Solitary Wave Solutions for a K(m,n,p,q+r) Equation with Generalized Evolution

Houria Triki¹, Abdul-Majid Wazwaz² *

¹Radiation Physics Laboratory, Department of Physics, Faculty of Sciences, Badjhi Mokhtar University, P.O. Box 12, 23000 Annaba, Algeria.
²Department of Mathematics, Saint Xavier University, Chicago, IL 60655.

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Abstract: Studying solitons and compactons is of important significance in nonlinear physics. In this work we study an extension of the K(n,n) equation and the resulting compactons that appear in the super deformed nuclei, the fission of liquid drops and the inertial fusion. The generalized equation, with the generalized evolution term, nonlinear convection terms, the fifth-order nonlinear dispersion and the higher-order nonlinear dispersion corrections will be examined. The sine-cosine ansatz is used to carry out the analysis for this equation and to study the structures of the obtained solutions. A variety of solutions of different structures that contain periodic solutions, solitons solutions, compactons solutions and solitary pattern solutions is obtained.

Keywords: K(n,n) equation; sine-cosine ansatz; solitary waves solutions; compactons

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1 Introduction

The celebrated KdV equation reads

\[ u_t + 6uu_x + u_{xxx} = 0. \] (1)

The KdV equation models a variety of nonlinear phenomena, including ion acoustic waves in plasmas, and shallow water waves. The derivative \( u_t \) characterizes the time evolution of the wave propagating in one direction, the nonlinear term \( uu_x \) describes the steepening of the wave, and the linear term \( u_{xxx} \) accounts for the spreading or dispersion of the wave. The KdV equation (1) gives rise to solitons due to the balance between the nonlinear convection \( uu_x \) and the linear dispersion \( u_{xxx} \). Soliton is a localized wave that has an infinite support or a localized wave with exponential wings [1–15]. Solitons retain their identities after mutual collision. This means that soliton has the property of a particle.

Moreover, the standard \( K(n,n) \) equation

\[ u_t + (u^n)_x + (u^n)_{xxx} = 0, \] (2)

where compactons arise as a result of the delicate balance between the nonlinear convection \( (u^n)_x \) with genuine nonlinear dispersion of \( (u^n)_{xxx} \). Compactons are solitary waves with exact compact support that are termed compactons. Unlike soliton that narrows as the amplitude increases, the compacton’s width is independent of the amplitude.

In modern physics, a suffix-on is used to indicate the particle property, for example phonon, photon, and soliton. For this reason, the solitary wave with compact support is called compacton to indicate that it has the property of a particle. The classical solitons are analytic solutions, whereas compactons are nonanalytic solutions. Compactons were proved to collide elastically [16–22] and vanish outside a finite core region.

However, the compacton concept has been studied by using many analytical and numerical methods, such as the pseudo spectral method, the tri-Hamiltonian operators, the finite difference method, and Adomian decomposition method. The compactons discovery motivated a considerable work to make compactons be practically realized in scientific applications, such as the super deformed nuclei, pre-formation of cluster in hydrodynamic models, the fission of liquid drops (nuclear physics), inertial fusion and others.

*Corresponding author. E-mail address: halimgamil@yahoo.com

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Compactons have multiple applications in physics. The $K(n,n)$ was discovered as a simplified model to study the role of nonlinear dispersion on pattern formation in liquid drops, being also proposed in the analysis of patterns on liquid surfaces. Equations with compacton solutions have also found applications such as the lubrication approximation for thin viscous films [22], semi classical models for Bose Einstein condensates, long nonlinear surface waves in a rotating ocean when the high-frequency dispersion is null, the pulse propagation in ventricleaorta system, dispersive models for magma dynamics, or, even, particle wave functions in nonlinear quantum mechanics. In nonlinear lattices the propagation of compacton-like kinks has been observed using mechanical, electrical, and magnetic analogs.

Nonlinear evolution equations supporting travelling wave solutions have been the subject of intense research in recent years. The investigation of exact travelling wave solutions to non-linear partial differential equations (NLPDEs) plays an important role in the study of non-linear physical phenomena [1]. Traveling waves appear in many physical structures in solitary wave theory such as solitons, kinks, peakons, and cuspons [2]. One of the localized travelling waves which has attracted a considerable interest in nonlinear science is envelope solitons because of their potential application in many physics areas. Solitons are defined as localized waves that propagate without change of its shape and velocity properties and stable against mutual collisions [3]. It needs to be noted that the formation of this kind of pulses is due to a perfect balance between weak nonlinearity and dispersion effects under specific conditions. From a mathematical point of view, there exists a certain class of NLPDEs that support soliton solutions in physical systems. The most famous ones are the nonlinear Schrödinger (NLS) equation, the Korteweg-de Vries (KdV) equation, the sine-Gordon (sG) equation and so on.

