{H}_9)\) There exists a continuous function \( p_3 : I \to [0, \infty] \) such that
\[
\|f(t, x, y, z)\| \leq p_3(t) \psi(\|x(s)\|) + \|y\| + \|z\|, t, s \in I \text{ and } x, y, z \in \mathcal{X},
\]
where \( \psi : [0, +\infty) \to (0, \infty) \) is a continuous non-decreasing function.

\((\text{H}_{10})\) The function \( I_k : \mathcal{X} \to \mathcal{X}, k = 1, 2, \ldots, m \), are completely continuous and uniformly bounded. Denote \( M_k = \sup\{\|I_k(x)\| : x \in \mathcal{X}\}, k = 1, 2, \ldots, m \).

\((\text{H}_{11})\) The operator families \( (S_n(t))_1 \geq 0 \) and \( (T_n(t))_1 \geq 0 \) are compact, where \( (T_n(t)) = t^{1-\alpha}T_n(t) \).

**Theorem 7** Assume that the hypotheses \((\text{H}_5)-(\text{H}_{11})\) holds. If \( A \in \mathcal{A}^\alpha(\theta_0, \omega_0) \), and the following conditions

\( \text{(a)} \) \( \bar{M}_s < 1; \)

\( \text{(b)} \) \( \frac{\bar{M}_s}{(1-M_s)} \int_0^T \bar{m}(s)ds < \int_0^\infty \frac{ds}{\psi(s)} \), where \( C = \max_{1 \leq s \leq m} \left\{ \frac{\bar{M}_s\|x_0\| + \bar{M}_s\bar{N}_1}{1 - M_s} \right\} \) are satisfied, then the system (1.4)-(1.6) has at least one mild solution defined on \( I \).

**Proof.** Define operator \( \Gamma : PC(I, \mathcal{X}) \to PC(I, \mathcal{X}) \) as by Theorem 3.1,
Note that $Γ$ is well defined on $PC(\mathcal{I}, \mathcal{X})$.

We prove the Theorem in the following five steps.

**Step 1.** $Γ$ is continuous.

Let $\{x_n\}$ be a sequence such that $x_n \to x$ in $PC(\mathcal{I}, \mathcal{X})$. Then for each $t \in \mathcal{I}$,

$$f(s, x_n(s), \int_0^s a(s, \tau, x_n(\tau))d\tau, \int_0^s b(s, \tau, x_n(\tau))d\tau) \to f(s, x(s), \int_0^s a(s, \tau, x(\tau))d\tau, \int_0^s b(s, \tau, x(\tau))d\tau) \quad \text{as} \ n \to \infty,$$

because the function $f$ is continuous on $\mathcal{I} \times \mathcal{X} \times \mathcal{X}$. Now, for every $t \in [0, t_1]$, we have

$$\|Γx_n(t) − Γx(t)\| \leq \frac{M_T T^\alpha}{\alpha} \left[\|f(s, x_n(s), \int_0^s a(s, \tau, x_n(\tau))d\tau, \int_0^s b(s, \tau, x_n(\tau))d\tau) − f(s, x(s), \int_0^s a(s, \tau, x(\tau))d\tau, \int_0^s b(s, \tau, x(\tau))d\tau)\|\right]$$

$$\leq \frac{\epsilon T^\alpha M_T}{\alpha}, \quad \epsilon > 0$$

$$\epsilon \to 0 \quad \text{as} \quad n \to \infty.$$

For $t \in (t_1, t_2]$, we have

$$\|Γx_n(t) − Γx(t)\| \leq \tilde{M}_s \left[\|x_n(t_1^-) − x(t_1^-)\| + \|I_1(x_n(t_1^-)) − I_1(x(t_1^-))\|\right]$$

$$+ \frac{M_T T^\alpha}{\alpha} \left[\|f(s, x_n(s), \int_0^s a(s, \tau, x_n(\tau))d\tau, \int_0^s b(s, \tau, x_n(\tau))d\tau) − f(s, x(s), \int_0^s a(s, \tau, x(\tau))d\tau, \int_0^s b(s, \tau, x(\tau))d\tau)\|\right]$$

