Existence of Global Solutions for Impulsive Abstract Partial Neutral Functional Differential Equations

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Abstract: In this paper we study the existence of global solutions for a class of impulsive abstract partial neutral functional differential equations. An application is provided to illustrate the theory.

Keywords: impulsive system; neutral differential equations; analytic semigroup theory; mild solutions; global solutions.

1 Introduction

Many evolution processes and phenomena experience abrupt changes of state through short-term perturbations. Since the durations of the perturbations are negligible in comparison with the duration of each process, it is quite natural to assume that these perturbations act in terms of impulses; see the monographs of Lakshmikantham et al. [17] and Samoilenko and Perestyuk [31]. It is to be noted that the recent progress in the development of the qualitative theory of impulsive differential equations has been stimulated primarily by a number of interesting applied problems [1–3, 7, 18, 19, 21]. A natural generalization of impulsive differential equations is abstract impulsive differential equations in Banach space. For general aspects of impulsive differential equations, monographs [29, 30] are recommended.

Recently in [8], the authors studied the existence of global solutions for an impulsive abstract functional differential equation of the form

\[
\begin{align*}
\frac{du}{dt}(t) &= A(t)u(t) + f(t, u(t), u(\rho(t))), \quad t \in I = [0, a] \text{ or } [0, \infty), \\
u(0) &= u_0, \\
\Delta u(t_i) &= J_i(u(t_i)), \quad i \in \mathbb{F} \subset \mathbb{N},
\end{align*}
\]

by using Leray-Schauder’s alternative theorem and in [4], the existence of solutions for impulsive neutral functional differential equations of the form

\[
\begin{align*}
\frac{d}{dt}(u(t) + F(t, u_t)) &= A(t)u(t) + G(t, u_t), \quad t \in I, \ t \neq t_i, \\
\Delta u(t_i) &= I_i(u_{t_i}), \\
u_0 &= \varphi \in \mathcal{B},
\end{align*}
\]

by using Leray-Schauder’s alternative theorem.

In this paper, we establish the existence of mild solutions for a class of impulsive neutral functional differential equations described by

\[
\begin{align*}
\frac{d}{dt}(u(t) + g(t, u_t)) &= Au(t) + f(t, u_t), \quad t \in I, \ t \neq t_i, \\
u_0 &= \varphi \in \mathcal{B}, \\
\Delta u(t_i) &= I_i(u_{t_i}),
\end{align*}
\]

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where $A$ is the infinitesimal generator of a $C_0$-semigroup of bounded linear operators $((T(t))_{t \geq 0}$ defined on a Banach space $(\mathcal{X}, \| \cdot \|)$; $I$ is an interval of the form $[0, T)$; $0 < t_1 < t_2 < \cdots < t_i < \cdots < T$ are prefixed numbers; the history $u_i : (-\infty, 0] \to \mathcal{X}, u_i(\theta) = u(t + \theta)$, belongs to some abstract phase space $\mathcal{B}$ defined axiomatically; $g, f : I \times \mathcal{B} \to \mathcal{X}$, \( \mathcal{I}_i : \mathcal{B} \to \mathcal{X} \) are appropriate functions and the symbol $\Delta \xi(t)$ represents the jump of the function $\xi$ at $t$, which is defined by $\Delta \xi(t) = \xi(t^+) - \xi(t^-)$.

The existence of global solutions for impulsive ordinary and partial differential system has been considered recently in the literature, see among others, [4, 8, 10–15, 23–26, 32–34]. To the best of our knowledge, the study of the existence of global solutions for the impulsive functional differential equations described in the general “abstract” form (1.1)-(1.3), is an untreated topic in the literature; this will be the main motivation of our paper.

We now turn to a summary of the work. The second section provides the definitions and preliminary results to be used in the theorems stated and proved in this article; we review some of the standard facts on phase spaces, mild solutions and certain useful fixed point theorems. In the third section, we focus our attention on the local existence of mild solutions for the problem (1.1)-(1.3), when $I = [0, a]$. The fourth section is dedicated to the study of the existence of global solutions for the problem (1.1)-(1.3). Finally, in the fifth section, we give an application.

## 2 Preliminaries

We introduce certain notations which will be used throughout the paper without any further mention. Let $(\mathcal{X}, \| \cdot \|)$ and $(\mathcal{Y}, \| \cdot \|)$ be Banach spaces, and $\mathcal{L}(\mathcal{Y}, \mathcal{X})$ be the Banach space of bounded linear operators from $\mathcal{Y}$ into $\mathcal{X}$ equipped with its natural topology; in particular, we use the notation $\mathcal{L}(\mathcal{X})$ when $\mathcal{Y} = \mathcal{X}$.

Throughout this work, $A$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $(T(t))_{t \geq 0}$ defined on a Banach space $(\mathcal{X}, \| \cdot \|)$ and a is a positive constant such that $\|T(t)\| \leq a$ for every $t \in I$. For the theory of strongly continuous semigroup, refer to Pazy [27].

In this paper, we employ an axiomatic definition of the phase space $\mathcal{B}$, which is similar to the one introduced in Hino et al [16] and suitably modified to treat retarded impulsive differential equations. More precisely, $\mathcal{B}$ is a linear space of functions mapping $(-\infty, 0]$ into $\mathcal{X}$ endowed with a seminorm $\| \cdot \|_B$ and we assume that $\mathcal{B}$ satisfies the following axioms:

**A** If $x : (-\infty, \sigma + a) \to \mathcal{X}$, $a > 0$, $\sigma \in \mathbb{R}$ such that $x_\sigma \in \mathcal{B}$, and $x_{[\sigma, \sigma + a]} \in \mathcal{P}(\mathcal{X}, \mathcal{Y})$, then for every $t \in [\sigma, \sigma + a)$ the following conditions hold:

(i) $x_t$ is in $\mathcal{B}$;

(ii) $\|x(t)\| \leq H|x_t|_B$;

(iii) $\|x_t\|_B \leq K(t - \sigma) \sup \{\|x(s)\|_\mathcal{X} : \sigma \leq s \leq t\} + M(t - \sigma)\|x_\sigma\|_B$.

where $H > 0$ is a constant; $K, M : [0, \infty) \to [1, \infty)$, $K$ is continuous, $M$ is locally bounded and $H, K, M$ are independent of $x(\cdot)$.

**B** The space $\mathcal{B}$ is complete.

**Remark 1** In impulsive functional differential systems, the map $[\sigma, \sigma + a] \to \mathcal{B}$, $t \mapsto x_t$ is in general discontinuous. For this reason, this property has been omitted from the description of phase space $\mathcal{B}$.

**Example 1** The Phase Space $\mathcal{P}_r \times L^2(g, \mathcal{X})$.

Let $r > 0$ and $g : (\infty, -r] \to \mathbb{R}$ be a non-negative, locally Lebesgue integrable function. Assume that there is a non-negative measurable, locally bounded function $\eta(\cdot)$ on $(-\infty, 0]$ such that $g(\xi + \theta) \leq \eta(\xi)g(\theta)$ for all $\xi \in (-\infty, 0]$ and $\theta \in (-\infty, -r) \setminus N$, where $N \subset (-\infty, -r]$ is a set with Lebesgue measure zero. We denote by $\mathcal{P}_r \times L^2(g, \mathcal{X})$ the set of all functions $\varphi : (-\infty, 0] \to \mathcal{X}$ such that $\varphi_{[-r, 0]} \in \mathcal{P}(\mathcal{X})$ and $\int_{-\infty}^{-r} g(\theta)\|\varphi(\theta)\|^2_\mathcal{X} d\theta < +\infty$. In $\mathcal{P}_r \times L^2(g, \mathcal{X})$, we consider the seminorm defined by

$$
\|\varphi\|_B = \sup_{\theta \in [-r, 0]} \|\varphi(\theta)\|_\mathcal{X} + \left(\int_{-\infty}^{-r} g(\theta)\|\varphi(\theta)\|_\mathcal{X}^2 d\theta\right)^{1/2}.
$$

From the proceeding conditions, the space $\mathcal{P}_r \times L^2(g, \mathcal{X})$ satisfies the axioms (A) and (B). Moreover, when $r = 0$, we can take $H = 1$, $K(t) = \left(1 + \int_0^t g(\theta)\right)^{1/2}$ and $M(t) = \eta(-t)$ for $t \geq 0$.