In many practical physics problems, the resulting nonlinear wave equations of interest are non integrable [4]. In some particular cases they may be close to an integrable one [4]. It is remarkable that non-integrability is not necessarily related to the nonlinear terms [5]. Higher order dispersions, for example, also can make the system to be non-integrable (while it remains Hamiltonian) [5].

The effort in finding exact solution to nonlinear equation is important for the understanding of most nonlinear physical phenomena [6]. In recent years, many powerful methods to construct exact solutions of NLPDEs have been established and developed, which lead to one of the most excited advances of nonlinear science and theoretical physics [7]. In fact, many kinds of exact soliton solutions have been obtained by using, for example, the homogeneous balance principle and F-expansion method [8], the Jacobi elliptic functions method [6], the sine-cosine and tanh methods [9], the Hirota’s bilinear method, the Bäcklund transformation method, and so on.

The concept of compactons: solitons with compact support, or strict localization of solitary waves appeared recently in the literature [10]. In [10], the authors introduced a genuinely nonlinear equation $K(m,n)$, a special type of the KdV equation, to examine the role played by the nonlinear dispersion in the formation of patterns in liquid drops. The proposed $K(m,n)$ equation takes the form [10]

\[ u_t + (u^m)_x + (u^n)_{xxx} = 0, \quad m > 0, \quad 1 < n \leq 3. \tag{3} \]

which is a generalization of the KdV equation. As important examples, we find that the $K(2,1)$ and $K(3,1)$ models are the well known KdV and mKdV equations. The delicate interaction between nonlinear convection $(u^m)_x$ with genuine nonlinear dispersion $(u^n)_{xxx}$ in the $K(m,n)$ equation (1) generates solitary waves with exact compact support that are termed compactons.

The issue of the existence of compactons solutions for the $K(m,n)$ models has been addressed by many authors since the analysis made so far in this regard. Wazwaz [11] discussed two generalized forms of the $K(n,n)$ and the KP equations that exhibit compactons. In [12], a study was conducted on $mK(n,n)$ equations in higher dimensional spaces and the construction of compact and noncompact solutions was shown. Lately, Wazwaz [13] used the Adomian decomposition method to construct exact special solutions with solitary patterns to the defocusing $K(m,n)$ equation:

\[ u_t - a (u^m)_x + (u^n)_{xxx} = 0, \quad m, n > 1. \tag{4} \]

The new solitary wave special solutions with compact support were also constructed in [10] and [14] for the nonlinear focusing dispersive $K(m,n)$ equation:

\[ u_t + a (u^m)_x + (u^n)_{xxx} = 0, \quad m, n > 1. \tag{5} \]

Recently, Biswas [15] obtained the 1- soliton solution of $K(m,n)$ equation with the generalized evolution term of the form

\[ (u^l)_t + au^m u_x + b (u^n)_{xxx} = 0, \tag{6} \]

where $a, b \in \mathbb{R}$ are constants, while $l, m, n \in \mathbb{Z}^+$. 

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More recently, Yin and Tian [16] have used the variational iteration method for solving the $K(2, 2, 1)$ and $K(3, 3, 1)$ equations that are particular forms of the following nonlinear dispersive $K(p, q, 1)$ equations

$$u_t + a (u^p)_x + (u^q)_{3x} + u_{5x} = 0,$$

In [17], the following two equations:

$$u_t + (u^m)_x + (u^m)_{3x} + \delta (u^m)_{5x} = 0,$$

and

$$u_t + (u^{m+1})_x + [u (u^{m})_{xx}]_x = 0,$$

were examined and a variety of explicit compact solitary waves is determined.

Very recently, in [18], the authors have studied the two and three dimensional, $N = 2, 3$, nonlinear dispersive equation $C_N(m, a + b)$:

$$u_t + (u^m)_x + \frac{1}{b} [u^a (\nabla^2 u^b)]_x = 0,$$

where the degeneration of the dispersion at the ground state induces cylindrically and spherically symmetric compactons convected in the $x$ direction.

