ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.12(2011) No.2,pp.245-256

Impulsive Partial Neutral Functional Differential Equation of Second-order with Infinite Delay

Runping Ye *
Education Department of Suqian College, Jiangsu, 223800, P R China
(Received 6 March 2011 , accepted 20 July 2011)

Abstract: This paper shows the existence of mild solutions for partial neutral functional differential equations of second-order with impulsive and infinite delay. We derive conditions in respect of the Hausdorff measure of noncompactness under which the mild solutions exist in Banach spaces. Our results improve and generalize some previous results.

Keywords: impulsive differential equation; partial neutral functional differential; mild Solution; Hausdorff measure of noncompactness; phase space

1 Introduction

The study of impulsive functional differential equation is linked to their utility in simulating processes and phenomena subject to short-time perturbations during their evolution. The perturbations are performed discretely and their duration is negligible in comparison with the total duration of the processes and phenomena. Therefore, the theory of impulsive differential equations has become an important area of investigation in the past two decades because of their applications to various problems arising in communications, control technology, impact mechanics, electrical engineering, medicine, and biology, etc. However, differential equations with delays are often more realistic to describe natural phenomena than those without delays, and neutral differential equations arise in many areas of applied mathematics. Relative to this matter, we refer the reader to [1-12].

In [5] , the authors studied existence of solutions for impulsive partial neutral functional differential equations with infinite delay in Banach space with the compactness assumption on associated family of operators.

It is well known, there are few compact semigroups in infinite dimensional space. In this paper, we study such question without the assumption of compactness on the family of operators.

Inspired by above mentioned works, in this paper, we investigate the existence of mild solutions for some impulsive partial neutral functional differential equations of second order. In this paper, we are concerned with second order equations modelled in the form

$$\frac{d}{dt}(x^{'}(t)+g(t,x_t))=Ax(t)+f(t,x_t),\ t\in J=[0,b], \tag{1}$$

$$x_0=\varphi\in\mathcal{B},x^{'}(0)=z\in X \tag{2}$$

$$\Delta x(t_i)=I_i^1(x_{t_i}),\quad i=1,2,...,n, \tag{3}$$

$$\Delta x(t_i)=I_i^2(x_{t_i}),\quad i=1,2,...,n, \tag{4}$$

Similarly, we also consider the second order problem

$$\frac{d}{dt}(x^{'}(t)+g(t,x_t,x^{'}(t)))=Ax(t)+f(t,x_t,x^{'}(t)),\quad t\in J=[0,b], \tag{5}$$

* *E-mail address*: yeziping168@sina.com

IJNS.2011.10.15/535

$$x_0 = \varphi \in \mathcal{B}, x^{'}(0) = z \in X \tag{6}$$

$$Deltax(t_i) = I_i^1(x_{t_i}), \quad i = 1, 2, ..., n, \tag{7}$$

$$\Delta x(t_i) = I_i^2(x_{t_i}), \quad i = 1, 2, ..., n, \tag{8}$$

where A is the infinitesimal generator of a strongly continuous cosine function of bounded linear operatorson a Banach space X. In both cases, the history $x_t : (-\infty, 0] \to X,\ x_t(\theta) = x(t+\theta)$, belongs to some abstract phase space $\mathcal{B}$ defined axiomatically; $g,\ f,\ I_i^j,\ i = 1, ..., n,\ j = 1, 2$, are appropriate functions; $0 < t_1 < ... < t_n < a$ are fixed numbers and the symbol $\Delta\xi(t)$ represent the jump of the function ξ at t , which is defined by $\Delta\xi(t) = \xi(t^+) - \xi(t^-)$.

We give the existence of mild solution of the initial value problem(1)-(4) and (5)-(6) under the conditions in respect of Hausdorff's measure of non-compactness.

2 Preliminaries

Now we introduce some definitions, notations and preliminary facts which are used throughout this paper.

We say that a family $\{C(t) : t \in \mathbf{R}\}$ of operators in $B(X)$ is a strongly continuous cosine family if

(i)$C(0) = I$ (I is the identity operator in X);

(ii)$C(t+s) + C(t-s) = 2C(t)C(s)$ for all $s, t \in \mathbf{R}$;

(iii)The map $t \to C(t)x$ is strongly continuous for each $x \in X$.

The strongly continuous sine family $\{S(t) : t \in \mathbf{R}\}$, associated to the given strongly continuous cosine family $\{C(t) : t \in \mathbf{R}\}$, is defined by

$$S(t)x = \int_0^t C(s)xds, x \in X, t \in \mathbf{R}.$$

For more details on strongly continuous cosine and sine families, we refer the reader to the books by Goldstein[14] and Fattorini[17].

Along of this paper, A is the infinitesimal generator of a strongly continuous cosine function of bounded linear operators, $(C(t))_{t\in R}$, on X and $S(t)$ is the sine function associated with $(C(t))_{t\in R}$. We designate by N, $\widetilde{N}$ certain constants such that $\|C(t)\| \leq N$ and $\|S(t)\| \leq \widetilde{N}$ for every $t \in J$. We refer the reader to [17] for the necessary concepts about cosine functions. Next we only mention a few results and notations needed to establish our results. As usual we denote by $D(A)$ the domain of A endowed with the graph norm $\|x\|_A = \|x\| + \|Ax\|, x \in D(A)$.

To describe appropriately our problems we say that a function $u : [\sigma, \tau] \to X$is a normalized piecewise continuous function on $[\sigma, \tau]$ if u is piecewise continuous and left continuous on $(\sigma, \tau]$. We denote by $\mathcal{PC}([\sigma, \tau]; X)$ the space formed by the normalized piecewise continuous functions from $[\sigma, \tau]$ into X. In particular, we introduce the space $\mathcal{PC}$ formed by all functions $u : [0, b] \to X$ such that u is continuous at $t \neq t_i, u(t_i^-) = u(t_i)$ and $u(t_i^+)$ exists, for all $i = 1, \cdots, n$. It is clear that $\mathcal{PC}$ endowed with the norm of the uniform convergence is a Banach space.

To establish an appropriate concept of solution for the problem (5)-(8) we must to precise the meaning of the derivative in (5)-(8). We say that $x \in \mathcal{PC}$ is piecewise smooth if x is continuously differentiable at $t \neq t_i,\ i = 0, 1, \cdots, n+1$, and for $t = t_i,\ i = 0, 1, \cdots, n$, there exists the right derivative $x^{'}_R(t) = \lim\limits_{s\to 0^+} \frac{x(t+s)-x(t^+)}{s}$, while for $t = t_i,\ i = 1, \cdots, n+1$, there exists the left derivative $x^{'}_L(t) = \lim\limits_{s\to 0^-} \frac{x(t+s)-x(t^-)}{s}$. Next $x^{'}(0)$ corresponds to $x^{'}_R(0)$ and we represent by $x^{'}(t)$ the left derivative of x at $t > 0$. Furthermore, we denote by $\mathcal{PC}^1$ the space of piecewise smooth functions in the sense above described endowed with the norm $\|u\|_1 = \|u\| + \|u^{'}\|$.

In this work we will employ an axiomatic definition of the phase space $\mathcal{B}$ which is similar to that introduced by Hale and Kato [8] and it is appropriated to treat retarded impulsive differential equations.