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Let $0 < t_1 < \cdots < t_n < a$ be pre-fixed numbers. We introduce the space $\mathcal{PC} = \mathcal{PC}([0, a]; \mathbb{X})$ formed by all functions $u : [0, a] \to \mathbb{X}$ such that are continuous at $t \neq t_i$, $u(t_i^-) = u(t_i^+)$ and $u(t_i^+)$ exists, for all $i = 1, \ldots, n$. In this paper, we assume that $\mathcal{PC}$ is endowed with the norm $\|u\|_{\mathcal{PC}} = \sup_{s \in [0, a]} \|u(s)\|_{\mathbb{X}}$. It is clear that $(\mathcal{PC}, \|\cdot\|_{\mathcal{PC}})$ is a Banach space; see [9] for details.

In what follows, we put $t_0 = 0$, $t_{n+1} = a$, and for $u \in \mathcal{PC}$, we denote by $\tilde{u}_i \in C([t_i, t_{i+1}]; \mathbb{X})$, $i = 0, 1, 2, \ldots, n$, the function given by

$$
\tilde{u}_i(t) = \begin{cases}
  u(t), & \text{for } t \in (t_i, t_{i+1}], \\
  u(t_i^+), & \text{for } t = t_i.
\end{cases}
$$

Moreover, for $B \subset \mathcal{PC}$, we employ the notation $\tilde{B}_i$, $i = 0, 1, 2, \ldots, n$, for the sets $\tilde{B}_i = \{ \tilde{u}_i : u \in B \}$.

**Lemma 2** ([9]). A Set $B \subset \mathcal{PC}$ is relatively compact in $\mathcal{PC}$ if and only if the set $\tilde{B}_i$ is relatively compact in the space $C([t_i, t_{i+1}]; \mathbb{X})$, for every $i = 0, 1, \ldots, n$.

In the following definition, we introduce the concept of a mild solution for the problem (1.1)-(1.3).

**Definition 1** ([28]). A function $u : (-\infty, 0] \cup I \to \mathbb{X}$ is called a mild solution of the abstract Cauchy problem (1.1)-(1.3), if $u_0 = \phi \in B; u|_I \in \mathcal{PC}(I; \mathbb{X})$; the function $s \to AT(t-s)g(s, u_s)$ is integrable in $[0, t)$ for all $t \in I$ and

$$
\begin{align*}
  u(t) &= T(t)(\phi(0) + g(0, \phi)) - g(t, u_t) - \int_0^t AT(t-s)g(s, u_s)ds + \int_0^t T(t-s)f(s, u_s)ds \\
  &\quad + \sum_{t_i < t} T(t-t_i)I_i(u_{t_i}) + \sum_{t_i < t} [g(t, u_t)|_{t_i^+} - g(t, u_t)|_{t_i^-}], \quad t \in I.
\end{align*}
$$

Motivated by the previous definition, we introduce the following assumptions. There exists a Banach space $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ continuously included in $\mathbb{X}$ such that

1. $\mathbb{S}_1$ The function $s \to T(t-s)y \in C([t, +\infty); \mathbb{Y}), \; t \geq 0$.
2. $\mathbb{S}_2$ The function $s \to AT(t-s)$ defined from $(t, +\infty), \; t > 0$ into $L(\mathbb{Y}, \mathbb{X})$ is continuous and there is a function $H(\cdot) \in L^1((0, \infty); \mathbb{R}^+)$ such that
   $$
   \|AT(t-s)\|_{L(\mathbb{Y}, \mathbb{X})} \leq H(t-s), \; \text{for all } t > s.
   $$

**Remark 3** Assume that $\mathbb{S}_1$ and $\mathbb{S}_2$ hold and $u \in C([0, t]; \mathbb{Y})$, then from the Bochner’s criterion for integrable functions and the estimate

$$
\|AT(t-s)u(s)\|_{\mathbb{X}} \leq \|AT(t-s)\|_{L(\mathbb{Y}, \mathbb{X})}\|u(s)\|_{\mathbb{Y}} \leq H(t-s)\|u(s)\|_{\mathbb{Y}},
$$

we have that the function $s \to AT(t-s)u(s)$ is integrable over $[0, t)$ for all $t > 0$. For additional remarks about these types of condition in partial neutral differential equations, see e.g.[20]. In general, we observe that, except in trivial cases, the operator function $s \to AT(t-s)$ is not integrable over $[0, t]$ (see e.g.[5]).

To obtain our results we need the following results.

**Theorem 4** ([16, Theorem 6.5.4]) Leray-Schauder’s Alternative Theorem. Let $D$ be a closed convex subset of a Banach space $(\mathbb{Z}, \|\cdot\|_{\mathbb{Z}})$ and assume that $0 \in D$. If $F : D \to D$ is a completely continuous map, then the set $\{ x \in D : x = \lambda F(x), \; 0 < \lambda < 1 \}$ is unbounded or the map $F$ has a fixed point in $D$.

**Theorem 5** ([22, Corollary 4.3.2]). Let $D$ be a closed, convex and bounded subset of a Banach space $(\mathbb{Z}, \|\cdot\|_{\mathbb{Z}})$. If $B, C : D \to \mathbb{Z}$ are continuous functions such that

1. $Bz + Cz \in D$ for all $z \in \mathbb{Z}$;
2. $C(D)$ is compact;
3. there exists $0 \leq \gamma < 1$ such that $\|Bz - Bw\| \leq \gamma \|z - w\|$ for all $z, w \in D$;

then there exists $x \in D$ such that $Bx + Cx = x$. 

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3 Local Existence

In this section, we study the existence of mild solutions for the impulsive systems (1.1)-(1.3), when \( I = [0, a] \). To obtain our results, we introduce the following conditions.

(H1) The function \( g : I \times \mathcal{B} \to \mathbb{X} \) is completely continuous, \( g(I \times \mathcal{B}) \subset \mathcal{Y} \), \( g \in C(I \times \mathcal{B}, \mathcal{Y}) \) and there exist positive constants \( c_1 \) and \( c_2 \) such that
\[
\| g(t, \phi) \|_{\mathcal{Y}} \leq c_1 \| \phi \|_{\mathcal{B}} + c_2, \quad t \in I, \quad \phi \in \mathcal{B}.
\]

(H2) The function \( f : I \times \mathcal{B} \times \mathbb{X} \to \mathbb{X} \) satisfies the following conditions.

(a) The function \( f(t, \cdot) : \mathcal{B} \to \mathbb{X} \) is continuous for all \( t \in I \).

(b) The function \( f(\cdot, \phi) : I \to \mathbb{X} \) is strongly measurable for all \( \phi \in \mathcal{B} \).

(c) There exists a continuous function \( m : I \to [0, \infty) \) and a continuous non-decreasing function \( W : [0, \infty) \to (0, \infty) \) such that
\[
\| f(t, \phi) \| \leq m(t)W(\| \phi \|_{\mathcal{B}}), \quad \text{for every } (t, \phi) \in I \times \mathcal{B}.
\]

(H3) The maps \( I_i : \mathcal{B} \to \mathbb{X} \) are completely continuous and uniformly bounded, \( i \in \mathbb{F} = \{1, 2, \ldots, N\} \). In what follows, we use the notation: \( N_i = \sup\{|I_i(\phi)| : \phi \in \mathcal{B}\} \).

(H4) There are positive constants \( L_i \), such that
\[
\| I_i(\psi_1) - I_i(\psi_2) \| \leq L_i \| \psi_1 - \psi_2 \|_{\mathcal{B}}, \quad \psi_1, \psi_2 \in \mathcal{B}, \quad i \in \mathbb{F}.
\]

(H6) The function \( g : I \times \mathcal{B} \to \mathcal{Y} \) is Lipschitz constant; that is, there is a constant \( L_g > 0 \) such that
\[
\| g(t, \phi_1) - g(t, \phi_2) \|_{\mathcal{Y}} \leq L_g \| \phi_1 - \phi_2 \|_{\mathcal{B}}, \quad t \in I, \quad \phi_i \in \mathcal{B}, \quad i = 1, 2.
\]

After these preparations, we can formulate the main result of this section.