2 The problem

In this work, we introduce the following $K(m, n, p, q + r)$ equation with the generalized evolution term, nonlinear convection terms, the fifth-order nonlinear dispersion and higher-order nonlinear dispersions corrections:

$$(u^l)_t + a_1 u^m u_x + a_2 u^{m+1}_x + a_3 (u^n)_{3x} + a_4 (u^p)_{5x} + a_5 [u^q (u^r)]_{xx} = 0.$$  \hspace{1cm} (11)

where $a_i \in R$ are constants, $i = 1, \ldots, 5$, with $l$, $m$, $n$, $p$, $q$, $r \in Z^+$. Note that the coefficients $a_i$ depend on the physical context. The finding of these coefficients is directly related to the environmental properties of the medium. Here in (11), the first term is the generalized evolution term, the second and third terms together represent nonlinear advection, the fourth and fifth terms represent the third-order and fifth-order nonlinear dispersions respectively, while the last term is a nonlinear dispersion correction term. Remarkably, the last term proportional to $a_5$ contains two nonlinear dispersion contributions $u^q (u^r)_{xxx}$ and $(u^q)_t (u^r)_{xx}$.

This equation is a generalized quintic extension of the $K(m, n)$ equation (4). In fact, when extremely short pulses are considered, the standard $K(m, n)$ equation (4) fails in the physical description of the propagation of solitary waves and higher-order effects must be taken into account. Thus, the dynamics of pulses is described by the $K(m, n)$ family of equations with higher order nonlinear dispersions terms. Attention has been focused on the role of additional terms (the third, fifth and last terms) which describe higher order nonlinear and dispersive effects and may have influence on the properties of the resulting solutions. As a matter of fact, this problem is important since the generalized versions of the $K(m, n)$ equation are of a great interest of both mathematical and physical point of view.

In the particular case where $a_2 = a_4 = a_5 = 0$, Eq. (11) reduces to the $K(m, n)$ equation (4) which has been interestingly studied by Biswas [15].

In the limit $l = m = n = p = q = r = 1$, Eq. (11) becomes

$$u_t + a_1 u u_x + a_2 u^2 u_x + a_3 u^3 x_x + a_4 u^5 x_x + a_5 [u (u)]_{xx} = 0.$$  \hspace{1cm} (12)

Formally, Eq. (10) is identical to the celebrated higher-order KdV equation that has been extensively studied by Marchant [19-20].

First, we should point out that Eq. (11) is not integrable because of the presence of the higher-order nonlinear and dispersive effects. By integrable [23–33] we mean that the equation has Lax pairs and gives multiple soliton solutions. Also, integrable equations have infinite laws of conservation of energy. The KdV equation, modified KdV equation, Schrodinger equation, Benjamin-Ono equation are examples of integrable equations. The fifth-order Lax equation, the fifth-order Sawada-Kotera equation, the fifth-order Kaup-Kuperschmidt equation, the Boussinesq equation, and the sixth-order Raman equations are also examples of integrable equations [23–33]. Moreover, the Burgers equation

$$u_t + a u u_x + \nu u_{xx},$$  \hspace{1cm} (13)

that includes nonlinear evolution term and the dissipative term $u_{xx}$ also is an integrable equation.
As stated before, the K(m,n) equation is not an integrable equation. The BBM equation, the generalized KdV equation, the Zakharov-Kuznetsov equation and the fifth-order Kawahara equation are examples of non-integrable equations. Multiple soliton solutions for these equations cannot be obtained in this case.

It is always useful and desirable to construct exact analytical solutions by using appropriate techniques. In this paper, we deal with the existence of exact solitary wave solutions of the $K(m,n,p,q+r)$ equation (9) as it appears, namely for general values of $l, m, n, p, q$ and $r$. Importantly, it is not possible to integrate (9) by the inverse scattering transform for any general values of the exponents $l, m, n, p, q$ and $r$ since the Painlevé test of integrability will fail in this situation. It is widely believed that possession of the Painlevé property is a sufficient criterion for integrability [5].