Definition 1 *[16]. Let $\mathcal{B}$ be a linear space of functions mapping $(-\infty, 0]$ into X endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$. We will assume that $\mathcal{B}$ satisfies the following axioms:*
(A) If $x : (-\infty, \sigma+b] \to X, b > 0$, such that $x_\sigma \in \mathcal{B}$ and $x|_{[\sigma,\sigma+b]} \in \mathcal{PC}([\sigma, \sigma+b] : X)$, then for every $t \in [\sigma, \sigma+b)$ the following conditions hold:

(i) x_t is in $\mathcal{B}$,

(ii) $\|x(t)\| \leq H\|x_t\|_{\mathcal{B}}$,

(iii) $\|x_t\|_{\mathcal{B}} \leq K(t-\sigma)\sup\{\|x(s)\| : \sigma \leq s \leq t\} + M(t+\sigma)\|x_\sigma\|_{\mathcal{B}}$,
where $H > 0$ *is a constant;* $K, M : [0,\infty) \to [1,\infty)$, K *is continuous,* M *is locally bounded and* H, K, M *are independent of* $x(\cdot)$.
(B) The space $\mathcal{B}$ *is complete.*

Definition 2 *[8]. The Hausdorff's measure of noncompactness* χ_{Y} *is defined by*
$\chi_{Y}(B) = \inf\{r > 0, B \text{ can be covered by finite number of balls with radius } r\}$,
for bounded set B *in any Banach space* Y.

Lemma 1 [8]. *Let* Y *be a real Banach space and* $B, C \subseteq Y$ *be bounded, the following properties are satisfied:*
(1) B is pre-compact if and only if $\chi_{Y}(B) = 0$;
(2) $\chi_{Y}(B) = \chi_{Y}(\overline{B}) = \chi_{Y}(convB)$, *where* $\overline{B}$ *and* $convB$ *are the closure convex hull of* B *respectively;*
(3) $\chi_{Y}(B) \leq \chi_{Y}(C)$ *when* $B \subseteq C$;
(4) $\chi_{Y}(B+C) \leq \chi_{Y}(B) + \chi_{Y}(C)$ *where* $B + C = \{x + y; x \in B, y \in C\}$;
(5) $\chi_{Y}(B \cup C) = \max\{\chi_{Y}(B), \chi_{Y}(C)\}$;
(6) $\chi_{Y}(\lambda B) = |\lambda|\chi_{Y}(B)$ *for any* $\lambda \in R$;
(7) If the map $Q : D(Q) \subseteq Y \to Z$ *is Lipschitz continuous with constant* k *then* $\chi_{Z}(QB) \leq k\chi_{Y}(B)$ *for any bounded subset* $B \subseteq D(Q)$, *where* Z *is a Banach space;*
(8) If $\{W_n\}_{n=1}^{+\infty}$ *is a decreasing sequence of bounded closed nonempty subset of* Y *and* $\lim_{n\to\infty} \chi_{Y}(W_n) = 0$, *then* $\cap_{n=1}^{+\infty} W_n$ *is nonempty and compact in* Y.

Definition 3 *[10]. The map* $Q : W \subseteq Y \to Y$ *is said to be a* $\chi_{Y} - contraction$ *if there exists a positive constant* $k < 1$ *such that* $\chi_{Y}(Q(C)) \leq k\chi_{Y}(C))$ *for any bounded close subset* $C \subseteq W$ *where* Y *is a Banach space.*

Lemma 2 [16]. *(Darbo) If* $W \subseteq Y$ *is closed and convex and* $0 \in W$, *the continuous map* $Q : W \to W$ *is a* $\chi_{Y} - contraction$, *if the set* $\{x \in W : x = \lambda Qx\}$ *is bounded for* $0 < \lambda < 1$, *then the map* Q *has at least one fixed point in* W.

Lemma 3 [8]. *(Darbo-Sadovskii) If* $W \subseteq Y$ *is bounded closed and convex ,the continuous map* $Q : W \to W$ *is a* $\chi_{Y} - contraction$, *then the map* Q *has at least one fixed point in* W.

In this paper we denote by χ the Hausdorff's measure of noncompactness of X, by χ_{C} the Hausdorff's measure of noncompactness of $C([0,b];X)$ and by $\chi_{\mathcal{PC}}$ the Hausdorff's measure of noncompactness of $\mathcal{PC}$. To discuss the existence we need the following auxiliary results.

Lemma 4 [11] *(1) If* $W \subset \mathcal{PC}([a,b];X)$ *is bounded, then* $\chi(W(t)) \leq \chi_{\mathcal{PC}}(W)$, *for any* $t \in [a,b]$, *where* $W(t) = \{u(t) : u \in W\} \subseteq X$;

(2) If W *is piecewise equicontinuous on* $[a,b]$, *then* $\chi(W(t))$ *is piecewise continuous for* $t \in [a,b]$, *and*

$$\chi_{\mathcal{PC}}(W) = \sup\{\chi(W(t)),\ t \in [a,b]\};$$

(3) If $W \subset \mathcal{PC}([a,b];X)$ *is bounded and piecewise equicontinuous, then* $\chi(W(t))$ *is piecewise continuous for* $t \in [a,b]$, *and*

$$\chi(\int_a^t W(s)ds) \leq \int_a^t \chi(W(s))ds, \forall t \in [a,b],$$

where $\int_a^t W(s)ds = \{\int_a^t x(s)ds : x \in W\}$.

The following Lemma is easy to get.

Lemma 5 *If the semigroup* $C(t)$ *is equicontinuous and* $\eta \in L \in ([0,b];R^+)$, *then the set*

$$\{\int_0^t S(t-s)u(s)ds, \|u(s)\| \leq \eta(s) \ for \ a.e. \ s \in [0,b]\}$$

is equicontinuous for $t \in [0,b]$.

Lemma 6 [15] *If $W \subset \mathcal{PC}^1(J,X)$ is bounded and the elements of W' are equicontinuous on each $J_k(k=1,2,\cdots,m)$, then*

$$\chi_{\mathcal{PC}^1}(W) = \max\{\sup_{t\in J}\chi(W(t)), \sup_{t\in J}\chi(W'(t))\},$$

where $\chi_{\mathcal{PC}^1}$ denotes the Hausdorff measure of noncompactness in space $\mathcal{PC}^1(J,X)$.

3 Main results

Now we define the mild solution for the initial value problem(1)-(4).

Definition 4 *A function $x:(-\infty,b]\to X$ is a mild solution of the initial value problem (1)-(4) if $x_0=\varphi$, $x(\cdot)|_J\in\mathcal{PC}$ and*

$$x(t) = C(t)\varphi(0) + S(t)(z+g(0,\varphi)) - \int_0^t C(t-s)g(s,x_s)ds + \int_0^t S(t-s)f(s,x_s)ds \\ + \sum_{o<t_i<t} C(t-t_i)I_i^1(x_{t_i}) + \sum_{o<t_i<t} s(t-t_i)I_i^2(x_{t_i}), \, t\in J.$$

For the system (1)-(4), we assume that the following hypotheses are satisfied:
(H_{1f}) $f: J\times\mathcal{B}\to X$ satisfy the following conditions:

(1) For each $x:(-\infty,b]\to X$, $x_0\in\mathcal{B}$ and $x|_J\in\mathcal{PC}$, the function $t\to f(t,x_t)$ is strongly measurable and $f(t,\cdot)$ is continuous for a.e., $t\in J$;

(2) There exist an integrable function $\alpha: J\to[0,+\infty)$ and a monotone continuous nondecreasing function $\Omega:[0,+\infty)\to(0,+\infty)$, such that

$$\|f(t,v)\|\le\alpha(t)\Omega(\|v\|_{\mathcal{B}}), \text{ for all } t\in J, v\in\mathcal{B};$$

(3) There exists an integrable function $\eta: J\to[0,+\infty)$, such that

$$\chi(S(s)f(t,D))\le\eta(t)\sup_{-\infty\le\theta\le0}\chi(D(\theta)) \text{ for a.e. } s,t\in J,$$

where $D(\theta)=\{v(\theta): v\in D\}$.