Theorem 6 Assume that the hypotheses (S1), (S2), (H1), (H2), (H3) and (H6) are fulfilled. Suppose, in addition, that the following properties hold.

(a) For all \( t, s \in [0, a] \), \( t > s \) and \( r > 0 \), the set \( \{T(t-s)f(s, \psi) : s \in [0, t], \| \psi \|_{\mathcal{B}} \leq r\} \) is relatively compact in \( \mathbb{X} \).

(b) \( \mu = c_1 K_a \left( \| i_e \|_{L(\mathcal{Y}, \mathbb{X})} + \int_0^a H(s)ds \right) < 1, \quad \frac{K_a}{1-\mu} \int_0^a m(s)ds < \int_0^\infty \frac{ds}{W(s)}, \) where \( i_e : \mathbb{Y} \to \mathbb{X} \) is the inclusion operator and
\[
c = \frac{K_a}{1-\mu} \left( M + c_1 M \| i_e \|_{L(\mathcal{Y}, \mathbb{X})} + K_a^{-1} M_a \| \phi \|_{\mathcal{B}} + c_2 \| i_e \|_{L(\mathcal{Y}, \mathbb{X})} (M + 1) + c_2 \int_0^a H(s)ds + \bar{M} \sum_{i=1}^N N_i \right),
\]

where \( H \) is a constant given by Axiom (A), \( K_a \) and \( M_a \) are given by \( K_a = \sup_{0 \leq t \leq a} K(t) \) and \( M_a = \sup_{0 \leq t \leq a} M(t) \), respectively.

Then there exists a mild solution of the initial value problem (1.1)-(1.3).

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Proof.

Let \( S(I) \) and \( y : (-\infty, a] \to \mathbb{X} \) be introduced in (H\(_6\)) and \( \Gamma : S(I) \to S(I) \) be the operator defined by \((\Gamma u)_0 = 0\) and

\[
\Gamma u(t) = T(t)g(0, \phi) - g(t, u_t + y_t) - \int_0^t A T(t-s)g(s, u_s + y_s)ds + \int_0^t T(t-s)f(s, u_s + y_s)ds + \sum_{0 < t_i < t} T(t-t_i)I_i(u_{t_i} + y_{t_i}).
\]  

(6)

To prove that the function \( \Gamma u \in S(I) \), we need to show that the following properties are satisfied:

(i) The function \( \Gamma u \) is continuous in \( t \neq t_i \).

(ii) The limit \( \lim_{t \to t_i^-} \Gamma u(t) = \Gamma u(t_i) \), for all \( i = 1, \ldots, N \).

(iii) The limit \( \lim_{t \to t_i^+} \Gamma u(t) \), exists, for all \( i = 1, \ldots, N \).

Taking into the account that \( g \) and \( t \to T(t)g(0, \phi) \) are continuous, it follows from (S\(_1\)), (H\(_2\)) and the Remark 2 that the function \( \Gamma u(t) \) is continuous in \( t \neq t_i \). On the other hand, from the definition of \( \Gamma \), the Dominated Convergence Theorem and the condition (S\(_2\)), we have that \( \lim_{t \to t_i^-} \Gamma u(t) = \Gamma u(t_i) \) and that \( \lim_{t \to t_i^+} \Gamma u(t) = \Gamma u(t_i) + I_i(u_{t_i}) \).

Next, we show that \( \Gamma \) is a continuous operator. Let \((u^n)_{n \in \mathbb{N}}\) be a sequence in \( S(I) \) and \( u \in S(I) \) such that \( u^n \to u \) in \( S(I) \). By using the continuity of \( g \) and conditions ((H\(_2\)) - (a)), (H\(_3\)) and (H\(_6\)), we prove that \( g(t, u^n_s + y^n_s) \to g(t, u_s + y_s) \) uniformly in \( I \), \( f(s, u^n_s + y^n_s) \to f(s, u_s + y_s) \) for all \( s \in I \), and \( I_i(u^n_s + y^n_s) \to I_i(u_s + y_s) \) uniformly on \( I \). Using conditions (S\(_2\)), ((H\(_2\)) - (b), c) and Lebesgue’s Dominated Convergence Theorem, we conclude that

\[
\int_0^t AT(t-s)g(s, u^n_s + y^n_s)ds \to \int_0^t AT(t-s)g(s, u_s + y_s)ds, \int_0^t T(t-s)f(s, u^n_s + y^n_s)ds \to \int_0^t T(t-s)f(s, u_s + y_s)ds,
\]

as \( n \to \infty \), which clearly implies that \( \Gamma \) is a continuous operator.

In order to use Theorem 2, we need to obtain a \textit{a priori} bound for the solutions of the integral equations \( \lambda \Gamma u = u \), \( \lambda \in (0,1) \). Let \( \lambda \in (0,1) \) and let \( x^\lambda \) be a solution of \( \lambda \Gamma u = u \), \( 0 < \lambda < 1 \); taking into account that \( \|y_t\|_{\mathbb{B}} \leq (K_aM_H + M_a)\|\phi\|_{\mathbb{B}} \) and using Remark 2, (H\(_2\)) and (H\(_4\)), we find that

\[
\|x^\lambda(t)\| \leq \hat{M} \|\phi\|_{\mathbb{B}} + \|c_1\|_{\mathbb{L}(Y, X)} \left( c_1 K_a \|x^\lambda\|_t + (K_aM_H + M_a)\|\phi\|_{\mathbb{B}} + c_2 \right) + \int_0^t \mathcal{H}(t-s) \left( c_1 K_a \|x^\lambda\|_s + (K_aM_H + M_a)\|\phi\|_{\mathbb{B}} + c_2 \right)ds + \hat{M} \int_0^t m(s)W(K_a\|x^\lambda\|_s + (K_aM_H + M_a)\|\phi\|_{\mathbb{B}} + c_2)ds + \sum_{i=1}^N \hat{M} N_i,
\]

where we are using the following notation: \( \|f\|_t = \sup_{0 \leq s \leq t} \|f(s)\|_X \). Next, putting

\[
\delta \lambda(t) = K_a \|x^\lambda\|_t + (K_aM_H + M_a)\|\phi\|_{\mathbb{B}};
\]

it follows that

\[
\|x^\lambda(t)\| \leq \|c_1\|_{\mathbb{L}(Y, X)} \left( \hat{M} \|\phi\|_{\mathbb{B}} + c_2 \right) + \|c_1\|_{\mathbb{L}(Y, X)} \left( c_1 \delta \lambda(t) + c_2 \right) + \int_0^t \mathcal{H}(t-s) \left( c_1 \delta \lambda(s) + c_2 \right)ds + \hat{M} \int_0^t m(s)W(\delta \lambda(s))ds + \sum_{i=1}^N \hat{M} N_i.
\]

We obtain after some simplification and a rearrangement of terms

\[
\delta \lambda(t) \leq \left[ K_aM_H + c_1 K_a \hat{M} \|c_1\|_{\mathbb{L}(Y, X)} + M_a \right] \|\phi\|_{\mathbb{B}} + c_2 K_a \|c_1\|_{\mathbb{L}(Y, X)} \hat{M} + 1 + K_a c_2 \int_0^\alpha \mathcal{H}(s)ds + c_1 K_a \left( \|c_1\|_{\mathbb{L}(Y, X)} \right) \int_0^t \mathcal{H}(s)ds \delta \lambda(t) + K_a \hat{M} \int_0^t m(s)W(\delta \lambda(s))ds + K_a \hat{M} \sum_{i=1}^N N_i.
\]

From the hypothesis \((b)\), we have that

\[
\delta \lambda(t) \leq c + \frac{\hat{M} K_a}{1 - \mu} \int_0^t m(s)W(\delta \lambda(s))ds.
\]
Denote by $\beta_\lambda(t)$ the right-hand side of the previous inequality. Computing $\beta'_\lambda(t)$ and observing that $\delta_\lambda(t) \leq \beta_\lambda(t)$ for $t \in I$, we arrive at

$$\beta'_\lambda(t) \leq \frac{\tilde{M}K_\lambda}{1-\mu} m(t) W(\beta_\lambda(t)),$$

hence

$$\int_{\beta_\lambda(0)}^{\beta_\lambda(t)} ds \leq \frac{\tilde{M}K_\lambda}{1-\mu} \int_0^t m(s) ds \leq \int_0^\infty ds \frac{M}{W(s)}.$$

This allows us to conclude that the set of functions $\{\beta_\lambda : \lambda \in (0, 1)\}$ is bounded. This implies that $\{u_\lambda : u_\lambda = \Gamma u_\lambda, \lambda \in (0, 1)\}$ is bounded in $S(I)$.