In this work, we plan to use the sine-cosine ansatz for the determination of exact analytical solitary wave solutions for the considered nonlinear evolution equation. All the physical parameters in the solitary wave solutions are obtained as functions of the model coefficients. To the best of our knowledge, studies of exact solutions for the $K(m,n)$ models including higher-order effects are still few. Moreover, the $K(m,n,p,q+r)$ model (9) with various important physical effects and general values of the exponents $l, m, n, p, q$ and $r$ was not introduced and studied before. It is worth noting that the existence or the non-existence of solitary wave solutions depends on the dependent model coefficients, and therefore on the specific nonlinear and dispersive features of the medium.

3 Exact solitary wave solutions

We begin our analysis by introducing the wave variable:

$$\xi = x - ct,$$

(14)

where $c$ is the wave speed. The transformation (11) converts the nonlinear equation (9) into

$$-c \left( u' \right) + a_{11} \left( u^{m+1} \right) + a_{22} \left( u^{m+2} \right) + a_{33} \left( u^n \right) + a_4 \left( u^p \right) + a_5 \left( u^q \right) = 0,$$

(15)

where $a_{11} = \frac{a_1}{m+1}$ and $a_{22} = \frac{a_2}{m+2}$. Here the prime represents the derivative of $u(\xi)$ with respect to the variable $\xi$.

Integrating Eq. (12) once give rise to the following reduced ODE equation

$$-cu' + a_{11} u^{m+1} + a_{22} u^{m+2} + a_{33} u^{n} + a_{44} \left( u^p \right) + a_{55} u^q = 0,$$

(16)

Note that we have set the integration constant to zero for the case of solitary wave solutions.

Compactons are compact solutions that are usually expressed by powers of trigonometric functions sine and cosine [21]. The sine–cosine ansatz admits the use of the assumption

$$u(\xi) = \left\{ \lambda \cos^\beta (\mu \xi) \right\}, \quad |\xi| \leq \frac{\pi}{2\mu}$$

(17)

or the assumption

$$u(\xi) = \left\{ \lambda \sin^\beta (\mu \xi) \right\}, \quad |\xi| \leq \frac{\pi}{\mu}$$

(18)

where $\lambda$, $\mu$ and $\beta$ are parameters that will be determined. Here $\lambda$ represents the amplitude of the compacton, while $\mu$ is the wave number. The exponent $\beta$ will be determined as a function of $l, m, n, p, q$ and $r$.

From the ansatz (15), one obtains

$$u'(\xi) = \lambda \cos^\beta (\mu \xi),$$

(19)

$$u^{m+1}(\xi) = \lambda^{m+1} \cos^{\beta(m+1)} (\mu \xi),$$

(20)

$$u^{m+2}(\xi) = \lambda^{m+2} \cos^{\beta(m+2)} (\mu \xi),$$

(21)

$$(u^n)' = -n\mu^2 \beta^2 \lambda^n \cos^{\beta n} (\mu \xi) + n\mu^2 \lambda^n \beta (n\beta - 1) \cos^{\beta n - 2} (\mu \xi),$$

(22)

$$(u^p)' = -\mu^4 \lambda^p \beta p (\beta p - 1) \left( \beta^2 p^2 - 2\beta p + 2 \right) \cos^{\beta p - 2} (\mu \xi),$$

(23)

$$u^q (u^r)' = \mu^2 \lambda^q r^2 (\beta r - 1) \cos^{\beta (q+r)-2} (\mu \xi) - \mu^2 \lambda^q r^2 \beta r^2 \cos^{\beta (q+r)} (\mu \xi).$$

(24)

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From the ansatz (16), we get
\begin{align*}
u' &= \lambda' \sin^{\beta l} (\mu \xi), \\
u'^{m+1} &= \lambda^{m+1} \sin^{\beta (m+1)} (\mu \xi), \\
u'^{m+2} &= \lambda^{m+2} \sin^{\beta (m+2)} (\mu \xi),
\end{align*}
(25)
(26)
(27)

\( (u^n)' = -n^2 \mu^2 \beta^2 \lambda^m \sin^{n \beta} (\mu \xi) + n \mu^2 \lambda^\beta (\beta n - 1) \sin^{n \beta - 2} (\mu \xi), \)
(28)

\( (u^p)' = \mu^4 \lambda^\beta \sin^{4 \beta} (\mu \xi) - 2 \mu^4 \lambda^\beta \sin^{2 \beta + 2} (\mu \xi), \)
(29)