$$\leq \tilde{M}_s \left[\|x_n(t_1^-) − x(t_1^-)\| + \|I_1(x_n(t_1^-)) − I_1(x(t_1^-))\|\right]$$

$$+ \frac{\epsilon M_T T^\alpha}{\alpha}, \quad \epsilon > 0$$

$$\epsilon \to 0 \quad \text{as} \quad n \to \infty.$$

Moreover, we have

$$\|Γx_n(t) − Γx(t)\| \leq \tilde{M}_s \left[\|x_n(t_1^-) − x(t_1^-)\| + \|I_1(x_n(t_1^-)) − I_1(x(t_1^-))\|\right]$$

$$+ \frac{\epsilon M_T T^\alpha}{\alpha}, \quad \epsilon > 0$$

$$\epsilon \to 0 \quad \text{as} \quad n \to \infty,$$

for all $t \in (t_i, t_{i+1}]$.

And

$$\|Γx_n(t) − Γx(t)\| \leq \tilde{M}_s \left[\|x_n(t_m^-) − x(t_m^-)\| + \|I_1(x_n(t_m^-)) − I_1(x(t_m^-))\|\right]$$

$$+ \frac{\epsilon M_T T^\alpha}{\alpha}, \quad \epsilon > 0$$

$$\epsilon \to 0 \quad \text{as} \quad n \to \infty,$$

for all $t \in (t_m, T]$.

Since $f$ and $I_k, k = 1, 2, \ldots, m$ are continuous functions, we have

$$\lim_{n \to \infty} \|Γx_n − Γx\|_{PC} = 0.$$
Step 2: $\Gamma$ maps bounded sets into bounded sets in $PC(I, X)$.

Indeed, it is enough to show that for any $r > 0$, there exists a positive constant $\gamma > 0$ such that $\|\Gamma x\|_{PC} \leq \gamma$ for each $x \in B_r = \{x \in PC(I, X) : \|x\|_{PC} \leq r\}$. By $(H_0)$ and $(H_{10})$, for each $t \in [0, t_1]$, we have

$$\|\Gamma x(t)\| \leq \tilde{M}_s \|x_0\| + \tilde{M}_T \int_0^t (t-s)^{\alpha - 1} \left\| f \left( s, x(s), \int_0^s (a, \tau, x(\tau)) d\tau, \int_0^T b(s, \tau, x(\tau)) d\tau \right) \right\| ds$$

$$\leq \tilde{M}_s \|x_0\| + \tilde{M}_T \int_0^t (t-s)^{\alpha - 1} \left[ p_3(s) \psi(\|x(s)\|) + p_1(s) \psi(\|x(s)\|) + p_2(s) \psi(\|x(s)\|) \right] ds$$

$$\leq \tilde{M}_s r + \tilde{M}_T T^{\alpha} \left[ p_3(t) + p_1(t) + p_2(t) \right] \psi(\|x(t)\|)$$

For $t \in (t_1, t_2)$, we have

$$\|\Gamma x(t)\| \leq \tilde{M}_s \|x(t^-)\| + \tilde{M}_s I_1(x(t^-)) + \tilde{M}_T T^{\alpha} \left[ p_3(t) + p_1(t) + p_2(t) \right] \psi(\|x(t)\|)$$

$$\leq \tilde{M}_s [r + M_1] + \tilde{M}_T T^{\alpha} \left[ p_3(t) + p_1(t) + p_2(t) \right] \psi(\|x(t)\|)$$

$$\leq \tilde{M}_s [r + M_1] + \tilde{M}_T T^{\alpha} \left[ p_3(t) + p_1(t) + p_2(t) \right] \psi(r) = \gamma.$$

Similarly, for $t \in (t_i, t_{i+1})$, $i = 1, 2, ..., m$,

$$\|\Gamma x(t)\| \leq \tilde{M}_s [r + M_1] + \tilde{M}_T T^{\alpha} \left[ p_3(t) + p_1(t) + p_2(t) \right] \psi(\|x(t)\|)$$

Step 3: $\Gamma(B_r)$ is bounded sets into equicontinuous sets, where $B_r$ is defined in Step 2.