It remains to show that $\Gamma$ is completely continuous. To this end, we introduce the decomposition $\Gamma = \Gamma_1 + \Gamma_2$, where $(\Gamma_i u)_0 = 0$, $i = 1, 2,$ and

$$\Gamma_1 u(t) = T(t)g(0, \phi) - g(t, u_t + y_t) - \int_0^t AT(t-s)g(s, u_s + y_s)ds + \int_0^t T(t-s)f(s, u_s + y_s)ds, \quad t \in [0, a],$$

$$\Gamma_2 u(t) = \sum_{0 < t_i < t} T(t-t_i)I_i(u_{t_i} + y_{t_i}), \quad t \in [0, a].$$

We now prove that $\Gamma_1$ is a completely continuous operator. We consider $u$ in $B_r(0, S(I))$, an open ball of radius $r$. Applying now the Mean Value Theorem for the Bochner integral (see [22, Lemma 2.1.3]), we infer that

$$(\Gamma_1 u)(t) \subset T(t)g(0, \phi) - g(t, u_t + y_t) - t\sup_{0 \leq \theta \leq t} |AT(t-\theta)g(\theta, u_\theta + y_\theta)|.$$  

Taking the advantage of assumptions (a) and (H4), we conclude that $\{(\Gamma_1 u)(t) : u \in B_r(0, S(I))\}$ is a compact subset of $S(I)$.

Now, we prove that the set of functions $\{(\Gamma_1 u) : u \in B_r(0, S(I))\}$ is equicontinuous in $I$. To this end, consider $0 < \varepsilon < t < t'$ for $t, t' \in [0, a]$; then there is $0 < \delta < \varepsilon$ such that

$$\|T(t)g(0, \phi) - T(t')g(0, \phi)\| < \varepsilon,$$

$$\|g(t, u_t + y_t) - g(t', u_{t'} + y_{t'})ds\| < \varepsilon,$$

$$\|T(t-s)f(u_s + y_s) - T(t'-s)G(u_s + y_s)\| < \varepsilon,$$

for all $s \in [0, t], |t - t'| < \delta$ and $u \in B_r(0, S(I))$. On the other-hand, using the continuity of the map $(t, s) \to AT(t-s)$ for $t > s$, we arrive at

$$\|AT(t-s)g(s, u_s + y_s) - AT(t'-s)g(s, u_s + y_s)\| < \varepsilon,$$

for all $s \in [0, t - \varepsilon], |t - t'| < \delta$ and $u \in B_r(0, S(I))$. Under these conditions, for $u \in B_r(0, S(I)), |t - t'| < \delta$, we infer that

$$\|\Gamma_1 u(t) - \Gamma_1 u(t')\| \leq \|(T(t) - T(t'))g(0, \phi)\| + \|g(t, u_t + y_t) - g(t', u_{t'} + y_{t'})\| + \int_{t-\varepsilon}^{t'} \|AT(t-s)g(s, u_s + y_s)\|ds$$

$$+ \int_0^{t-\varepsilon} \|AT(t-s) - AT(t'-s)\|g(s, u_s + y_s)\|ds + \int_{t-\varepsilon}^{t'} \|AT(t-s) - AT(t'-s)\|g(s, u_s + y_s)\|ds$$

$$+ \int_0^t \|(T(t-s) - T(t'-s))f(s, u_s + y_s)\|ds + \int_{t-\varepsilon}^{t'} \|(T(t'-s) - T(t'-s))f(s, u_s + y_s)\|ds.$$

From the above estimate, one can deduce the following inequality:

$$\|\Gamma_1 u(t) - \Gamma_1 u(t')\| \leq 2\varepsilon + \varepsilon(t - \varepsilon) + 2\varepsilon t + (c_1 K_a r + K_a \tilde{M} H + M_a) \int_{t-\varepsilon}^{t'} \mathcal{H}(t-s)ds + \int_{t-\varepsilon}^{t'} \mathcal{H}(t-s)ds$$

$$+ \tilde{M}W(c_1 K_a r + K_a \tilde{M} H + M_a) \int_{t-\varepsilon}^{t'} m(s)ds,$$
which shows the equicontinuity at $t \in I$. So, to conclude the proof, we show that $\Gamma_2$ is completely continuous. We observe that the continuity of $\Gamma_2$ is obvious. On the other hand, for $r > 0$, $t \in [t_i, t_{i+1}], i \geq 1$, and $u \in B_r = B_r(0, PC([0, a]; \mathbb{X}))$, we have that there exists $\tilde{r} > 0$ such that

$$\Gamma_2^2 u_i(t) \in \begin{cases} \sum_{j=1}^i T(t - t_j)I_j(B_r(0; B)), & t \in (t_i, t_{i+1}), \\ \sum_{j=1}^{t_i - 1} T(t - t_j)I_j(B_r(0; B)), & t = t_{i+1}, \\ \sum_{j=1}^{t_i} T(t - t_j)I_j(B_r(0; B)) + I_i(B_r(0; B)), & t = t_i, \end{cases}$$

where $B_r(0; B)$ is an open ball of radius $\tilde{r}$. From condition (H$_4$) it follows that $[\Gamma_2^2(B_r)]_i(t)$ is relatively compact in $\mathbb{X}$, for all $t \in [t_i, t_{i+1}], i \geq 1$. Moreover, using the fact that the operators $\{I_i\}_i$ are compact and the strong continuity of $(T(t))_{t \geq 0}$, we conclude that for $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|T(t - t_i)z - T(s - t_i)z\| \leq \varepsilon, \quad z \in \bigcup_{i=1}^N I_i(B_r(0, B)), \quad (3.3)$$

for all $t_i, i = 1, \ldots, N, t, s \in (t_i, t_{i+1}]$ with $|t - s| < \delta$. Thus for $u \in B_r, t \in [t_i, t_{i+1}], i \geq 0, \text{ and } 0 < |h| < \delta$ with $t + h \in [t_i, t_{i+1}]$, we have that

$$\|\Gamma_2^2 u_i(t + h) - \Gamma_2^2 u_i(t)\| \leq N\varepsilon, \quad i = 1, \ldots, N. \quad (3.4)$$

We have shown that the set $[\Gamma_2^2(B_r)]_i$ is uniformly equicontinuous for $i \geq 0$. Now, from Lemma 1, we conclude that $\Gamma_2$ is completely continuous.

We have proven that $\Gamma$ satisfies the conditions of Theorem 2, which allows us to infer the existence of a mild solution of the problem (1.1)-(1.3). This completes the proof of the theorem. \(\blacksquare\)

If the map $F$ and $I_i, i = 1, \ldots, N$ fulfill some Lipschitz condition instead of the compactness properties considered in Theorem 4, we also can establish an existence result.

**Theorem 7** Suppose that the assumptions (S$_1$), (S$_2$), (H$_2$), (H$_4$) and (H$_5$) are satisfied and that the condition (a) of Theorem 4 is fulfilled. If

$$\left[ K_a L_g \left( \|i_e\|_{L(Y, \mathbb{X})} + \int_0^T \mathcal{H}(s) ds \right) + M K_a \sum_{i=1}^N L_i + M H \lim_{r \to \infty} \inf_{\tau \geq r} \frac{W(\tau)}{r} \int_0^a m(s) ds \right] < 1,$$

where $i_e$ denotes the inclusion operator from $\mathbb{Y}$ into $\mathbb{X}$, then there exists a mild solution of the impulsive problem (1.1)-(1.3).