\( -\mu^4 \lambda^\beta \sin^{4 \beta - 2} (\mu \xi), \)
(30)

Substituting (17)–(22) into the reduced ODE (14) gives
\begin{align*}
-c \lambda^l \cos^{\beta l} (\mu \xi) + a_{11} \lambda^{l+1} \cos^{\beta (l+1)} (\mu \xi) + a_{22} \lambda^{l+2} \cos^{\beta (l+2)} (\mu \xi) \\
+ a_3 \left\{ -n^2 \mu^2 \beta^2 \lambda^m \cos^{n \beta} (\mu \xi) + n \mu^2 \lambda^\beta (\beta n - 1) \cos^{n \beta - 2} (\mu \xi) \right\} \\
+ a_4 \left\{ -\mu^4 \lambda^\beta \sin^{4 \beta} (\mu \xi) - 2 \mu^4 \lambda^\beta \sin^{2 \beta + 2} (\mu \xi) \right\} \\
- \mu^4 \lambda^\beta \sin^{4 \beta - 2} (\mu \xi),
\end{align*}
(31)

To obtain solitary wave solutions for the \( K(m, n, p, q + r) \) equation (9), we need to impose some restrictions on the dependent exponents so that Eq. (29) satisfies the homogeneous balance principle. In fact, a judicious choice of exponents \( l, m, n, p, q, r \) leads to a specific closed-form solution that is physically meaningful. This may be a complicated task since we are concerned by a \( K(m, n, p, q + r) \) model with several dependent exponents. In what follows we analyze the resulting Eq. (29) in the framework of two interesting cases depending on the exponents of the considered model.

3.1 Case I: \( l = n = p = m + 1 = q + r \)

In this case, Eq. (29) becomes
\begin{align*}
-c \lambda^l \cos^{\beta l} (\mu \xi) + a_{11} \lambda^{l+1} \cos^{\beta (l+1)} (\mu \xi) + a_{22} \lambda^{l+2} \cos^{\beta (l+2)} (\mu \xi) \\
+ a_3 \left\{ -n^2 \mu^2 \beta^2 \lambda^m \cos^{n \beta} (\mu \xi) + n \mu^2 \lambda^\beta (\beta n - 1) \cos^{n \beta - 2} (\mu \xi) \right\} \\
+ a_4 \left\{ -\mu^4 \lambda^\beta \sin^{4 \beta} (\mu \xi) - 2 \mu^4 \lambda^\beta \sin^{2 \beta + 2} (\mu \xi) \right\} \\
- \mu^4 \lambda^\beta \sin^{4 \beta - 2} (\mu \xi),
\end{align*}
(32)

Equating the exponents and the coefficients of like powers of cosine functions leads to
\begin{align*}
\beta l (\beta l - 1) (\beta l - 2) (\beta l - 3) &= 0, \\
\beta l - 4 &= \beta (l + 1), \\
-c + a_{11} - a_{31}^2 \mu^2 \beta^2 + a_{44} \mu^4 \beta^4 - a_{55} \mu^2 \beta^2 r &= 0, \\
a_{33} \mu^2 \beta (\beta l - 1) - 2a_{44} \mu^4 \beta l (\beta l - 1) (\beta l - 2) + a_{55} \mu^2 \beta (\beta r - 1) &= 0, \\
a_{22} \lambda^{l+1} - a_{44} \mu^4 \beta l (\beta l - 1) (\beta l - 2) (\beta l - 3) &= 0.
\end{align*}
(33)
(34)
(35)
(36)
(37)

Solving this system yields
\begin{align*}
\beta l &= 0, 1, 2, 3, \\
\beta &= -4,
\end{align*}
(38)
(39)

\( \mu = \frac{1}{2} \sqrt{\frac{1}{a_4 (4l^2 + 4l + 1)} \left\{ a_3 + a_5 \frac{r (4r + 1)}{l (4l + 1)} \right\} } \)
(40)

\( c = a_{11} - 16 (l^2 a_3 + r^2 a_5) \mu^2 + 256a_4 \mu^4 l^4, \)
(41)

\( \lambda = \frac{4a_4 \mu^4 l (4l + 1) (4l + 2) (4l + 3)}{a_{22}}. \)
(42)

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In Eqs. (39) and (40), \( \mu \) is given by Eq. (38).

