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On the Cauchy Problem for the Generalized Weak Dissipative Camassa-Holm Equation

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Abstract: In this paper we investigate the local well-posedness for the generalized weak dissipative Camassa-Holm system. We also derive a blow-up mechanism for strong solutions. In addition,we determine the exact blow-up rate of such solutions to the equation.

Keywords: Weak dissipative Camassa-Holm equation;Local well-posedness; Blow-up; Blow-up rate

1 Introduction

The Camassa-Holm equation

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, t > 0, x \in R \tag{1}$$

is a model for wave motion on shallow water,where $u\,(t, x)$ represents the fluid's free surface above a flat bottom (or equivalently,the fluid velocity at time $t \geq 0$ in the spatial x direction).

In the recent years, equation (1) has attracted the attention of a large number of researchers with two remarkable properties. The first one is the equation possesses the solutions in the form of peaked solitons or 'peakons'[1,2]. The peakon $u(t, x) = ce^{-|x-ct|}, c \neq 0$ is smooth except at its crest and the tallest among all waves of fixed energy. It is a feature observed for the traveling waves of largest amplitude which solve the governing equations for water waves[3,4,5,6]. Another remarkable property is the equation has breaking waves[1,7], that is, the solution remains bounded while its slope becomes unbounded in finite time. After wave breaking the solutions can be continued uniquely as either global conservative[8,9] or global dissipative solutions[10].

The Cauchy problem of the Camassa-Holm equation has been studied extensively. It has been shown that this equation is locally well-posed[7,11,12,13,14,15,16,17] for initial data $u_0 \in H^s\,(R)$ with $s > \frac{3}{2}$. More interestingly, it has not only global strong solutions modelling permanent waves [3,7,14,16,17,18], but also blow-up solutions modelling wave breaking [7,14,19,20,21]. On the other hand, it has global weak solutions with initial data $u_0 \in H^1\,(R)$[8,10,18,22]. Moreover, the initial-boundary value problem for the Camassa-Holm equation on the half-line and on a finite interval were studied recently in[23]. The advantage of the Camassa-Holm equation in comparison with the KdV equation lies in the fact that the Camassa-Holm equation has peaked solitons and models wave breaking[1,7,24].In recent years,the two-component Camassa-Holm equation has aroused the concern of many scholars.There have been a lot of researchs on the two-component Camassa-Holm system[25,26,27].

In general, it is difficult to avoid energy dissipation mechanisms in a real world. Wu and Yin have derived dissipative Camassa-Holm equation

$$u_t - u_{txx} + 3uu_x + L(u) = 2u_x u_{xx} + uu_{xxx} \tag{2}$$

Where $L(u)$is a dissipative term,$L$is a differential operator or the pseudo- differential operator[28,29].When $L(u) = \lambda(u - u_{xx})(\ \lambda > 0)$,Wu and Yin have obtained the global existence and blow-up of the strong solution of equation(2)[29]. Applying Kato theorem,when $L(u) = \lambda(u - u_{xx})(\ \lambda > 0)$,they have derived the local well-posedness of the solution of (2) .Moreover, they also give necessary and sufficient conditions of the blow-up and blow-up rate of the solution[28].

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In this paper, we consider the generalized weak dissipative Camassa-Holm equation

$$u_t - u_{txx} + ku_x + 3uu_x + \lambda u^{2n+1} - \beta u_{xx} = 2u_x u_{xx} + uu_{xxx}, t > 0, x \in R \quad (3)$$

where $k, \lambda \geq 0$ and $\beta \geq 0$are constants,$n \geq 0$is a integer. We find that the behaviors of the equation (3) are similar to the Camassa-Holm equation in a finite interval of time, such as, the local well-posedness and the blow-up phenomena. Because of the presence of the nonlinear terms $u^{2n+1}$and the dissipative term u_{xx}, the equation (3) has not the following conservation laws: $E_1 = \int_s udx$, $E_2 = \int_s (u^2 + u_x^2)dx$,which play an important role in the study of the Camassa-Holm equation,so it makes estimate more difficult.

The basic elementary framework is as follwos.Firstly,we prove the local well- posedness of the initial value problem associated with equation (3) by applying Kato's theorem.And then,we give a blow-up mechanism for strong solution-s.Finally, we determine the exact blow-up rate of such solutions to the equation(3).