**Proof.** Let $\Gamma$ be the function given in the proof of Theorem 4 and consider the decomposition $\Gamma = \Gamma_1 + \Gamma_2$, where $(\Gamma_1 u) = 0$ on $(-\infty, 0], i = 1, 2,$ and

$$\Gamma_1 u(t) = T(t)g(0, \phi) - g(t, u_+ + y_1) - \int_0^t A T(t - s)g(s, u_+ + y_1) ds$$

$$+ \sum_{0 < t_i < t} T(t - t_i)I_i(u_+ + y_1), \quad t \in [0, a],$$

$$\Gamma_2 u(t) = \int_0^T T(t - s)f(s, u_+ + y_1) ds, \quad t \in [0, a].$$

We claim that there exists $r > 0$ such that $\Gamma(B_r(0, S(I))) \subset B_r(0, S(I))$. In fact, if we assume that this assertion is false, then for all $r > 0$ we can choose $x' \in B_r = B_r(0, S(I))$ and $t' \in [0, a]$ such that $\|\Gamma x'(t')\| > r$. Observe that standard computations involving the phase space axioms yield

$$r \leq \|\Gamma x'(t')\| \leq \tilde{M}\|g(0, \phi)\| + \|i_e\|_{L(Y, \mathbb{X})} \sup_{0 \leq t \leq a} \|g(t, 0)\| \|Y\| + L_g \|i_e\|_{L(Y, \mathbb{X})} (K_a r + (K_a \tilde{M} H + M_a)\|\phi\|_{B})$$

$$+ \int_0^a \mathcal{H}(s) ds \left( L_g (K_a r + (K_a \tilde{M} H + M_a)\|\phi\|_{B}) + \sup_{0 \leq \tau \leq a} \|g(\tau, 0)\| \|Y\| \right)$$

$$+ \tilde{M} \int_0^a m(s) W((K_a r + (K_a \tilde{M} H + M_a)\|\phi\|_{B}) ds + \tilde{M} \sum_{i=1}^N (L_i (K_a r + (K_a \tilde{M} H + M_a)\|\phi\|_{B}) + \|I_i(0)\|).$$

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Thus,
\[
1 \leq \left[ K_a L_2 \left( \| \nu \|_{\mathcal{L}(\mathcal{Y}, \mathcal{X})} + \int_0^a \mathcal{H}(s) ds \right) + \bar{M}_a \sum_{i=1}^N L_i + \bar{M}_a \lim_{r \to \infty} \inf \frac{W(r)}{r} \int_0^a m(s) ds \right],
\]
which is contrary to our assumptions.

Let \( r > 0 \) such that \( \Gamma(B_r(0, S(I))) \subset B_r(0, S(I)) \). It follows from the proof of Lemma 3.1 in [10] that \( \Gamma_2 \) is continuously discontinuous in \( B_r(0, S(I)) \). Moreover, the estimate
\[
\| \Gamma_1 u - \Gamma_1 v \|_{\infty} \leq \left[ K_a L_F \left( \| \nu \|_{\mathcal{L}(\mathcal{Y}, \mathcal{X})} + \int_0^a \mathcal{H}(s) ds \right) + \bar{M}_a \sum_{i=1}^N L_i \right] \| u - v \|_{\infty},
\]
u, v \in B_r, shows that \( \Gamma_1 \) is a contraction on \( B_r \). Consequently, \( \Gamma \) is a condensing operator from \( B_r \) into \( B_r \). Then the existence of a mild solution of (1.1)-(1.3) is a consequence of Theorem 3. This completes the proof. 

## 4 Global solutions

In this section, we study the existence of mild solutions for the impulse problem
\[
\frac{d}{dt}(u(t) + g(t, u_i)) = Au(t) + f(t, u_i), \quad t \in I = [0, \infty),
\]
u_0 = \varphi \in \mathcal{B}
\]
\[
\Delta u(t_i) = I_i(u_{t_i}), \quad 0 < t_1 < t_2 < \cdots < t_n < \cdots
\]
where \((t_i)_{i \in \mathbb{N}}\) is an increasing sequence of positive numbers.

Let \( h : [0, \infty) \to [0, \infty) \) be a positive, non-decreasing, continuous function such that \( h(0) = 1 \) and \( \lim_{t \to \infty} h(t) = +\infty \). In addition, suppose that the map \((t, s) \to T(t-s)\) is uniformly bounded. Moreover, \( \mathcal{PC}([0, \infty); \mathcal{X}) \) and \( \mathcal{PC}_h^0(\mathcal{X}) \) denote the spaces
\[
\mathcal{PC}([0, \infty); \mathcal{X}) = \left\{ x : [0, \infty) \to \mathcal{X} : \begin{array}{c} x|_{[0, a]} \in \mathcal{PC}, \forall a \in (0, \infty) \setminus \{ t_i : i \in \mathbb{N} \}, \\
\| x \|_{\infty} = \sup_{t \geq 0} \| x(t) \| < \infty
\end{array} \right\}
\]
and
\[
\mathcal{PC}_h^0(\mathcal{X}) = \left\{ x \in \mathcal{PC}([0, \infty); \mathcal{X}) : \begin{array}{c} \| x(t) \| \leq \lim_{t \to \infty} \frac{\| x(t) \|}{h(t)} = 0
\end{array} \right\},
\]
edowed with the norms \( \| x \|_{\infty} = \sup_{t \geq 0} \| x(t) \| \) and \( \| x \|_{\mathcal{PC}_h^0} = \sup_{t \geq 0} \frac{\| x(t) \|}{h(t)} \), respectively.

To get the next results, we need a very detailed knowledge of the relativity compact sets of the space \( \mathcal{PC}_h^0(\mathcal{X}) \). We will use the following result.

**Lemma 8** A bounded set \( \mathcal{B} \subset (\mathcal{PC}_h^0) \) is relatively compact in \( (\mathcal{PC}_h^0) \) if, and only if:

(a) The set \( \mathcal{B}_a = \left\{ u|_{[0, a]} : u \in \mathcal{B} \right\} \) is relatively compact in \( \mathcal{PC}([0, a]; \mathcal{X}) \), for all \( a \in (0, \infty) \setminus \{ t_i : i \in \mathbb{N} \} \).

(b) \( \lim_{t \to \infty} \frac{\| x(t) \|}{h(t)} = 0 \), uniformly for \( x \in \mathcal{B} \).

We introduce the following concept of global solution of system (4.1)-(4.3).

**Definition 2** A function \( u : \mathbb{R} \to \mathcal{X} \) is called a global solution of the problem (4.1)-(4.3), if the conditions (4.2) and (4.3) are verified; \( u|_{[0, a]} \in \mathcal{PC}([0, a]; \mathcal{X}) \) for all \( a \in (0, +\infty) \setminus \{ t_i : i \in \mathbb{N} \} \) and
\[
u(t) = T(t)(\phi(0) + g(0, \phi)) - g(t, u_t) - \int_0^t AT(t-s)g(s, u_s) ds + \int_0^t T(t-s)f(s, u_s) ds + \sum_{t_i < t} T(t-t_i)I_i(u_{t_i}), t \in I = [0, \infty).
\]
We have to prove the following theorem.

**Theorem 9** Assume that the hypotheses \((S_1), (S_2), (H_2), (H_3), (H_0)\) and the condition \((a)\) of Theorem 4 are valid in \(I = [0, a]\) for all \(a > 0\). Suppose, in addition, that functions \(M(\cdot)\) and \(K(\cdot)\) given by Axiom \((A)\) are bounded and that the following conditions hold.

\((a)\) For all \(J > 0\), \(\lim_{t \to \infty} \frac{1}{\eta(t)} \int_{0}^{t} m(s)W(Jh(s))ds = 0\).