(H_{1g}) The function $g(\cdot)$ is continuous for all $(t,v)\in J\times\mathcal{B}$, and $g(t,\cdot)$ satisfies the Lipschitz condition, that is, there exists a positive constants L_g, such that,

$$\|g(t,v_1)-g(t,v_2)\|\le L_g\|v_1-v_2\|_{\mathcal{B}}, \forall v_1,v_2\in\mathcal{B}.$$

(H_{1I}) (1) There exist positive constants L_i^j such that

$$\|I_i^j(u)-I_i^j(v)\|\le L_i^j\|u-v\|_{\mathcal{B}},\ i=1,...,n;\ j=1,2.$$

(2) There exist positive constants $C^1_{I_i^j}$, $C^2_{I_i^j}$, $i=1,...,n$, $j=1,2$, such that

$$\|I_i^j(v)\| = C^1_{I_i^1}\|v\|_{\mathcal{B}} + C^2_{I_i^j}.$$

(H_1) (1) If $\frac{K_bM}{1-\mu_2}\int_0^t\alpha(s)ds < \int_c^{+\infty}\frac{ds}{\Omega(s)}$, where $\mu_1 = K_b(\widetilde{N}(\|z\|+\|g(0,\varphi)\|)+N\int_0^b\|g(s,0)\|ds+\sum_{i=1}^n(NC^2_{I_i^1}+\widetilde{N}C^2_{I_i^2}))+(K_bNH+M_b)\|\varphi\|_{\mathcal{B}}$, $\mu_2 = K_b(NL_gb+\sum_{i=1}^n(NC^1_{I_i^1}+\widetilde{N}C^1_{I_i^2}))<1$, $c=\frac{\mu_1}{1-\mu_2}$;

(2) $K_b(NL_gb+\sum_{i=1}^n(NL_i^1+\widetilde{N}L_i^2))+\int_0^b\eta(s)ds<1$.

In this section, $y:(-\infty,b]\to X$ is the function defined by $y_0=\varphi$ and $y(t)=(T(t)\varphi(0)+S(t)(z+g(0,\varphi))$ on J. Clearly ,$\|y_t\|_{\mathcal{B}}\le(K_bMH+M_b)\|\varphi\|_{\mathcal{B}}$, where $K_b=\sup_{0\le t\le b}K(t)$, $M_b=\sup_{0\le t\le b}M(t)$.

Now we are in conditions to establish our main results.

Theorem 7 *If the hypotheses (H_{1f}) (H_{1g}) (H_{1I}) and (H_1) are satisfied, then the initial value problem(1)-(4) has at least one mild solution.*

Proof. Let $S(b)$ be the space $S(b)=\{x:(-\infty,b]\to X|x_0=0,\ x|_J\in\mathcal{PC}\}$ endowed with supremum norm $\|\cdot\|_b$. Let $\Gamma: S(b)\to S(b)$ be the map defined by

$$(\Gamma x)(t)=\begin{cases}0, & t\in(-\infty,0],\\ -\int_0^t C(t-s)g(s,x_s+y_s)ds+\int_0^t S(t-s)f(s,x_s+y_s)ds & \\ +\sum\limits_{o<t_i<t} C(t-t_i)I_i^1(x_{t_i}+y_{t_i})+\sum\limits_{o<t_i<t} s(t-t_i)I_i^2(x_{t_i}+y_{t_i}),\ t\in J. & \end{cases} \quad (9)$$

It is easy to see that $\|x_t+y_t\|_{\mathcal{B}}\leq K_b\widetilde{N}(\|z\|+\|g(0,\varphi)\|)+(K_bNH+M_b)\|\varphi\|_{\mathcal{B}}+K_b\|x\|_t$, where $\|x\|_t=\sup\limits_{0\leq s\leq t}\|x(s)\|$. Thus, Γ is well defined and with values in $S(b)$. In addition, from the axioms of phase space, the Lebesgue dominated convergence Theorem and the conditions (H_{1f}), (H_{1g}) and (H_{1I}), we can show that Γ is continuous.

Step 1. For $0<\lambda<1$, set $\{x\in\mathcal{PC}: x=\lambda\Gamma x\}$ is bounded.

Let x_λ be a solution of $x=\lambda\Gamma x$ for $0<\lambda<1$. Then

$$\|x_{\lambda t}+y_t\|_{\mathcal{B}}\leq K_b\widetilde{N}(\|z\|+\|g(0,\varphi)\|)+(K_bNH+M_b)\|\varphi\|_{\mathcal{B}}+K_b\|x_\lambda\|_t.$$

Let $v_\lambda(t)=K_b\widetilde{N}(\|z\|+\|g(0,\varphi)\|)+(K_bNH+M_b)\|\varphi\|_{\mathcal{B}}+K_b\|x_\lambda\|_t$, for each $t\in[0,b]$. Then

$$\begin{aligned}\|x_\lambda(t)\|&=\|\lambda\Gamma x_\lambda(t)\|=\lambda\|\Gamma x(t)\|\\ &\leq N\int_0^t(L_g\|x_{\lambda s}+y_s\|_{\mathcal{B}}+\|g(s,0)\|)ds+\widetilde{N}\int_0^t\alpha(s)\Omega(v_\lambda(s))ds\\ &\quad+N\sum_{0<t_i<t}(C^1_{I_i^1}\|x_{t_i}+y_{t_i}\|_{\mathcal{B}}+C^2_{I_i^1})+\widetilde{N}\sum_{0<t_i<t}(C^1_{I_i^2}\|x_{t_i}+y_{t_i}\|_{\mathcal{B}}+C^2_{I_i^2}).\end{aligned}$$

We get

$$\begin{aligned}\|x_\lambda\|_t&\leq(N\int_0^b\|g(s,0)\|ds+\sum_{i=1}^n(NC^2_{I_i^1}+\widetilde{N}C^2_{I_i^2}))\\ &\quad+(NL_gb+\sum_{i=1}^n(NC^1_{I_i^1}+\widetilde{N}C^1_{I_i^2}))v_\lambda(t)+\widetilde{N}\int_0^t\alpha(s)\Omega(v_\lambda(s))ds,\end{aligned}$$

which implies that

$$\begin{aligned}v_\lambda(t)&\leq K_b\widetilde{N}(\|z\|+\|g(0,\varphi)\|)+(K_bNH+M_b)\|\varphi\|_{\mathcal{B}}+K_b((N\int_0^b\|g(s,0)\|ds\\ &\quad+\sum_{i=1}^n(NC^2_{I_i^1}+\widetilde{N}C^2_{I_i^2}))+(NL_gb+\sum_{i=1}^n(NC^1_{I_i^1}+\widetilde{N}C^1_{I_i^2}))v_\lambda(t)+\widetilde{N}\int_0^t\alpha(s)\Omega(v_\lambda(s))ds)\\ &\leq K_b(\widetilde{N}(\|z\|+\|g(0,\varphi)\|)+N\int_0^b\|g(s,0)\|ds+\sum_{i=1}^n(NC^2_{I_i^1}+\widetilde{N}C^2_{I_i^2}))\\ &\quad+(K_bNH+M_b)\|\varphi\|_{\mathcal{B}}+K_b(NL_gb+\sum_{i=1}^n(NC^1_{I_i^1}+\widetilde{N}C^1_{I_i^2}))v_\lambda(t)+K_b\widetilde{N}\int_0^t\alpha(s)\Omega(v_\lambda(s))ds.\end{aligned}$$

Consequently,

$$v_\lambda(t)\leq c+\frac{K_bM}{1-\mu_2}\int_0^t\alpha(s)\Omega(v_\lambda(s))ds, \quad (10)$$

where $\mu_1=K_b(\widetilde{N}(\|z\|+\|g(0,\varphi)\|)+N\int_0^b\|g(s,0)\|ds+\sum_{i=1}^n(NC^2_{I_i^1}+\widetilde{N}C^2_{I_i^2}))+(K_bNH+M_b)\|\varphi\|_{\mathcal{B}}$, $\mu_2=K_b(NL_gb+\sum_{i=1}^n(NC^1_{I_i^1}+\widetilde{N}C^1_{I_i^2}))<1$, $c=\frac{\mu_1}{1-\mu_2}$.