Now, we introduce some notation. All spaces of functions are over R and for simplicity, we drop R in our notation of function spaces if there is no ambiguity. Additionally, if A is an unbounded operator, $D(A)$denotes the domain of the operator A.$[A, B]$denotes the commutator of linear operator$A$and$B$(i.e. $[A, B] = AB - BA$).$\|\cdot\|_X$denotes the norm of Banach space$X$.For convenience, let $\|\cdot\|_s$ and $(\cdot,\cdot)_s$denote the norm and the inner product of $H^s, s \in R$,respectively.

2 Local well-posedness

We consider the generalized weak dissipative Camassa-Holm system

$$\begin{cases} u_t - u_{txx} + ku_x + 3uu_x + \lambda u^{2n+1} - \beta u_{xx} = 2u_x u_{xx} + uu_{xxx} & t > 0, x \in R \\ u(0, x) = u_0(x) & x \in R \end{cases} \quad (4)$$

where $k, \lambda \geq 0$ and $\beta \geq 0$are constants,$n \geq 0$is a integer.

Set $y = u - u_{xx}$,then Eq (4) becomes the form of a quasi-linear evolution equation of hyperbolic type

$$\begin{cases} y_t + uy_x + 2yu_x = -ku_x - \lambda u^{2n+1} + \beta u_{xx} & t > 0, x \in R \\ y = u - u_{xx} & t > 0, x \in R \\ u(0, x) = u_0(x), y_0(x) = u_0(x) - u_{0xx}(x) & x \in R \end{cases} \quad (5)$$

Note that if $g(x) = \frac{1}{2}e^{-|x|}, x \in R$,then $(1 - \partial_x^2)^{-1} f = g * f$for all $f \in L^2(R)$ and $g * y = u$,where$*$denotes the spatial convolution. Using this identity, and applying the pseudo-differential operator$(1 - \partial_x^2)^{-1}$to Eq (5), one can rewrite Eq (5) as a quasi-linear nonlocal evolution system of hyperbolic type

$$\begin{cases} u_t + uu_x = -\partial_x g * (u^2 + \frac{1}{2}u_x^2 + ku) - \lambda(g * u^{2n+1}) + \beta(g * u) - \beta u & t > 0, x \in R \\ u(0, x) = u_0(x) & x \in R \end{cases} \quad (6)$$

Next, we will apply Kato's theory to establish the local well-posedness for the Cauchy problem (6). For convenience, we recall Kato's theorem as follows.

Consider the abstract quasi-linear evolution equation

$$\frac{dv}{dt} + A(v)v = f(v) \quad t \geq 0, v(0) = v_0 \quad (7)$$

Let$X$and$Y$be Hilbert spaces such that$Y$is continuously and densely embedded in$X$and let$Q : Y \to X$ be a topological isomorphism. Let $L(Y, X)$denote the space of all bounded linear operator from $Y$to$X(L(X)$,if $X = Y$). Assume that

(i)$A(y) \in L(Y, X)$for$y \in X$with

$$\|(A(y) - A(z))w\|_X \leq \mu_1 \|y - z\|_X \|w\|_Y, \quad y, z, w \in Y$$

And $A(y) \in G(X, 1, \beta)$,i.e.$A(y)$ is quasi-m-accretive, uniformly on bounded sets in$Y$.

(ii)$QA(y)Q^{-1} = A(y) + B(y)$,where$B(y) \in L(X)$is bounded, uniformly on bounded sets in$Y$. Moreover,

$$\|(B(y) - B(z))w\|_X \leq \mu_2 \|y - z\|_Y \|w\|_X, \quad y, z \in Y \ , w \in X$$

(iii) $f : Y \to Y$ extends to a map from$X$into$X$.$f$is bounded on bounded sets in$Y$, and

$$\begin{matrix} \|f(y) - f(z)\|_Y \leq \mu_3 \|y - z\|_Y & y, z \in Y \\ \|f(y) - f(z)\|_X \leq \mu_4 \|y - z\|_X & y, z \in Y \end{matrix},$$

here $\mu_1, \mu_2, \mu_3, \mu_4$depend only on max $\{\|y\|_Y, \|z\|_Y\}$.