Let $s_1, s_2 \in I = [0, T], s_1 < s_2$ by $\|S_{\omega}\| \leq Me^{\omega}$ and Theorem 2.3, we have

$$\|(\Gamma x)(s_2) - (\Gamma x)(s_1)\| \leq \|Me^{\omega_2} - Me^{\omega_1}\| \|x_0\|$$

$$+ \left[ p_3(t) \psi(\|x(s)\|) + p_1(t) \psi(\|x(s)\|) + p_2(t) \psi(\|x(s)\|) \right]$$

$$(\times) \tilde{M}_T \left( \int_0^{s_2} (s_2 - s)^{\alpha - 1} ds - \int_0^{s_1} (s_1 - s)^{\alpha - 1} ds \right)$$

$$\leq M |e^{\omega_2} - e^{\omega_1}| \|x_0\|$$

$$+ \tilde{M}_T (s_2 - s_1) \left[ p_3(t) + p_1(t) + p_2(t) \right] \psi(\|x(t)\|)$$

Similarly, for all $s_1, s_2 \in (t_i, t_{i+1}]$, with $s_1 < s_2$, $i = 1, 2, ..., m$, we have,

$$\|(\Gamma x)(s_2) - (\Gamma x)(s_1)\| \leq M(r + M_1) |e^{\omega_2} - e^{\omega_1}| + \tilde{M}_T (s_2 - s_1)$$

$$(\times) \left[ p_3(t) + p_1(t) + p_2(t) \right] \psi(\|x(t)\|)$$

Then, from the above inequalities, we have

$$\lim_{s_1 \to s_2} \| (\Gamma x)(s_2) - (\Gamma x)(s_1) \| = 0,$$

So $\Gamma(B_r)$ is equicontinuous.
Step 4: Γ maps \( B_r \) into a compact set in \( X \).

To this end, we decompose \( \Gamma \) by \( \Gamma = \Gamma_1 + \Gamma_2 \), where

\[
\Gamma_1 x(t) = \begin{cases} 
\int_0^t T_\alpha(t-s) f \left( s, x(s), \int_0^s a(s, \tau, x(\tau)) d\tau, \int_0^t b(s, \tau, x(\tau)) d\tau \right) ds, & t \in (0, t_1] \\
\int_1^t T_\alpha(t-s) f \left( s, x(s), \int_0^s a(s, \tau, x(\tau)) d\tau, \int_0^t b(s, \tau, x(\tau)) d\tau \right) ds, & t \in (t_1, t_2] \\
\vdots \\
\int_{t_{m-1}}^t T_\alpha(t-s) f \left( s, x(s), \int_0^s a(s, \tau, x(\tau)) d\tau, \int_0^t b(s, \tau, x(\tau)) d\tau \right) ds, & t \in (t_m, T] 
\end{cases}
\]

and

\[
\Gamma_2 x(t) = \begin{cases} 
S_\alpha(t)x_0, & t \in [0, t_1] \\
S_\alpha(t-t_1)(x(t_1^-) + I_1(x(t_1^-))), & t \in (t_1, t_2] \\
\vdots \\
S_\alpha(t-t_m)(x(t_m^-) + I_1(x(t_m^-))), & t \in (t_m, T]. 
\end{cases}
\]

We now prove that \( \Gamma_1 \) is completely continuous operator on \( X \). For \( t \in [0, t_1] \), applying the mean value theorem for Bochner integral [41] and the Young’s inequality [10], we have

\[
\{\langle \Gamma_1 x(t) \rangle \} \subset \frac{t_2^{1+\alpha}}{\alpha} \co {T_\alpha(t-s) f \left( s, x(s), \int_0^s a(s, \tau, x(\tau)) d\tau, \int_0^t b(s, \tau, x(\tau)) d\tau \right) d\tau : s \in [0, t_1], s \in B_r}.
\]