\((b)\) The function \(g : X \times B \to Y\) is completely continuous and there exists a positive continuous function \(c_1 : [0, \infty) \to [0, \infty)\) with \(c_1(t) = 0\) when \(t \to \infty\) and a positive constant \(d_2\) such that \(\|g(t, \psi)\|_Y \leq c_1(t)\|\psi\|_B + d_2\) for all \(t > 0\), \(\psi \in B\) and

\[\lim_{t \to \infty} \frac{1}{h(t)} \int_{0}^{t} \mathcal{H}(t-s)h(s)ds = 0.\]

\((c)\) \(\eta = \left(\|i_\epsilon\|_{\mathcal{L}(Y,X)}K_{\infty}c_{1,\infty} + sup_{t \geq 0} K_{\infty} \int_{0}^{t} \mathcal{H}(t-s)c_1(s)ds\right) < 1.\)

\((d)\) \(\frac{MK_{\infty}}{1-\eta} \int_{0}^{\infty} m(s)ds < \int_{0}^{\infty} \frac{1}{W(s)}ds\), with \(c = (1 - \eta)^{-1} \left(\frac{MK_{\infty}H + M_{\infty} + \tilde{M}K_{\infty}\|i_\epsilon\|_{\mathcal{L}(Y,X)}c_{1,\infty}}{1-\eta} + \|i_\epsilon\|_{\mathcal{L}(Y,X)}\|\phi\|_B + \|i_\epsilon\|_{\mathcal{L}(Y,X)}\|\phi\|_B + d_2\right)\)

\[\geq d_2K_{\infty} \int_{0}^{\infty} \mathcal{H}(s)ds < \infty,\]

where \(c_{1,\infty} = sup_{t \geq 0} c_1(t), K_{\infty} = sup_{t \geq 0} K(t), M_{\infty} = sup_{t \geq 0} M(t).\)

Then there exists a global solution of \((4.1)-(4.3).\)

**Proof.** Let \(y : X \to \mathbb{R}\) be the extension of \(\phi\) such that \(y(t) = T(t)\phi(0), t \geq 0\) and \(S(\infty)\) be the set defined by

\[S(\infty) = \left\{ x : \mathbb{R} \to \mathbb{X} : x_0 = 0 \text{ and } x_{\mid_{[0, \infty)}} \in \mathcal{PC}_h(\mathbb{X}) \right\};\]

endowed with the norm \(\|x\|_{\mathcal{PC}_h} = sup_{t \geq 0} \frac{x_0(t)}{h(t)}\) and \(\Gamma : S(\infty) \to S(\infty)\) be the operator defined by \((\Gamma u)_0 = 0\) and

\[\Gamma u(t) = T(t)g(0, \phi) - g(t, y_0 + u_0) - \int_{0}^{t} AT(t-s)g(s, y_s + u_s)ds + \int_{0}^{t} T(t-s)f(s, y_s + u_s)ds + \sum_{t_i < t} T(t-t_i)I_i(y_{t_i} + u_{t_i}), t \geq 0.\]

Since \(\|u(t)\| \leq \|u\|_{\mathcal{PC}_h} h(t)\) for all \(t \geq 0\), we have that

\[\frac{\|\Gamma u(t)\|}{h(t)} \leq \frac{\tilde{M}\|\phi(0)\|}{h(t)} + \frac{\|i_\epsilon\|_{\mathcal{L}(Y,X)}c_1(t)K_{\infty}\|u\|_{\mathcal{PC}_h}(t)}{h(t)} + \frac{\|i_\epsilon\|_{\mathcal{L}(Y,X)}c_1(t)(\tilde{M}K_{\infty}H + M_{\infty})\|\phi\|_B + d_2}{h(t)} + \frac{\tilde{M}K_{\infty}}{h(t)} \sum_{i=1}^{\infty} N_i.
\]

so by \((a)\) and \((b)\), we have that \(\Gamma u \in S(\infty).\)

Next, we show that \(\Gamma(B_{r})\), where \(B_{r} = B_{r}(0, \mathcal{PC}_h(\mathbb{X})),\) satisfies the conditions of Lemma 2. To do this, we show the continuity of the operator \(\Gamma\). Let \((u^n)_{n \in \mathbb{N}}\) be a sequence in \(S(\infty)\) and \(u \in S(\infty)\) such that \(u^n \to u\) in \(S(\infty)\). Take \(C = sup\{\|u^n\|_{\mathcal{PC}_h}, \|u\|_{\mathcal{PC}_h}; n \in \mathbb{N}\}, J = (\tilde{M}K_{\infty}H + M_{\infty})\|\phi\|_B + CK_{\infty}\) and the function \(\mu : [0, \infty) \to \mathbb{R}\) defined by \(\mu(t) = c_1(t)J + d_2\).

From the conditions \((a), (b), (c), (e)\), there exists \(L_1 > 0\) such that

\[\frac{2}{h(t)} \int_{0}^{t} \mathcal{H}(t-s)\mu(s)h(s)ds + \frac{2\tilde{M}}{h(t)} \int_{0}^{t} m(s)W(Jh(s))ds + \frac{2\tilde{M}}{h(t)} \sum_{i=1}^{\infty} N_i < \frac{\varepsilon}{2},\]
for \( t \geq L_1 \). From the proof of Theorem 4, we have that \( g(t, u^n + y_t) \to g(t, u_t + y_t) \) uniformly for \( t \in [0, L_1] \), when \( n \to \infty \). From Lebesgue’s Dominated Convergence Theorem, we can fix a positive number \( \varepsilon \) such that

\[
\frac{1}{h(t)}\|g(t, u^n + y_t) - g(t, u_t + y_t)\| + \frac{M}{h(t)} \int_0^{L_1} \|f(s, u^n_s + y_s) - f(s, u_s + y_s)\| ds
\]

\[
+ \frac{1}{h(t)} \int_0^t \|H(t-s)(g(s, u^n_s + y_s) - g(s, u_s + y_s))\| ds
\]

\[
+ \frac{M}{h(t)} \sum_{t_i \leq L_1} \|I_i(u^n_{t_i} + y_{t_i}) - I_i(u_{t_i} + y_{t_i})\| < \frac{\varepsilon}{2}
\]

for all \( t \in [0, L_1] \) and \( n \geq N_\varepsilon \). Using the above inequality, for \( t \in [0, L_1] \) and \( n \geq N_\varepsilon \), we have that

\[
\sup \left\{ \frac{\|\Gamma u^n(t) - \Gamma u(t)\|}{h(t)} : t \in [0, L_1], \; n \geq N_\varepsilon \right\} \leq \varepsilon
\]

(11)

On the other hand, for \( t \geq L_1 \) and \( n \geq N_\varepsilon \), we get

\[
\frac{\|\Gamma u^n(t) - \Gamma u(t)\|}{h(t)} \leq \frac{1}{h(t)} \int_0^{L_1} \|AT(t-s)(g(s, u^n_s + y_s) - g(s, u_s + y_s))\| ds
\]

\[
+ \frac{1}{h(t)} \int_0^t \|AT(t-s)(g(s, u^n_s + y_s) - g(s, u_s + y_s))\| ds
\]

\[
+ \frac{M}{h(t)} \sum_{t_i \leq L_1} \|I_i(u^n_{t_i} + y_{t_i}) - I_i(u_{t_i} + y_{t_i})\| + \frac{2\bar{M}}{h(t)} \sum_{i=1}^{L_1} N_i
\]

\[
\leq \frac{2}{h(t)} \int_0^t \|H(t-s)\| ds + \frac{2\bar{M}}{h(t)} \sum_{i=1}^{L_1} N_i + \frac{\varepsilon}{2},
\]

and thus

\[
\sup \left\{ \frac{\|\Gamma u^n(t) - \Gamma u(t)\|}{h(t)} : t \geq L_1, \; n \geq N_\varepsilon \right\} \leq \varepsilon.
\]

(12)

Using inequalities (4.5) and (4.6), we conclude that \( \Gamma \) is continuous.