Lemma 1 *([31])Assume that(i), (ii) and (iii) hold. Given $v_0 \in Y$, there is a $T > 0$ depending only on $\|v_0\|_Y$ and a unique solution v to Eq (7) such that*

$$v = v(\cdot, v_0) \in C([0,T); Y) \cap C^1([0,T); X).$$

Moreover, the map $v_0 \mapsto v(\cdot, v_0)$ is continuous from Y to $C([0,T); Y) \cap C^1([0,T); X)$.

Set

$$A(u) = u\partial_x, f(u) = -\partial_x g * (u^2 + \frac{1}{2}u_x^2 + ku) - \lambda(g * u^{2n+1}) + \beta(g * u) - \beta u, Y = H^s, X = H^{s-1},$$

and $Q = \Lambda = (1 - \partial_x^2)^{\frac{1}{2}}$.Obviously, Q is an isomorphism of H^sontoH^{s-1}.

Lemma 2 *Assume $u_0(x) \in H^s, s > \frac{3}{2}$, and $u\,(t,x)$ is the solution of Eq (4), then*

$$\|u\|_{H^1}^2 \le \|u_0\|_{H^1}^2 .$$

Proof. Taking the scalar product of Eq (4) with u in $L^2\,(R)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_R (u^2 + u_x^2)dx + \int_R (\lambda u^{2n+2} + \beta u_x^2)dx$$

$$\frac{d}{dt}\int_R \left(u^2 + u_x^2 + 2\int_0^t \left(\lambda u^{2n+2} + \beta u_x^2\right) d\tau\right)dx = 0$$

Then

$$\int_R \left(u^2 + u_x^2 + 2\int_0^t \left(\lambda u^{2n+2} + \beta u_x^2\right) d\tau\right)dx = \int_R (u_0^2 + u_{0x}^2)dx$$

From the above equalities, we have

$$\|u\|_{H^1}^2 \le \int_R \left(u^2 + u_x^2 + 2\int_0^t \left(\lambda u^{2n+2} + \beta u_x^2\right) d\tau\right)dx = \int_R (u_0^2 + u_{0x}^2)dx = \|u_0\|_{H^1}^2$$

Hence, the lemma is proved. ■

Lemma 3 *([32]) Let$X$and$Y$be two Banach spaces and Y be continuously and densely embedded in X. Let $-A$ be the infinitesimal generator of the C_0 -semigroup $T(t)$ on X and S be an isomorphism from$Y$onto$X$.Y is –A-admissible(i.e. $T(t)Y \subset Y, \forall t \ge 0$,and the restriction of $T(t)$ to$Y$ is a C_0 -semigroup on Y) if and only if $-A_1 = -SAS^{-1}$ is the infinitesimal generator of the C_0 -semigroup $T_1(t) = ST(t)S^{-1}$ on X. Moreover, if Y is –A-admissible then the part of –A in Y is the infinitesimal generator of the restriction of $T(t)$ to$Y$.*

Lemma 4 ([30]*)The operator$A(u) = u\partial_x$,with $u \in H^s, s > \frac{3}{2}$,belongs to $G(L^2, 1, \beta)$.*

Lemma 5 ([30]*)The operator$A(u) = u\partial_x$,with $u \in H^s, s > \frac{3}{2}$,belongs to $G(H^{s-1}, 1, \beta)$*

Lemma 6 ([30]*)The operator$A(u) = u\partial_x$,with $u \in H^s, s > \frac{3}{2}$, then $A(u) \in L(H^s, H^{s-1})$ Moreover,*

$$\|(A(u) - A(z))w\|_{s-1} \le c_1 \|u - z\|_{s-1} \|w\|_s , \quad u, z, w \in H^s.$$

Lemma 7 ([30]*) $B(u) = [\Lambda^1, u\partial_x]\Lambda^{-1} \in L(H^{s-1})$,for $u \in H^s, s > \frac{3}{2}$. Moreover,*

$$\|(B(u) - B(z))w\|_{s-1} \le c_2 \|u - z\|_s \|w\|_{s-1} , u, z \in H^s, w \in H^{s-1}.$$

Lemma 8 *Let$f(u) = -\partial_x g * (u^2 + \frac{1}{2}u_x^2 + ku) - \lambda(g * u^{2n+1}) + \beta(g * u) - \beta u$. Then f is bounded on bounded sets in H^s and satisfies for all $s > \frac{3}{2}$. Moreover,*