Denoting by $\beta_\lambda(t)$ the right-hand side of (10), we get

$$\frac{\beta'_\lambda(t)}{\Omega(\beta_\lambda(t))}\leq\frac{K_bM}{1-\mu_2}\alpha(s). \quad (11)$$

Integrating (11) and applying our hypothesis $(H)(1)$, we obtain

$$\int_c^{\beta_\lambda(t)}\frac{ds}{\Omega(s)}\leq\frac{K_bM}{1-\mu_2}\int_0^t\alpha(s)ds<\int_c^{+\infty}\frac{ds}{\Omega(s)},$$

which implies that the functions $\beta_\lambda(t)$ are bounded on J. So are $v_\lambda(t)$ and $x_\lambda(\cdot)$.

Step 2. Next we show that Γ is $\chi - contraction$. To clarify this, we decompose Γ in the form $\Gamma = \Gamma_1 + \Gamma_2$, For $t \geq 0$, where

$$\Gamma_1 x(t) = -\int_0^t C(t-s)g(s, x_s + y_s)ds + \sum_{o<t_i<t} C(t-t_i)I_i^1(x_{t_i} + y_{t_i}) + \sum_{o<t_i<t} S(t-t_i)I_i^2(x_{t_i} + y_{t_i}),\ t \in [0, b],$$

$$\Gamma_2 x(t) = \int_0^t S(t-s)f(s, x_s + y_s)ds.$$

First, we show the Γ_1 is $Lipschitz$ continuous. For any $x_1, x_2 \in B_k$, from definition 1 and hypotheses conditions, we get

$$\|\Gamma_1 x_1(t) - \Gamma_1 x_2(t)\| \leq NL_g b\|x_{1t} - x_{2t}\|_{\mathcal{B}} + \sum_{o<t_i<t} NL_i^1\|x_{1t_i} - x_{2t_i}\|_{\mathcal{B}} + \sum_{o<t_i<t} \widetilde{N}L_i^2\|x_{1t_i} - x_{2t_i}\|_{\mathcal{B}}.$$

$$\leq K_b(NL_g b + \sum_{i=1}^n (NL_i^1 + \widetilde{N}L_i^2))\|x_{1t} - x_{2t}\|_b,$$

that is,

$$\|\Gamma_1 x_1(t) - \Gamma_1 x_2(t)\|_b \leq K_b(NL_g b + \sum_{i=1}^n (NL_i^1 + \widetilde{N}L_i^2))\|x_{1t} - x_{2t}\|_b.$$

Hence, Γ_1 is $Lipschitz$ continuous with $Lipschitz$ constant

$$L' = K_b(NL_g b + \sum_{i=1}^n (NL_i^1 + \widetilde{N}L_i^2)).$$

Next, taking $W \subset B_k$, Obviously, $S(t)$ is equicontinuous. From Lemma5, W is piecewise equicontinuous. As $\chi_{\mathcal{PC}}(W) = \sup\{\chi(W(t)),\ t \in [a, b]\}$, we have

$$\chi(\Gamma_2 w(t)) = \chi(\int_0^t S(t-s)f(s, W_s + y_s)ds)$$
$$\leq \int_0^t \eta(s) \sup_{-\infty<\theta\leq 0} \chi(W(s+\theta) + y(s+\theta))ds$$
$$\leq \int_0^t \eta(s) \sup_{0\leq\tau\leq s} \chi W(\tau)ds \leq \chi_{\mathcal{PC}}(W)\int_0^t \eta(s)ds,$$

for each bounded set $W \in \mathcal{PC}$. Since

$$\chi_{\mathcal{PC}}(\Gamma W) = \chi_{\mathcal{PC}}(\Gamma_1 W + \Gamma_2 W) \leq \chi_{\mathcal{PC}}(\Gamma_1 W) + \chi_{\mathcal{PC}}(\Gamma_2 W)$$
$$\leq (L' + \int_0^t \eta(s)ds))\chi_{\mathcal{PC}}(W) \leq \chi_{\mathcal{PC}}(W),$$

Γ is $\chi - contraction$. In view of Lemma 2 $Darbo$ fixed point Theorem, we conclude that Γ has at least one fixed point in W. Let x is a fixed of Γ on $S(b)$, then $z = x + y$ is a mild solution of (1)-(4). So we deduce the existence of a mild solution of (1)-(4). ■

Theorem 8 *If* (H_{1f}), (H_{1g}), (H_{1I}), $(H_1)(2)$ *are satisfied with*

$$K_b(NL_g b + \sum_{i=1}^n (NC^1_{I_i^1} + \widetilde{N}C^1_{I_i^2}) + \widetilde{N}\int_0^b \alpha(s)ds \limsup_{\tau\to\infty} \frac{\Omega(\tau)}{\tau}) < 1,$$

then the initial value problem(1)-(4) has at least one mild solution.

Proof. Proceeding as in the proof of Theorem 7, we infer that the map Γ given by (9) is continuous from $S(b)$ into $S(b)$. Furthermore, there exists $k > 0$ such that $\Gamma(B_k) \subset B_k$, where $B_k = \{x \in S(b) : \|x\|_b \leq k\}$. In fact, if we assume that the assertion is false, then for $k > 0$ there exists $x_k \in B_k$ and $t_k \in J$ such that $k < \|\Gamma x_k(t_k)\|$. This yields that

$$k < \|\Gamma x_k(t_k)\| \leq N\int_0^{t_k} (L_g\|x_{ks} + y_s\|_{\mathcal{B}} + \|g(s, 0)\|)ds + \widetilde{N}\int_0^{t_k} \alpha(s)\Omega(\|x_{ks} + y_s\|_{\mathcal{B}})ds$$
$$+N\sum_{0<t_i<t} (C^1_{I_i^1}\|x_{t_i} + y_{t_i}\|_{\mathcal{B}} + C^2_{I_i^1}) + \widetilde{N}\sum_{0<t_i<t} (C^1_{I_i^2}\|x_{t_i} + y_{t_i}\|_{\mathcal{B}} + C^2_{I_i^2}).$$
$$\leq N\int_0^b L_g(K_b\widetilde{N}(\|z\| + \|g(0, \varphi)\| + (K_b NH + M_b)\|\varphi\|_{\mathcal{B}} + K_b k) + \|g(s, 0)\|)ds$$
$$+\widetilde{N}\int_0^b \alpha(s)ds\Omega(K_b\widetilde{N}(\|z\| + \|g(0, \varphi)\|) + (K_b NH + M_b)\|\varphi\|_{\mathcal{B}} + K_b k),$$

$$+N\sum_{i=1}^{n}(C^1_{I^1_i}(K_b\widetilde{N}(\|z\|+\|g(0,\varphi)\|+(K_bNH+M_b)\|\varphi\|_{\mathcal{B}}+K_bk)+C^2_{I^1_i})$$
$$+\widetilde{N}\sum_{i=1}^{n}(C^1_{I^2_i}(K_b\widetilde{N}(\|z\|+\|g(0,\varphi)\|+(K_bNH+M_b)\|\varphi\|_{\mathcal{B}}+K_bk)+C^2_{I^2_i}),$$

which implies that

$$1<K_b(NL_gb+\sum_{i=1}^{n}(NC^1_{I^1_i}+\widetilde{N}C^1_{I^2_i}))$$
$$+\widetilde{N}\int_0^b\alpha(s)ds\limsup_{k\to\infty}\frac{\Omega(K_b\widetilde{N}(\|z\|+\|g(0,\varphi)\|)+(K_bNH+M_b)\|\varphi\|_{\mathcal{B}}+K_bk)}{k}$$
$$\le K_b(NL_gb+\sum_{i=1}^{n}(NC^1_{I^1_i}+\widetilde{N}C^1_{I^2_i})+\widetilde{N}\int_0^b\alpha(s)ds\limsup_{\tau\to\infty}\frac{\Omega(\tau)}{\tau})<1,$$

a contradiction.