Clearly, two sorts of solitary wave solutions can be obtained in the wave number expression (38). When

\[
\frac{1}{a_4 (8l^2 + 4l + 1)} \left\{ a_3 + a_5 \frac{r (4r + 1)}{l (4l + 1)} \right\} < 0,
\]

the solitary wave solutions take the forms of the \( \sec \) and \( \csc \) functions giving rise to the following periodic solutions:

\[
u_1(x, t) = \lambda \sec^4 \left[ \frac{1}{2} \sqrt{-\frac{1}{a_4 (8l^2 + 4l + 1)} \left\{ a_3 + a_5 \frac{r (4r + 1)}{l (4l + 1)} \right\} (x - ct)} \right],
\]

and

\[
u_2(x, t) = \lambda \csc^4 \left[ \frac{1}{2} \sqrt{-\frac{1}{a_4 (8l^2 + 4l + 1)} \left\{ a_3 + a_5 \frac{r (4r + 1)}{l (4l + 1)} \right\} (x - ct)} \right],
\]

However, in the opposite case where

\[
\frac{1}{a_4 (8l^2 + 4l + 1)} \left\{ a_3 + a_5 \frac{r (4r + 1)}{l (4l + 1)} \right\} > 0,
\]

the soliton solutions are obtained in the form

\[
u_3(x, t) = -\lambda \sec h^4 \left[ \frac{1}{2} \sqrt{-\frac{1}{a_4 (8l^2 + 4l + 1)} \left\{ a_3 + a_5 \frac{r (4r + 1)}{l (4l + 1)} \right\} (x - ct)} \right],
\]

and

\[
u_4(x, t) = \lambda \csc h^4 \left[ \frac{1}{2} \sqrt{-\frac{1}{a_4 (4l^2 + 4l + 1)} \left\{ a_3 + a_5 \frac{r (4r + 1)}{l (4l + 1)} \right\} (x - ct)} \right],
\]

where \( \lambda \) and \( c \) are determined from Eqs. (39) and (40).

### 3.2 Case II: \( p = q + r \)

In this case, Eq. (29) becomes

\[
-c^4 \cos \beta t (\mu \xi) + a_{11} \lambda^{m+1} \cos \beta (m+1) (\mu \xi) + a_{22} \lambda^{m+2} \cos \beta (m+2) (\mu \xi) + a_3 \left\{-n^2 \mu \beta^2 \lambda \cos \beta \right\} + a_4 \left\{ \mu^4 \lambda^2 \beta^4 p^4 \cos \beta (\mu \xi) - 2 \mu^4 \lambda^2 \beta \beta (\beta p - 1) \left( \beta^2 p^2 - 2 \beta p + 2 \right) \cos \beta (\mu \xi) \right\} + a_4 \left\{ \mu^4 \lambda^2 \beta \beta (\beta p - 1) \left( \beta^2 p - 2 \right) (\beta p - 3) \cos \beta (\mu \xi) \right\} + a_5 \left\{ \mu^2 \lambda^2 \beta r \beta r - 1 \cos \beta (\mu \xi) - \mu^2 \lambda^2 \beta r^2 \cos \beta (\mu \xi) \right\} = 0.
\]

To determine \( \beta \), we usually balance the highest power of \( \cos \beta (m+2) (\mu \xi) \) and \( \cos \beta (p-4) (\mu \xi) \) terms in Eq. (45) so that one obtains

\[
\beta = \frac{4}{p - m - 2}.
\]

for

\[
p \neq m + 2.
\]

It is important to note that the compacton solution (15) exists only when \( \beta > 0 \). This condition implies that \( p > m + 2 \) in Eq. (46).

Now setting the coefficients of \( \cos \beta (p-2) (\mu \xi) \) terms to zero in Eq. (45) yields

\[
-2a_4 \mu^4 \lambda^4 \beta (\beta p - 1) \left( \beta^2 p^2 - 2 \beta p + 2 \right) + a_5 \mu^2 \lambda^2 \beta r (\beta r - 1) = 0,
\]

which gives

\[
\mu = \sqrt{\frac{a_5 r (\beta r - 1)}{2a_4 \beta (\beta p - 1) \left( \beta^2 p^2 - 2 \beta p + 2 \right)}}.
\]
From equating the exponents of $\cos^n \beta - 2 (\mu \xi)$ and $\cos^m \beta (\xi)$ terms in Eq. (45), we get

$$\beta = \frac{2}{n - l},$$

(52)

for

$$n \neq l.$$  

(53)

In this case, since $\beta$ is positive for compactons solutions it follows directly that $n > l$.