(i) $\|f(y) - f(z)\|_s \le c_3 \|y - z\|_s \quad y, z \in H^s$

(ii) $\|f(y) - f(z)\|_{s-1} \le c_4 \|y - z\|_{s-1} \quad y, z \in H^s$

Proof. Let $y, z \in H^s, s > \frac{3}{2}$, Because of $H^{s-1}$is a Banach algebra. We have

$$\|f(y) - f(z)\|_s = \left\| -\partial_x \left(1 - \partial_x^2\right)^{-1} \left[\left(y^2 - z^2\right) + \frac{1}{2}\left(y_x^2 - z_x^2\right) + k(y - z)\right]\right.$$

$$\left. -\lambda \left(1 - \partial_x^2\right)^{-1} \left(y^{2n+1} - z^{2n+1}\right) + \beta \left(1 - \partial_x^2\right)^{-1} (y - z) - \beta(y - z)\right\|_s$$

$$\leq \left\|(y^2 - z^2) + \frac{1}{2}(y_x^2 - z_x^2) + k(y - z)\right\|_{s-1}$$

$$+\lambda \left\|y^{2n+1} - z^{2n+1}\right\|_{s-2} + \beta \|y - z\|_{s-2} + \beta \|y - z\|_s$$

$$\leq \|(y + z)(y - z)\|_{s-1} + \frac{1}{2} \|(y_x + z_x)(y_x - z_x)\|_{s-1} + |k| \|y - z\|_{s-1}$$

$$+\lambda \left\|(y - z) \sum_{i=0}^{2n} \left(y^{2n-i} z^i\right)\right\|_{s-2} + \beta \|y - z\|_{s-2} + \beta \|y - z\|_s$$

$$\leq \|y - z\|_s \left(\|y\|_s + \|z\|_s\right) + \frac{1}{2} \|y - z\|_s \left(\|y\|_s + \|z\|_s\right) + |k| \|y - z\|_s$$

$$+\lambda \|y - z\|_s \left(\sum_{i=0}^{2n} \left(\|y\|_s^{2n-i} \|z\|_s^i\right)\right) + 2\beta \|y - z\|_s$$

Note $c_3 = \frac{3}{2}\|y\|_s + \frac{3}{2}\|z\|_s + |k| + \lambda \sum_{i=0}^{2n} \left(\|y\|_s^{2n-i} \|z\|_s^i\right) + 2\beta$

Therefore ,

$$\|f(y) - f(z)\|_s \leq c_3 \|y - z\|_s \quad y, z \in H^s.$$

This proves(i).In addition, taking $z = 0$ in the above inequality, we obtain that f is bounded on bounded sets in H^s. And a similar argument as in the above inequality, we can proves (ii). ■

With the above preparations, we can obtain the the local well-posedness of the initial value problem associated with Eq (4).

Theorem 9 *Assume $u_0 \in H^s, s > \frac{3}{2}$, there exist a maximal $T = T(\lambda, n, \beta, k, \|u_0\|_s) > 0$ and a unique solution u to Eq(4), such that*

$$u = u(\cdot, u_0) \in C([0, T); H^s) \cap C^1([0, T); H^{s-1})$$

and the solution depends continuously on the initial data, i.e., the mapping $u_0 \mapsto u(\cdot, u_0) : H^s \mapsto C([0, T); H^s) \cap C^1([0, T); H^{s-1})$ is continuous. Moreover, T may be chosen independent of s in the following sense: if $u \in C([0, T); H^s) \cap C^1([0, T); H^{s-1})$ to Eq(4), and if $u_0 \in H^{s'}$for some $s' \neq s, s' > \frac{3}{2}$, then $u \in C([0, T); H^{s'}) \cap C^1([0, T); H^{s'-1})$ with the same T. In particular, if $u_0 \in H^\infty = \cap_{s \geq 0} H^s$, then $u \in C([0, T); H^\infty)$.

Proof. Combining Lemma 1-8 and Theorem 2.3 ([30]), we can get the statement of Theorem 9. ■

3 Blow-up

In this section, attention is turned to investigate conditions of blow-up. By using the local well-posedness result of Theorem 9 and energy estimates, one can establish the following precise blow-up scenario of strong solutions to Eq(4).