By Lemma 3, as the proof of Theorem 7, we conclude that (1)-(4) has at least a mild solution. ■

For the system (5)-(8), it is possible to establish similar results as those given in the first part of this section. Furthermore, we denote by $\mathcal{PC}^1$ the space of piecewise smooth functions in the sense above described endowed with the norm $\|u\|_1=\|u\|+\|u^{'}\|$.

Definition 5 *A function $x:(-\infty,b]\to X$ is a mild solution of the initial value problem (5)-(8) if $x_0=\varphi$, $x(\cdot)|_J\in\mathcal{PC}^1$ and for $t\in J$,*

$$x(t)=C(t)\varphi(0)+S(t)(z+g(0,\varphi,z))-\int_0^t C(t-s)g(s,x_s,x^{'}(t))ds$$
$$+\int_0^t S(t-s)f(s,x_s,x^{'}(t))ds+\sum_{o<t_i<t}C(t-t_i)I^1_i(x_{t_i},x^{'}(t_i))+\sum_{o<t_i<t}s(t-t_i)I^2_i(x_{t_i},x^{'}(t_i)).$$

For the system (5)-(8), we assume that the family $C(t):t\in R$ is equicontinuous and the following hypotheses are satisfied:

(H_{2f}) $f:J\times\mathcal{B}\times X\to X$ satisfy the following conditions:

(1) For each $x:(-\infty,b]\to X$, $x_0\in\mathcal{B}$ and $x|_J\in C^1([0,b];X)$, the function $t\to f(t,x_t,x^{'}(t))$ is strongly measurable and $f(t,\cdot,\cdot)$ is continuous for a.e. $t\in J$;

(2) There exist an integrable function $\alpha:J\to[0,+\infty)$ and a monotone continuous nondecreasing function $\Omega:[0,+\infty)\to(0,+\infty)$, such that

$$\|f(t,v,w)\|\le\alpha(t)\Omega(\|v\|_{\mathcal{B}}+\|w\|),\text{ for all } t\in J,\ (v,w)\in\mathcal{B}\times X;$$

(3) There exist integrable functions $\eta_i:J\to[0,+\infty),\ i=1,2$, such that

$$\chi(S(s)f(t,D_1,D_2))\le\eta_1(t)\sup_{-\infty\le\theta\le0}\chi(D_1(\theta))$$

and

$$\chi(C(s)f(t,D_1,D_2))\le\eta_2(t)\sup_{-\infty\le\theta\le0}\chi(D_2(\theta))\text{ for a.e., } s,t\in J,$$

where $D_i(\theta)=\{D_i(\theta):v\in D\},\ i=1,2$.

(H_{2g}) The function $g(\cdot)$ is continuous, and $g(t,\cdot,\cdot)$ satisfies the Lipschitz condition, that is, there exists a positive constant L_g such that

$\|g(t,v_1,w_1)-g(t,v_2,w_2)\|\le L_g(\|v_1-v_2\|_{\mathcal{B}}+\|w_1-w_2\|),(t,v_i,w_i)\in J\times\mathcal{B}\times X,\ i=1,2.$

(H_{2I}) (1)There exist positive constants $L^j_i,\ i=1,\cdots,n;\ j=1,2$, such that

$\|I^j_i(u_1,v_1)-I^j_i(u_2,v_2)\|\le L^j_i(\|u_1-u_2\|_{\mathcal{B}}+\|v_1-v_2\|),(u_1,v_1),\ (u_2,v_2)\in\mathcal{B}\times X;$

(2) For each $t\in J$, there exist positive constants $C^1_{I^j_i},C^2_{I^j_i},i=1,...,n,j=1,2$, such that

$$\|I^j_i(v,w)\|=C^1_{I^1_i}\|v\|_{\mathcal{B}}+C^2_{I^j_i}\|w\|,(v,w)\in\mathcal{B}\times X;$$

(H_2) (1) $L_g(K_b+1)(Nb+1+\|A\|\widetilde{Nb})+\sum_{i=1}^{n}((N+\|A\|\widetilde{N})(C^1_{I^1_i}K_b+C^2_{I^1_i})$

$$+(N+\widetilde{N})(C^1_{I^2_i}K_b+C^2_{I^2_i}))+(K_b+1)(N+\widetilde{N})\int_0^b\alpha(s)ds\limsup_{\tau\to\infty}\frac{\Omega(\tau)}{\tau}<1;$$

(2) $L_g(Nb+1+\|A\|\widetilde{N}b)+\max\{\int_0^b\eta_1(s)ds,\ \int_0^b\eta_2(s)ds\}<1.$

In this section, $y:(-\infty,b]\to X$ is the function defined by $y_0=\varphi$ and $y(t)=C(t)\varphi(0)+S(t)(z+g(0,\varphi,z))$ on J. Clearly, $\|y_t\|_{\mathcal{B}}\le K_b\|y\|_b+M_b\|\varphi\|_{\mathcal{B}}$, where $K_b=\sup\limits_{0\le t\le b}K(t)$, $M_b=\sup\limits_{0\le t\le b}M(t)$, $\|y\|_b=\sup\limits_{0\le t\le b}\|y(t)\|$.

Now we are in the position to establish our main results.

Theorem 9 *If the hypotheses (H_2f),(H_2g), (H_2I) and (H_2) are satisfied, then the initial value problem(5)-(8) has at least one mild solution.*

Proof. Let $S^1(b)$ be the space $S^1(b)=\{x:(-\infty,b]\to X|\ x_0=0,\ x|_J\in\mathcal{PC}^1,\ x^{'}(0)=-g(0,\varphi,z)\}$ endowed with supremum norm $\|\cdot\|_{1b}$. Let $\Gamma:S^1(b)\to S^1(b)$ be the map defined by

$$(\Gamma x)(t)=\begin{cases}0, & t\in(-\infty,0],\\ -\int_0^t C(t-s)g(s,x_s+y_s,x^{'}(s)+y^{'}(s))ds & \\ +\int_0^t S(t-s)f(s,x_s+y_s,x^{'}(t)+y^{'}(t))ds & \\ +\sum\limits_{o<t_i<t}C(t-t_i)I_i^1(x_{t_i}+y_{t_i},x^{'}(t_i)+y^{'}(t_i)) & \\ +\sum\limits_{o<t_i<t}s(t-t_i)I_i^2(x_{t_i}+y_{t_i},x^{'}(t_i)+y^{'}(t_i)), & t\in J,\end{cases}\tag{12}$$

where $y_0=\varphi$ and $y(t)=C(t)\varphi(0)+S(t)(z+g(0,\varphi,z))$ on J. Thus, Γ is well defined and with values in $S^1(b)$, and that

$$(\Gamma x)^{'}(t)=-g(t,x_t+y_t,x^{'}(t)+y^{'}(t))-\int_0^t AS(t-s)g(s,x_s+y_s,x^{'}(s)+y^{'}(s))ds$$
$$+\int_0^t C(t-s)f(s,x_s+y_s,x^{'}(s)+y^{'}(s))ds+\sum_{o<t_i<t}AS(t-t_i)I_i^1(x_{t_i}+y_{t_i},x^{'}(t_i)+y^{'}(t_i))$$
$$+\sum_{o<t_i<t}C(t-t_i)I_i^2(x_{t_i}+y_{t_i},x^{'}(t_i)+y^{'}(t_i)),\ t\in J.$$

In addition, from the axioms of phase space, the Lebesgue dominated convergence Theorem and the conditions (H_2f), (H_2g) and (H_2I), we can show that Γ is continuous.