Similarly, for \( t \in (t_i, t_{i+1}], \ i = 1, 2, \ldots, m \), we obtain,

\[
\{\langle \Gamma_1 x(t) \rangle \} \subset \frac{(t-t_i)^{1+\alpha}}{\alpha} \co {T_\alpha(t-s) f \left( s, x(s), \int_0^s a(s, \tau, x(\tau)) d\tau, \int_0^t b(s, \tau, x(\tau)) d\tau \right) : s \in [t_i, t_{i+1}], s \in B_r}.
\]

By assumptions (H_9) and (H_11), we conclude that \( \{\langle \Gamma_1 x(t) \rangle \} \) is a compact subset of \( X \), for \( t \in I, s \in B_r \). So \( \Gamma_1 \) is compact.

Next, we show that \( \{\langle \Gamma_2 x(t) \rangle : t \in I, s \in B_r \} \) is compact in \( X \), for all \( t \in [0, t_1] \), since \( \langle \Gamma_2 x(t) \rangle = S_\alpha(t)x_0 \), by (H_11), it follows that \( \{\langle \Gamma_2 x(t) \rangle : t \in [0, t_1], s \in B_r \} \) is a compact subset of \( X \). On the other hand, for \( t \in (t_i, t_{i+1}], i \geq 1 \) and \( s \in B_r \), there exists \( r > 0 \) such that

\[
\Gamma_2^{-1} x(t) \in \begin{cases} 
S_\alpha(t-t_i)(y(t_i^-) + I_1(y(t_i^-))), & t \in (t_i, t_{i+1}], y \in B_r \\
S_\alpha(t_{i+1} - t_i)(y(t_i^-) + I_1(y(t_i^-))), & t = t_{i+1}, y \in B_r \\
(y(t_i^-) + I_1(y(t_i^-))), & t = t_i, y \in B_r,
\end{cases}
\]

where \( B_r \) is an open ball of radius \( r \). From (H_10) and (H_11), it follows that \( \Gamma_2 x\vert_{[t_i, t_{i+1}]} \) is relatively compact in \( X \), for all \( t \in (t_i, t_{i+1}], i \geq 1 \). Moreover, by the compactness of \( I_1, (i=1, 2, \ldots, m) \) and the continuity of the evolution operator \( S_\alpha(t) \), for all \( t \in [0, T] \), we conclude that operator \( \Gamma_2 \) is also compact.

Step 5: We show that the set

\[ E = \{ x \in PC(I, X) : x = \lambda F x \text{ for some } 0 < \lambda < 1 \} \]
is bounded in $PC(I, \mathcal{X})$.

Let $x_\lambda \in E$, then $x_\lambda(t) = \lambda(\Gamma x_\lambda)(t)$ for some $0 < \lambda < 1$. Thus,

$$
\|x_\lambda(t)\| \leq \begin{cases}
\begin{aligned}
\lambda \left[ M_S \|x_0\| + \tilde{M}_T \int_0^t (t-s)^{\alpha-1} |p_3(s)+p_1(s)+p_2(s)| \psi(\|x_\lambda(s)\|) ds \right], & t \in (0, t_1]; \\
\lambda \left[ M_S (\|x_\lambda\| + M_1) + \tilde{M}_T \int_1^t (t-s)^{\alpha-1} |p_3(s)+p_1(s)+p_2(s)| \psi(\|x_\lambda(s)\|) ds \right], & t \in (t_1, t_2]; \\
\lambda \left[ M_S (\|x_\lambda\| + M_m) + \tilde{M}_T \int_m^t (t-s)^{\alpha-1} |p_3(s)+p_1(s)+p_2(s)| \psi(\|x_\lambda(s)\|) ds \right], & t \in (t_m, T].
\end{aligned}
\end{cases}
$$