Next, we prove that the set \( \Gamma(B_r) \) is relatively compact. Let \( r > 0 \) be a positive real number, \( J = (\bar{M}K_\infty H + M_\infty)||\phi||_B + rK_\infty \). From the proof of Theorem 4, it follows that the set \( \Gamma(B_r) \cap [0, a] = \{\Gamma u|_{[0, a]} : u \in B_r\} \) is relatively compact in \( PC([0, a]; \mathcal{X}) \) for all \( a \in (0, \infty) \setminus \{i, i \in \mathbb{N}\} \). Moreover, for \( x \in B_r \), we have that

\[
\frac{\|\Gamma u(t)\|}{h(t)} \leq \frac{1}{h(t)} \left[ M \|\phi\|_{\mathcal{L}(\mathcal{Y}, \mathcal{X})} (c_1(0)||\phi||_B + d_2) \right] + \frac{M}{h(t)} \left[ \|\phi\|_{\mathcal{L}(\mathcal{Y}, \mathcal{X})} c_1(1) J h(t) + d_2 \right] + \frac{d_2}{h(t)}
\]

\[
+ \frac{\bar{M}}{h(t)} \int_0^t m(s) W(J h(s)) ds + \frac{M}{h(t)} \sum_{i=1}^{\infty} N_i + \frac{1}{h(t)} \int_0^t \|H(t-s)\| ds + \|\phi\|_{\mathcal{L}(\mathcal{Y}, \mathcal{X})} c_1(t) J
\]

\[
+ \frac{d_2}{h(t)} + \frac{\bar{M}}{h(t)} \int_0^t m(s) W(J h(s)) ds + \frac{M}{h(t)} \sum_{i=1}^{\infty} N_i
\]

which enables us to conclude that \( \frac{\|\Gamma u(t)\|}{h(t)} \to 0 \), when \( t \to \infty \), uniformly for \( x \in B_r \).

By Lemma 4.1, we infer that \( \Gamma(B_r) \) is relatively compact in \( (PC)^0_0(\mathcal{X}) \). Thus, \( \Gamma \) is completely continuous. To finish the proof, we need to obtain an \textit{a priori} estimate for the solutions of the integral equations \( \lambda \Gamma u = u, \lambda \in (0, 1) \). To this

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end, let \( u_\lambda \) be the solution of the equation \( \lambda \Gamma u_\lambda = u_\lambda \). For \( t \geq 0 \), we have that
\[
\|u_\lambda(t)\| \leq \|i_0\|_{L(\mathcal{Y}, \mathcal{X})} \left[ c_{1,\infty} (\tilde{M} + 1) \|i_0\|_{L(\mathcal{Y}, \mathcal{X})} \|\phi\|_B + K_\infty \|u_\lambda\|_s \right] + \|i_0\|_{L(\mathcal{Y}, \mathcal{X})} (\tilde{M} + 1) d_2
+ \int_0^t \mathcal{H}(t-s) \left( c_1(s) (\tilde{M} K_\infty H + M_\infty) \|\phi\|_B + K_\infty \|u_\lambda\|_s \right) + d_2 \right) ds
+ \tilde{M} \int_0^t m(s) W ((\tilde{M} K_\infty H + M_\infty) \|\phi\|_B + K_\infty \|u_\lambda\|_s) ds + M \sum_{i=1}^\infty N_i.
\]
Putting
\[
\delta^\lambda(t) = (\tilde{M} K_\infty H + M_\infty) \|\phi\|_B + K_\infty \|u_\lambda\|_s,
\]
we have that
\[
\delta^\lambda(t) \leq (1 - \eta)^{-1} \left( (\tilde{M} K_\infty H + M_\infty + \tilde{M} K_\infty \|i_0\|_{L(\mathcal{Y}, \mathcal{X})} \|\phi\|_B \|i_0\|_{L(\mathcal{Y}, \mathcal{X})} K_\infty (\tilde{M} + 1) d_2 + d_2 K_\infty \int_0^\infty \mathcal{H}(s) ds \right)
+ \tilde{M} K_\infty \int_0^t m(s) W (\delta^\lambda(s)) ds + \tilde{M} K_\infty \sum_{i=1}^\infty N_i,
\]
where \( \eta = \left( \|i_0\|_{L(\mathcal{Y}, \mathcal{X})} c_{1,\infty} + \sup_{s \geq 0} K_\infty \int_0^t \mathcal{H}(t-s) c_1(s) ds \right) \). Denoting the right-hand side of the previous expression as \( \beta^\lambda(t) \), we see that
\[
\beta^\lambda(t) \leq \frac{\tilde{M} K_\infty}{1 - \mu} m(t) W (\beta^\lambda(t)),
\]
and subsequently, upon integrating over \([0, t]\), we obtain
\[
\int_{\beta^\lambda(0)}^{\beta^\lambda(t)} \frac{ds}{W(s)} \leq \frac{\tilde{M} K_\infty}{1 - \mu} \int_0^a m(s) ds \leq \int_c^\infty \frac{ds}{W(s)}
\]
The above inequality together with condition (d) enables us to conclude that the set of functions \( \{ u_\lambda : u_\lambda = \lambda \Gamma u_\lambda \} \) is bounded in \( \mathcal{PC}([0, \infty); \mathcal{X}) \). Note that, if \( x \in \mathcal{PC}([0, \infty); \mathcal{X}) \), then \( \|x\|_{\mathcal{PC}} \leq \|x\|_\infty \). This allows us to conclude that the set \( \{ u_\lambda : u_\lambda = \lambda \Gamma u_\lambda, \lambda \in (0, 1) \} \) is bounded in \( S(\infty) \). Finally, from Theorem 2 we infer the existence of a fixed point of \( \Gamma \), and consequently, the existence of a global solution of (4.1)-(4.3). This completes the proof.

If we suppose that the functions \( I_i, i \in \mathbb{N} \) are Lipschitz continuous, then we have the following result.

**Theorem 10** Suppose that the hypotheses (S1), (S2), (H2), (H4) and (H0) are valid for all \( \alpha > 0 \). Assume also that condition (a) of Theorem 4 and condition (b) of Theorem 6 are satisfied and that \( \sum_{i=1}^\infty \|I_i(0)\| < +\infty \). If
\[
eq 1 \tag{14}
\]
where \( T(r, s) = (\tilde{M} K_\infty H + M_\infty) \|\phi\|_B + K_\infty r h(s), \) then there exists a global solution of the problem (4.1)-(4.3).

**Proof.** Let \( \Gamma \) be the operator introduced in the Theorem 6 and consider the decomposition \( \Gamma = \Gamma_1 + \Gamma_2 \), where \( \langle \Gamma_i u \rangle_0 = 0, i = 1, 2, \) and
\[
\Gamma_1 u(t) = T(t) g(0, \phi) - g(t, y_t + u_t) - \int_0^t A T(t-s) g(s, y_s + u_s) ds + \int_0^t T(t-s) f(s, y_s + u_s) ds, \quad t \geq 0,
\]
\[
\Gamma_2 u(t) = \sum_{t_i < t} T(t-t_i) I_i(y_{t_i} + u_{t_i}), \quad t \geq 0.
\]
Proceeding as in the proof of the Theorem 6, we infer that the map \( \Gamma_1 \) is completely continuous. Moreover, it is easy to see that
\[
\|\Gamma_2 u - \Gamma_2 v\|_{\mathcal{PC}} \leq \tilde{M} K_\infty \sum_{i=1}^\infty L_i \|u - v\|_{\mathcal{PC}}, \quad u, v \in S(\infty),
\]