Step 1. There exists $k>0$ such that $\Gamma(B_k)\subset B_k$, where $B_k=\{x\in S^1(b):\|x\|_{1b}\le k\}$. In fact, if we assume that the assertion is false, then for $k>0$ there exists $x_k\in B_k$ and $t_k\in I$ such that $k<\|\Gamma x_k(t_k)\|$. This yields that

$$\begin{aligned}k&<\|\Gamma x_k(t_k)\|_1\le\|\Gamma x_k(t_k)\|+\|(\Gamma x_k)^{'}(t_k)\|\\
&\le N\int_0^{t_k}(L_g(\|x_{ks}+y_s\|_{\mathcal{B}}+\|x_k^{'}(s)+y^{'}(t)\|)+\|g(s,0,0)\|)ds\\
&\quad+\int_0^{t_k}\widetilde{N}\alpha(s)\Omega(\|x_{ks}+y_s\|_{\mathcal{B}}+\|x_k^{'}(s)+y^{'}(t)\|)ds\\
&\quad+N\sum_{0<t_i<t_k}(C^1_{I_i^1}\|x_{kt_i}+y_{t_i}\|_{\mathcal{B}}+C^2_{I_i^1}\|x_k^{'}(t_i)+y^{'}(t_i)\|)\\
&\quad+\widetilde{N}\sum_{0<t_i<t_k}(C^1_{I_i^2}\|x_{kt_i}+y_{t_i}\|_{\mathcal{B}}+C^2_{I_i^2}\|x_k^{'}(t_i)+y^{'}(t_i)\|)\\
&\quad+L_g(\|x_{kt_k}+y_{t_k}\|_{\mathcal{B}}+\|x_k^{'}(t_k)+y^{'}(t_k)\|)+\|g(t_k,0,0)\|\\
&\quad+\int_0^{t_k}\|A\|\widetilde{N}(L_g(\|x_{ks}+y_s\|_{\mathcal{B}}+\|x_k^{'}(s)+y^{'}(s)\|)+\|g(s,0,0)\|)ds\\
&\quad+\int_0^{t_k}N\alpha(s)\Omega(\|x_{ks}+y_s\|_{\mathcal{B}}+\|x_k^{'}(s)+y^{'}(s)\|)ds\\
&\quad+\|A\|\widetilde{N}\sum_{0<t_i<t_k}(C^1_{I_i^1}\|x_{kt_i}+y_{t_i}\|_{\mathcal{B}}+C^2_{I_i^1}\|x_k^{'}(t_i)+y^{'}(t_i)\|)\\
&\quad+N\sum_{0<t_i<t_k}(C^1_{I_i^2}\|x_{kt_i}+y_{t_i}\|_{\mathcal{B}}+C^2_{I_i^2}\|x_k^{'}(t_i)+y^{'}(t_i)\|)\\
&\le bNL_g(K_bk+K_b\|y\|_b+M_b\|\varphi\|_{\mathcal{B}}+k+\|y^{'}\|_b)+N\int_0^b\|g(s,0,0)\|ds\\
&\quad+\widetilde{N}\int_0^b\alpha(s)ds\Omega(K_bk+K_b\|y\|_b+M_b\|\varphi\|_{\mathcal{B}}+k+\|y^{'}\|_b)\end{aligned}$$

$$+\sum_{i=1}^{n} N(C^1_{I^1_i}(K_b k + K_b\|y\|_b + M_b\|\varphi\|_{\mathcal{B}}) + C^2_{I^1_i}(k + \|y'\|_b))$$
$$+\sum_{i=1}^{n} \widetilde{N}(C^1_{I^2_i}(K_b k + K_b\|y\|_b + M_b\|\varphi\|_{\mathcal{B}}) + C^2_{I^2_i}(k + \|y'\|_b))$$
$$+L_g(K_b k + K_b\|y\|_b + M_b\|\varphi\|_{\mathcal{B}} + k + \|y'\|_b) + \|g(t_k,0,0)\|$$
$$+b\|A\|\widetilde{N}L_g(K_b k + K_b\|y\|_b + M_b\|\varphi\|_{\mathcal{B}} + k + \|y'\|_b) + \|A\|\widetilde{N}\int_0^b \|g(s,0,0)\|ds$$
$$+N\int_0^b \alpha(s)ds\Omega(K_b k + K_b\|y\|_b + M_b\|\varphi\|_{\mathcal{B}} + k + \|y'\|_b)$$
$$+\sum_{i=1}^{n} \|A\|\widetilde{N}(C^1_{I^1_i}(K_b k + K_b\|y\|_b + M_b\|\varphi\|_{\mathcal{B}}) + C^2_{I^1_i}(k + \|y'\|_b))$$
$$+\sum_{i=1}^{n} N(C^1_{I^2_i}(K_b k + K_b\|y\|_b + M_b\|\varphi\|_{\mathcal{B}}) + C^2_{I^2_i}(k + \|y'\|_b)),$$

which implies that

$$1 < L_g(K_b+1)(Nb+1+\|A\|\widetilde{Nb}) + \sum_{i=1}^{n}((N+\|A\|\widetilde{N})(C^1_{I^1_i}K_b + C^2_{I^1_i}) + (N+\widetilde{N})(C^1_{I^2_i}K_b + C^2_{I^2_i}))$$
$$+(N+\widetilde{N})\int_0^b \alpha(s)ds \limsup_{k\to\infty}\frac{\Omega(K_b k + K_b\|y\|_b + M_b\|\varphi\|_{\mathcal{B}} + k + \|y'\|_b)}{k})$$
$$\leq L_g(K_b+1)(Nb+1+\|A\|\widetilde{Nb}) + \sum_{i=1}^{n}((N+\|A\|\widetilde{N})(C^1_{I^1_i}K_b + C^2_{I^1_i})$$
$$+(N+\widetilde{N})(C^1_{I^2_i}K_b + C^2_{I^2_i})) + (K_b+1)(N+\widetilde{N})\int_0^b \alpha(s)ds\limsup_{\tau\to\infty}\frac{\Omega(\tau)}{\tau} < 1,$$

a contradiction.

Step 2. We show that Γ is $\chi - contraction$.

To clarify this, we decompose Γ in the form $\Gamma = \Gamma_1 + \Gamma_2$, for $t \geq 0$, where

$$\Gamma_1 x(t) = -\int_0^t C(t-s)g(s, x_s + y_s, x'(s) + y'(s))ds$$
$$+\sum_{o<t_i<t} C(t-t_i)I^1_i(x_{t_i} + y_{t_i}, x'(t_i) + y'(t_i)) + \sum_{o<t_i<t} s(t-t_i)I^2_i(x_{t_i} + y_{t_i}, x'(t_i) + y'(t_i)),$$
$$\Gamma_2 x(t) = \int_0^t S(t-s)f(s, x_s + y_s, x'(s) + y'(s))ds.$$

Then

$$(\Gamma_1 x)'(t) = -g(t, x_t + y_t, x'(t) + y'(t)) - \int_0^t AS(t-s)g(s, x_s + y_s, x'(s) + y'(s))ds$$
$$+\sum_{o<t_i<t} AS(t-t_i)I^1_i(x_{t_i} + y_{t_i}, x'(t_i) + y'(t_i)) + \sum_{o<t_i<t} C(t-t_i)I^2_i(x_{t_i} + y_{t_i}, x'(t_i) + y'(t_i)).$$