Then, we find from setting their corresponding coefficients to zero that

$$-c \lambda^l + a_3n \mu^2 \lambda^\alpha (\beta n - 1) = 0,$$

which leads to

$$\lambda = \left\{ \frac{2c \alpha_p (\beta p - 1) (\beta^2 p^2 - 2\beta p + 2)}{a_5a_3n \beta (\beta n - 1) (\beta r - 1)} \right\} \frac{1}{\lambda^l},$$

(55)

for

$$c \neq 0, \beta n - 1 \neq 0, \beta r - 1 \neq 0, \beta p - 1 \neq 0, n \neq l.$$  

(56)

Also, equating the two values of $\beta$ from (46) and (50) leads to an algebraic relationship between the dependent model exponents:

$$2n + m + 2 = p + 2l,$$

(57)

which serves as a condition for the compactons to exist for the model (9).

### 3.2.1 Compactons solutions

By using Eqs. (49), (50) and (53), we obtain a family of compactons solutions

$$u(x, t) = \left\{ \frac{2c \alpha_p (\beta p - 1) (\beta^2 p^2 - 2\beta p + 2)}{a_5a_3n \beta (\beta n - 1) (\beta r - 1)} \cos^2 \left[ \sqrt{\frac{\alpha_p (\beta r - 1)}{2c \alpha_p (\beta p - 1) (\beta^2 p^2 - 2\beta p + 2)}} (x - ct) \right] \right\} \frac{1}{\lambda^l}, \quad |x - ct| \leq \frac{\pi}{2n},$$

(58)

and

$$u(x, t) = \left\{ \frac{2c \alpha_p (\beta p - 1) (\beta^2 p^2 - 2\beta p + 2)}{a_5a_3n \beta (\beta n - 1) (\beta r - 1)} \sin^2 \left[ \sqrt{\frac{\alpha_p (\beta r - 1)}{2c \alpha_p (\beta p - 1) (\beta^2 p^2 - 2\beta p + 2)}} (x - ct) \right] \right\} \frac{1}{\lambda^l}, \quad |x - ct| \leq \frac{\pi}{2n}.$$  

(59)

### 3.2.2 Solitary pattern solutions

By using Eqs. (49), (50), and (53), we obtain a family of solitary pattern solutions

$$\begin{array}{l}
u(x, t) = \left\{ \frac{2c \alpha_p (\beta p - 1) (\beta^2 p^2 - 2\beta p + 2)}{a_5a_3n \beta (\beta n - 1) (\beta r - 1)} \cosh^2 \phi \right\} \frac{1}{\lambda^l}, \\
\end{array}$$

(60)

and

$$\begin{array}{l}
\phi = \sqrt{\frac{\alpha_p (\beta r - 1)}{2c \alpha_p (\beta p - 1) (\beta^2 p^2 - 2\beta p + 2)}} (x - ct). \\
\end{array}$$

(61)
4 Conclusion

Compactons have multiple applications in physics and many scientific applications. The K(n,n) was discovered as a simplified model to study the role of nonlinear dispersion on pattern formation in liquid drops [22], being also proposed in the analysis of patterns on liquid surfaces. Equations with compacton solutions have also found applications such as the lubrication approximation for thin viscous films [22], semi classical models for Bose Einstein condensates, long nonlinear surface waves in a rotating ocean when the high-frequency dispersion is null, the pulse propagation in ventricleaorta system, dispersive models for magma dynamics, or, even, particle wave functions in nonlinear quantum mechanics [22].

In nonlinear lattices the propagation of compacton-like kinks has been observed using mechanical, electrical, and magnetic analogs [22].

We have considered a $K(m, n, p, q+r)$ equation with the generalized evolution term, nonlinear convection terms, fifth-order nonlinear dispersion and nonlinear dispersions corrections. The examined model is a generalized quintic extension of the $K(m, n)$ equation. By means of sine-cosine method, we have obtained a variety of physical solutions, including periodic solutions, solitons solutions, compactons solutions and solitary patterns solutions under two sets of parametric conditions. These solutions may be useful to explain some physical phenomena in genuinely nonlinear dynamical systems supporting higher order nonlinear dispersions.

References