By the Young’s inequality [10], for $t \in (t_i, t_{i+1}]$, $i = 1, 2, ..., m$, we get that,

$$
\|x_\lambda(t)\| \leq \tilde{M}_S \|x_\lambda(t)\| + \tilde{M}_S M_i + \frac{\tilde{M}_T T^\alpha}{\alpha} \int_{t_i}^t \left[ p_3(s) + p_1(s) + p_2(s) \right] \psi(\|x_\lambda(s)\|) ds
$$

and for $t \in (0, t_1]$, we have

$$
\|x_\lambda(t)\| \leq \tilde{M}_S \|x_0\| + \frac{\tilde{M}_T T^\alpha}{(1 - M_S)\alpha} \int_0^t \left[ p_3(s) + p_1(s) + p_2(s) \right] \psi(\|x_\lambda(s)\|) ds.
$$

Let us take the right hand side of the above inequality as $\beta_\lambda(t)$. Then, for all $t \in [0, T]$,

we have

$$
\|x_\lambda(t)\| \leq \beta_\lambda(t) = C + \frac{\tilde{M}_T T^\alpha}{(1 - M_S)\alpha} \int_0^t \left[ p_3(s) + p_1(s) + p_2(s) \right] \psi(\|x_\lambda(s)\|) ds.
$$

where $C = \max_{1 \leq i \leq m} \left\{ \frac{\tilde{M}_S \|x_0\| + \tilde{M}_S M_i}{1 - M_S} \right\}$, then

$$
\beta_\lambda(t) \leq C + \frac{\tilde{M}_T T^\alpha}{(1 - M_S)\alpha} \int_0^t \left[ p_3(s) + p_1(s) + p_2(s) \right] \psi(\|x_\lambda(s)\|) ds.
$$

Computing $\beta'_\lambda(t)$ for $t \in [0, T]$, we arrive at

$$
\beta'_\lambda(t) \leq \frac{\tilde{M}_T T^\alpha}{(1 - M_S)\alpha} \left[ p_3(t) + p_1(t) + p_2(t) \right] \psi(\|x_\lambda(t)\|)
$$

$$
\beta'_\lambda(t) \leq -\frac{\tilde{M}_T T^\alpha}{(1 - M_S)\alpha} \left[ p_3(t) + p_1(t) + p_2(t) \right] \psi(\|\beta_\lambda(t)\|)
$$

Integrating from 0 to $t$, we obtain

$$
\int_0^t \frac{\beta'_\lambda(s)}{\psi(\|\beta_\lambda(s)\|)} ds \leq \frac{\tilde{M}_T T^\alpha}{(1 - M_S)\alpha} \int_0^t \tilde{m}(s) ds
$$

$$
\int_{\beta'(0)=C} \frac{ds}{\psi(s)} \leq \frac{\tilde{M}_T T^\alpha}{(1 - M_S)\alpha} \int_0^T \tilde{m}(s) ds < \int_C^\infty \frac{ds}{\psi(s)},
$$

where $\tilde{m}(t) = \max \left\{ p_3(t) + p_1(t) + p_2(t) \right\}$, $\beta_\lambda(0) = C$, $\beta_\lambda(t)$ is positive and non-decreasing. Hence, by the above inequality, we conclude that the set of functions $\{ \beta_\lambda : \lambda \in (0, 1) \}$ is bounded. This implies that $E = \{ x \in PC(I, \mathcal{X}) : x = \lambda \Gamma x, 0 < \lambda < 1 \}$ is bounded in $\mathcal{X}$. Since we have already prove that, $\Gamma$ is continuous and compact, by Theorem 3.2, $\Gamma$ has a fixed point which is a mild solution of (1.4)-(1.6) defined on $I$. This completes the proof.
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References


IJNS homepage: http://www.nonlinearscience.org.uk/


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