First, we show the Γ_1 is $Lipschitz$ continuous. Let $B_k = \{x \in S^1(b) : \|x\|_{1b} \leq k\}$, for arbitrary $x_1, x_2 \in B_k$. From Definition 1 and hypotheses conditions, we get

$$\|\Gamma_1 x_1(t) - \Gamma_1 x_2(t)\|_1 \leq \|\Gamma_1 x_1(t) - \Gamma_1 x_2(t)\| + \|(\Gamma_1 x_1)'(t) - (\Gamma_1 x_2)'(t)\|$$
$$\leq \|\int_0^t C(t-s)(g(s, x_{1s} + y_s, x_1'(s) + y'(s)) - g(s, x_{2s} + y_s, x_2'(s) + y'(s)))ds\|$$
$$+\|\sum_{o<t_i<t} C(t-t_i)(I^1_i(x_{1t_i} + y_{t_i}, x_1'(t_i) + y'(t_i)) - I^1_i(x_{2t_i} + y_{t_i}, x_2'(t_i) + y'(t_i)))\|$$
$$+\|\sum_{o<t_i<t} S(t-t_i)(I^2_i(x_{1t_i} + y_{t_i}, x_1'(t_i) + y'(t_i)) - I^2_i(x_{2t_i} + y_{t_i}, x_2'(t_i) + y'(t_i)))$$
$$+\|g(t, x_{1t} + y_t, x_1'(t) + y'(t)) - g(t, x_{2t} + y_t, x_2'(t) + y'(t))\|$$
$$+\|\int_0^t AS(t-s)(g(s, x_{1s} + y_s, x_1'(s) + y'(s)) - g(s, x_{2s} + y_s, x_2'(s) + y'(s)))ds\|$$
$$+\|\sum_{o<t_i<t} AS(t-t_i)(I^1_i(x_{1t_i} + y_{t_i}, x_1'(t_i) + y'(t_i)) - I^1_i(x_{2t_i} + y_{t_i}, x_2'(t_i) + y'(t_i)))\|$$

$$+\|\sum_{o<t_i<t} C(t-t_i)(I_i^2(x_{1t_i}+y_{t_i},x_1^{'}(t_i)+y^{'}(t_i))-I_i^2(x_{2t_i}+y_{t_i},x_2^{'}(t_i)+y^{'}(t_i)))$$

$$\le NL_g\int_0^t(\|x_{1s}-x_{2s}\|_{\mathcal{B}}+\|x_1^{'}(s)-x_2^{'}(s)\|)ds$$

$$+N\sum_{o<t_i<t}L_i^1(\|x_{1t_i}-x_{2t_i}\|_{\mathcal{B}}+\|x_1^{'}(t_i)-x_2^{'}(t_i)\|)$$

$$+\widetilde{N}\sum_{o<t_i<t}L_i^2(\|x_{1t_i}-x_{2t_i}\|_{\mathcal{B}}+\|x_1^{'}(t_i)-x_2^{'}(t_i)\|)$$

$$+L_g(\|x_{1t}-x_{2t}\|_{\mathcal{B}}+\|x_1^{'}(t)-x_2^{'}(t)\|)$$

$$+\|A\|\widetilde{N}L_g\int_0^t(\|x_{1s}-x_{2s}\|_{\mathcal{B}}+\|x_1^{'}(s)-x_2^{'}(s)\|)ds$$

$$+\|A\|\widetilde{N}\sum_{o<t_i<t}L_i^1(\|x_{1t_i}-x_{2t_i}\|_{\mathcal{B}}+\|x_1^{'}(t_i)-x_2^{'}(t_i)\|)$$

$$+N\sum_{o<t_i<t}L_i^2(\|x_{1t_i}-x_{2t_i}\|_{\mathcal{B}}+\|x_1^{'}(t_i)-x_2^{'}(t_i)\|)$$

$$\le (K_b+1)(L_g(Nb+1+\|A\|\widetilde{N}b)+\sum_{i=1}^n((N+\|A\|\widetilde{N})L_i^1+(N+\widetilde{N})L_i^2))\|x_{1s}-x_{2s}\|_{1b},$$

that is,

$$\|\Gamma_1x_1(t)-\Gamma_1x_2(t)\|_{1b}\le (K_b+1)(L_g(Nb+1+\|A\|\widetilde{N}b)$$
$$+\sum_{i=1}^n((N+\|A\|\widetilde{N})L_i^1+(N+\widetilde{N})L_i^2))\|x_{1s}-x_{2s}\|_{1b}.$$

Hence, Γ_1 is $Lipschitz$ continuous with $Lipschitz$ constant

$$L^{'}=(K_b+1)(L_g(Nb+1+\|A\|\widetilde{N}b)+\sum_{i=1}^n((N+\|A\|\widetilde{N})L_i^1+(N+\widetilde{N})L_i^2)).$$

Next, taking $W\subset B_k$. Obviously, $S(t)$ is equicontinuous. From Lemma 5, W is piecewise equicontinuous. As $\chi_{\mathcal{PC}}(W)=\sup\{\chi(W(t)),t\in[a,b]\}$, we have

$$\chi_{\mathcal{PC}^1}(\Gamma_2w(t))=\chi_{\mathcal{PC}^1}(\int_0^t S(t-s)f(s,W_s+y_s,W^{'}(s)+y^{'}(s))ds)$$

$$=\max\{\sup_{t\in J}\chi_{\mathcal{PC}}(\int_0^t S(t-s)f(s,W_s+y_s,W^{'}(s)+y^{'}(s))ds),$$

$$\sup_{t\in J}\chi_{\mathcal{PC}}(\int_0^t C(t-s)f(s,W_s+y_s,W^{'}(s)+y^{'}(s))ds)\}$$

$$\le\max\{\sup_{t\in J}\int_0^t\eta_1(s)\sup_{-\infty<\theta\le 0}\chi_{\mathcal{PC}}(W(s+\theta)+y(s+\theta))ds),$$

$$\sup_{t\in J}\int_0^t\eta_2(s)\sup_{-\infty<\theta\le 0}\chi_{\mathcal{PC}}(W^{'}(s+\theta)+y^{'}(s+\theta))ds)\}$$

$$\le\max\{\sup_{t\in J}\int_0^t\eta_1(s)\sup_{0\le\tau\le s}\chi_{\mathcal{PC}}(W(\tau))ds,\sup_{t\in J}\int_0^t\eta_2(s)\sup_{0\le\tau\le s}\chi_{\mathcal{PC}}(W^{'}(\tau))ds\}$$

$$\le\max\{\int_0^b\eta_1(s)ds,\int_0^b\eta_2(s)ds\}\max\{\sup_{0\le\tau\le b}\chi_{\mathcal{PC}}(W(\tau)),\ \sup_{0\le\tau\le b}\chi_{\mathcal{PC}}(W^{'}(\tau))\}$$

$$\le\max\{\int_0^b\eta_1(s)ds,\int_0^b\eta_2(s)ds\}\chi_{\mathcal{PC}^1}(W),$$

for each bounded set $W\in\mathcal{PC}^1$. Since

$$\chi_{\mathcal{PC}^1}(\Gamma W)=\chi_{\mathcal{PC}^1}(\Gamma_1W+\Gamma_2W)\le\chi_{\mathcal{PC}^1}(\Gamma_1W)+\chi_{\mathcal{PC}^1}(\Gamma_2W)$$

$$\le(L_g(Nb+1+\|A\|\widetilde{N}b)+\max\{\int_0^b\eta_1(s)ds,\int_0^b\eta_2(s)ds\})\chi_{\mathcal{PC}^1}(W)\le\chi_{\mathcal{PC}^1}(W),$$

Γ is $\chi-contraction$. In view of Lemma 3, we conclude that Γ has at least one fixed point in W. Let x be a fixed of Γ on $S^{'}(b)$. Then $z=x+y$ is a mild solution of (4)-(8). So we deduce the existence of a mild solution of (4)-(8). ■

Remark Our results improve and generalize [5] Theorem 3.1, 3.2 and 3.3.