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which implies that $\Gamma_2$ is a contraction in $S(\infty)$. We prove that there is $r > 0$ such that $\Gamma B_r \subset B_r$, where $B_r = B_r(0, S(\infty))$. In fact, if we assume that the assertion is false, then, for every $r > 0$, there exists $u' \in B_r$ and $t' \geq 0$ such that $\|v_{u'}(t')\| > r$. This yields that

$$r \leq \frac{1}{h(t')^2} \hat{M} \|g(0, \phi)\| + \frac{\hat{M}}{h(t')} \sum_{i, \leq t'} \|I_i(u_i', y_i)\| + \frac{\|g(t', u_i', y_i)\|}{h(t')}$$

$$+ \frac{1}{h(t')} \int_0^{t'} ||AT(t' - s)||_{L(\mathbb{X}, \mathbb{Y})} ||g(s, u_i' + y_i)||_{\mathbb{Y}} ds + \frac{1}{h(t')} \int_0^{t'} \hat{M} f(s, u_i' + y_i) ||ds$$

$$\leq \frac{1}{h(t')} \hat{M} \|g(0, \phi)\| + \frac{||i||_{L(\mathbb{X}, \mathbb{Y})} c_{1, \infty} (\hat{M} K_{\infty} H + M_{\infty}) \|\phi\|_{B} + K_{\infty} rh(t'))}{h(t')}$$

$$+ \frac{1}{h(t')} \int_0^{t'} \hat{M} m(s) W((\hat{M} K_{\infty} H + M_{\infty}) \|\phi\|_{B} + K_{\infty} rh(s)) ds$$

$$+ \frac{\hat{M}}{h(t')} \sum_{i = 1}^\infty L_i ((\hat{M} K_{\infty} H) \|\phi\| + K_{\infty} rh(t_i)) + \frac{\|i\|_{L(\mathbb{X}, \mathbb{Y})} d_2}{h(t')} + \frac{\hat{M}}{h(t')} \sum_{i = 1}^\infty \|I_i(0)\|,$$

which implies that

$$1 \leq c_{1, \infty} K_{\infty} (\|i\|_{L(\mathbb{X}, \mathbb{Y})} + \int_0^{\infty} \mathcal{H}(s) ds) + K_{\infty} \hat{M} \sum_{i = 1}^\infty L_i + \lim_{r \to \infty} \hat{M} \int_0^{\infty} m(s) W(T(r, s)) \frac{ds}{r},$$

which is contrary to our assumptions. This prove that there exists $r > 0$ such that $\Gamma$ is a condensing operator from $B_r$ into $B_r$. This completes the proof. ■

5 An application

Let $\mathbb{X} = L^2([0, \pi])$ be the space of functions which are square integrable and $B = \mathcal{P} C \times L^2(g, \mathbb{X})$. Now, we consider the operator $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ defined by $Ax = x''$, where $D(A) = \{x \in \mathbb{X} : x'' \in \mathbb{X}, x(0) = x(\pi) = 0\}$.

Is well known that $A$ is the infinitesimal generator of an analytic and compact semigroup $(T(t))_{t \geq 0}$ on $\mathbb{X}$. Furthermore, $A$ has a discrete spectrum, the eigenvalues are $-n^2$, $n \in \mathbb{N}$, with corresponding normalized eigenfunctions $z_n(\xi) = (\frac{\xi}{\pi})^{1/2} \sin(n \xi)$, and the following properties hold.

(a) $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis for $\mathbb{X}$.

(b) If $x \in D(A)$ then $Ax = -\sum_{n=1}^{\infty} n^2 (x, z_n) z_n$.

(c) For $x \in \mathbb{X}$, $T(t)x = \sum_{n=1}^{\infty} e^{-n^2 t} (x, z_n) z_n$. In particular, we have that $(T(t))_{t \geq 0}$ is a uniformly stable semigroup with $\|T(t)\| \leq e^{-t}$ for all $t \geq 0$. Moreover, it is possible to define the fractional power of $A$; see e.g. [5, 15, 20].

(d) For each $x \in \mathbb{X}$ and each $\beta \in (0, 1)$, $(-A)^{-\beta} x = \sum_{n=1}^{n} \frac{1}{n^{2\beta}} (x, z_n) z_n$. In particular, $\|(-A)^{-1/2}\| = 1$.

(e) For each $x \in \mathbb{X}$ and $\beta \in (0, 1)$, $(-A)^{\beta} x = \sum_{n=1}^{\infty} n^{2\beta} (x, z_n) z_n$. Moreover,

$$D((-A)^{\beta}) = \left\{ x : x \in \mathbb{X}, \sum_{n=1}^{n} n^{2\beta} (x, z_n) z_n \in \mathbb{X} \right\}.$$
\[ w(t, 0) = w(t, \pi) = 0, \quad t \geq 0, \]  
\[ w(\tau, \xi) = \phi(\tau, \xi), \quad \tau \leq 0, \quad 0 \leq \xi \leq \pi, \]  
\[ \Delta w(t_i, \cdot) = w(t_i^+, \cdot) - w(\cdot, t_i^-) = \int_{0}^{\pi} p_i(\xi, w(t_i, s)) ds, \]  
where \((t_i)_{i \in \mathbb{N}}\) is a strictly increasing sequence of positive real numbers. To treat this system, we assume the following conditions.

(a) The functions \(v : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}\), is continuous and there is a continuous and integrable function \(m : \mathbb{R} \rightarrow [0, \infty)\) such that
\[ |v(t, x)| \leq m(t)(|x|), \quad t \geq 0, \quad x \in \mathbb{R}. \]

(b) The function \(a_1 : \mathbb{R} \rightarrow [0, \infty)\) is continuous and \(\lim_{t \rightarrow +\infty} a_1(t) = 0\).

(c) The functions \(b(s, \eta, \xi), \frac{\partial}{\partial \eta} b(s, \eta, \xi)\) are integrable, \(b(s, \eta, \pi) = b(s, \eta, 0) = 0\) and
\[ L_F = \sup \left\{ \int_{0}^{\pi} \int_{-\infty}^{0} \int_{0}^{\pi} \left( \frac{\partial}{\partial \xi} b(s, \eta, \xi) \right)^2 d\eta d\xi : i = 0, 1 \right\} < +\infty. \]

(d) The functions \(p_i : [0, \pi) \times \mathbb{R} \rightarrow \mathbb{R}, i \in \mathbb{N}\), are continuous and there are positive constants \(L_i\) such that
\[ |p_i(\xi, s) - p_i(\xi, \pi)| \leq L_i |s - \pi|, \quad \xi \in [0, \pi], \quad s, \pi \in \mathbb{R}. \]

We now define the functions \(g, f : [0, \infty) \times \mathcal{B} \rightarrow \mathcal{X}, I_i : \mathcal{X} \rightarrow \mathcal{X}\) by
\[ g(t, \phi)(\xi) = \int_{-\infty}^{0} \int_{0}^{\pi} a_1(t) b(s, \eta, \xi) \phi(s, \eta) d\eta ds, \quad t \geq 0, \quad \xi \in [0, \pi], \]
\[ f(t, \phi)(\xi) = v(t, \phi(0, \xi)) \quad t \geq 0, \quad \xi \in [0, \pi], \]
\[ I_i(\phi)(\xi) = \int_{0}^{\pi} p_i(\xi, \phi(0, s)) ds, \quad i \in \mathbb{N}, \quad \xi \in [0, \pi], \]
and
\[ \mathcal{Y} = D(A^{1/2}) = \left\{ x(\cdot) \in \mathcal{X} : \sum_{i=1}^{\infty} n < x, z_n > z_n \in \mathcal{X} \right\}. \]

Using the properties of semigroup \((T(t))_{t \geq 0}\), we have that \(\mathcal{Y}\) verifies the conditions \((H_1)\) \((H_2)\) and \((H_3)\). Moreover, in this case
\[ \|(-A)^{1/2} T(t)\|_{\mathcal{X}} \leq C t^{1/2} \left( \frac{t}{\sqrt{2}} \right)^n, \quad \text{for } t > 0. \]

It easy to see that problem (5.1)-(5.4) can be modeled as the abstract impulsive Cauchy problem (4.1)-(4.3).

The next result follows directly from Theorem 7.

**Proposition 11** Assume that the previous conditions hold. Moreover, suppose that
\[ \sum_{i=1}^{\infty} \left( \int_{0}^{\pi} |p_i(\xi, 0)|^2 d\xi \right)^{1/2} < \infty. \]
\[ K_{\infty} \left[ L_g \left( 1 + \Gamma(1/2) \right) + \hat{M} \int_{0}^{\infty} m(s) ds + \sum_{i=1}^{\infty} L_i \right] < 1, \]
where \(\Gamma(\cdot) : (0, \infty) \rightarrow \mathbb{R}\) denotes the Gamma function and \(c_{1, \infty} = L^{1/2} \sup_{t \geq 0} a_1(t)\), then there is a global solution of the impulsive system (5.1)-(5.4).
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