4 Examples

In the next applications, $\mathcal{B}$ will be the phase space $C_0 \times L^2(h, X)$ (see [5, 7]).

Now we discuss the existence of solutions for the second order neutral differential equation

$$\frac{\partial}{\partial t}(\frac{\partial u(t,\xi)}{\partial t} + \int_{-\infty}^{t}\int_{0}^{\pi} b(t-s,\eta,\xi)u(s,\eta)d\eta ds) = \frac{\partial^2 u(t,\xi)}{\partial \xi^2} + \int_{-\infty}^{t} F(t,t-s,\xi,u(s,\xi))ds, t\in[0,a], \xi\in[0,\pi], \quad (13)$$

$$u(t,0) = u(t,\pi) = 0, t \in [0,a], \quad (14)$$

$$u(\tau,\xi) = \varphi(\tau,\xi), \tau \leq 0, 0 \leq \xi \leq \pi, \quad (15)$$

$$\Delta u(t_i)(\xi) = \int_{-\infty}^{t_i} a_i(t_i - s)u(s,\xi)ds, \quad (16)$$

$$\Delta u^{'}(t_i)(\xi) = \int_{-\infty}^{t_i} \widetilde{a_i}(t_i - s)u(s,\xi)ds. \quad (17)$$

where $\varphi \in C_0 \times L^2(h; X), 0 < t_1 < \cdots < t_n < a$ and

(a) The functions $b(s,\eta,\xi)$, $\frac{\partial b(s,\eta,\xi)}{\partial \xi}$ are measurable, $b(s,\eta,\pi) = b(s,\eta,0) = 0$ and $L_g := \max\{(\int_0^{\pi}\int_{-\infty}^{0}\int_0^{\pi} \frac{1}{h(s)}(\frac{\partial^i b(s,\eta,\xi)}{\partial \xi})^2 d\eta ds d\xi)^{1/2} : i = 0,1\} < \infty$;

(b) The function $F : \mathbf{R}^4 \to \mathbf{R}$ is continuous and there is continuous function $\mu : \mathbf{R}^2 \to \mathbf{R}$ such that $|F(t,s,\xi,x)| \leq \mu(t,s)|x|, (t,s,\xi,x) \in \mathbf{R}^4$;

(c) The functions $a_i \in C([0,\infty);\mathbf{R})$ and $L_i^1 := (\int_{-\infty}^{0} \frac{(a_i(s))^2}{h(s)} ds)^{1/2} < \infty$ for all $i = 1, 2, \cdots, n$.

(d) The functions $\widetilde{a_i} \in C(\mathbf{R};\mathbf{R})$ and $L_i^2 := (\int_{-\infty}^{0} \frac{\widetilde{a_i}^2(s)}{h(s)} ds)^{1/2} < \infty$ for each $i = 1, \cdots, n$.

Assuming that conditions (a)-(d) are verified, the problem (13)-(17) can be modelled as the abstract impulsive Cauchy problem (1)-(4) by defining

$$g(t,\psi)(\xi) := \int_{-\infty}^{0}\int_{0}^{\pi} b(s,\nu,\xi)\psi(s,\nu)d\nu ds, \quad (18)$$

$$f(t,\psi)(\xi) := \int_{-\infty}^{0} F(t,s,\xi,\psi(s,\xi))ds, \quad (19)$$

$$I_i^1(\psi)(\xi) = \int_{-\infty}^{0} a_i(s)\psi(s,\xi)ds. \quad (20)$$

$$I_i^2(\psi) = \int_{-\infty}^{0} \widetilde{a_i}\psi(s,\xi)ds \quad (21)$$

Moreover, $g(t,\cdot), I_i^j, i = 1,\cdots,n, j = 1,2$ are bounded linear operators, $\|I_i^j\| \leq L_i^j$, $i = 1,\cdots,n$, $j = 1,2$ and $\|f(t,\psi)\| \leq d(t)\|\psi\|_{\mathcal{B}}$ for every $t \in [0,a]$, where $d(t) := (\int_{-\infty}^{0} \frac{\mu(t,s)^2}{h(s)} ds)^{1/2}$.

The next result is consequence of Theorem 8.

Proposition 10 *Let the previous conditions be satisfied. If*

$$(1 + (\int_{-a}^{0} h(\tau)d\tau)^{1/2})(aL_g + \sum_{i=1}^{n}(L_i^1 + L_i^2) + \int_0^a d(t)dt) < 1,$$

then there exists a mild solution of (13)-(17).

References

[1] M. Benchohra, J.Henderson, S.K.Ntouyas. Existence results for impulsive multivalued semilinear neutral functional inclusions in Banach spaces.*J. Math. Anal. Appl.* ,263 (2001):763-780.

[2] S. Baghlt, M. Benchohra. Perturbed functional and neutral functional evolution equations with infinite delay in $Fr\acute{e}chet$ spaces. *Electron. J. Differential Equations*, 2008(69)(2008):1-19.

[3] X. Fu, K. Ezzinbi. Existence of solutions for neutral functional differential evolution equations with nonlocal conditions. *Nonlinear Anal.* 54(2)(2003):215-227.

[4] E. Hernández. A second-order impulsive Cauchy problem. *Int. J. Math. Math. Sci.*,31(8)(2002):451-461.

[5] E. Hernández M,M.Rabello,H. Henrłquez.Existence of solutions for impulsive partial neutral functional differential equations.*J. Math. Anal. Appl.*, 331(2007):1135-1158.

[6] E.Hernández, H.R. Henríquez.Existence results for partial neutral functional differential equations with unbounded delay. *J. Math. Anal. Appl.* 221(1998):452-475.

[7] Y. Hino, S. Murakami, T. Naito.Functional-Differential Equations with Infinite Delay. Lecture Notes in Math., vol. 1473, Springer-Verlag, Berlin, 1991.

[8] Agarwal R, Meehan M,O'regan D.Fixed point theory and applications [M]. Cambridge Tracts in Mathematics. New York: Cambridge University Press,2001:178-179.

[9] Yuri V. Rogovchenko.Impulsive evolution systems: Main results and new trends. *Dyn. Contin. Discrete Impuls. Syst.*, 3(1)(1997):57-88.

[10] Yuri V. Rogovchenko.Nonlinear impulse evolution systems and applications to population models. *J. Math. Anal. Appl.*,207(2)(1997):300-315.

[11] R.Ye. Existence of solutions for impulsive partial neutral functional differential equation with infinite delay. *Nonlinear Anal.*, 73(2010):155-162.

[12] R. Ye, G. Zhang. Neutral functional differential equation of second-order with infinite delay. *Electron. J. Differential Equations*, 2010(36)(2010):1-12.

[13] R. Ye, Q. Dong, G. Li. Existence of solutions for double perturbed neutral functional evolution equation. *Intern. J. Nonlinear Sci.*, 8(3)(2009):360-367.

[14] J.A.Goldstein. Semigroups of Linear Operators and Applications. New York: Oxford Univ. Press, 1985.

[15] Dajun Guo, V. Lakshmikantham, Xinzhi Liu. Nonlinear Integral Equations in Abstrct Spaces. Kluwer Acadmic Publishers, Dordrecht, Boston, London,1996.

[16] J. K. Hale, J. Kato. Phase space for retarded equations with infinite delay [J]. *Funkcial Ekvac*,21(1978):11-41.

[17] H.O. Fattorini, Second Order Linear Differential Equations in Banach Spaces, North-Holland Math. Stud., vol. 108, North-Holland, Amsterdam, 1985.