Theorem 10 *Assume $u_0 \in H^s, s > \frac{3}{2}$, then the solution of Eq(4) blows up in a finite time $T > 0$, if and only if*

$$\lim_{t \to T^-} \inf_{x \in R} u_x(t, x) = -\infty. \quad (8)$$

Proof. The local well-posedness theorem and a density argument implies that it suffices to prove the desired estimates for $s \geq 3$. Thus we take $s = 3$ in the proof.

Multiplying the equation (4) by u_{xx} and integrating by parts, we get

$$\frac{d}{dt}\int_R \left(u_x^2 + u_{xx}^2\right) dx + 3\int_R u_x^3 dx + 2\lambda \int_R (2n+1)\, u^{2n} u_x^2 dx + 2\beta \int_R u_{xx}^2 dx = -3\int_R u_x u_{xx}^2 dx$$

$$\frac{d}{dt}\int_R \left(u_x^2 + u_{xx}^2\right) dx \leq -3\int_R u_x u_{xx}^2 dx - 2\beta \int_R u_{xx}^2 dx - 3\int_R u_x^3 dx \tag{9}$$

Assume that $T < +\infty$and there exists $M > 0$, such that

$$u_x\,(t,x) \geq -M, \forall\,(t,x) \in [0,T) \times R \tag{10}$$

It then follows from (9) that

$$\frac{d}{dt}\int_R \left(u_x^2 + u_{xx}^2\right) dx \leq 3M\int_R u_{xx}^2 dx - 2\beta\int_R u_{xx}^2 dx + 3M\int_R u_x^2 dx \leq 3M\int_R \left(u_x^2 + u_{xx}^2\right) dx \tag{11}$$

Applying Gronwall's inequality to (11), then

$$\|u\|_{H^2}^2 \leq e^{3MT}\,\|u_0\|_{H^2}^2 \quad \forall t \in [0,T) \tag{12}$$

Differentiating the equation (4) with respect to$x$, and multiplying the result equation by u_{xxx}, then integrating by parts, we get

$$\begin{aligned}&\frac{d}{dt}\int_R \left(u_{xx}^2 + u_{xxx}^2\right) dx + 15\int_R u_x u_{xx}^2 dx + 2\lambda\int_R (2n+1)\, u^{2n} u_{xx}^2 dx + 2\beta\int_R u_{xxx}^2 dx \\ &- \frac{4\lambda n(2n+1)(2n-1)}{3}\int_R u^{2n-2} u_x^4 dx = -5\int_R u_x u_{xxx}^2 dx\end{aligned} \tag{13}$$

By using Sobolev embedding inequalities and the assumption(10), together with (11) (12)(13) and lemma2, then we have

$$\begin{aligned}&\frac{d}{dt}\int_R \left(u_x^2 + 2u_{xx}^2 + u_{xxx}^2\right) dx \leq -15\int_R u_x u_{xx}^2 dx - 2\beta\int_R u_{xxx}^2 dx - 5\int_R u_x u_{xxx}^2 dx \\ &+ \frac{4\lambda n(2n+1)(2n-1)}{3}\int_R u^{2n-2} u_x^4 dx + 3M\int_R \left(u_x^2 + u_{xx}^2\right) dx \quad \leq 18M\int_R u_{xx}^2 dx + (5M - 2\beta)\int_R u_{xxx}^2 dx \\ &+ \left[\frac{4\lambda n(2n+1)(2n-1)}{3}\left(\frac{1}{2}\,\|u_0\|_{H^1}^2\right)^{n-1}\left(e^{3MT}\,\|u_0\|_{H^2}^2\right) + 3M\right]\int_R u_x^2 dx\end{aligned} \tag{14}$$

Note $M' = \max\left\{18M, \frac{4\lambda n(2n+1)(2n-1)}{3}\left(\frac{1}{2}\,\|u_0\|_{H^1}^2\right)^{n-1}\left(e^{3MT}\,\|u_0\|_{H^2}^2\right) + 3M\right\}$, then

$$\|u\|_{H^3}^2 \leq e^{M'T}\,\|u_0\|_{H^3}^2 \quad \forall t \in [0,T) \tag{15}$$

Which contradicts the assumption the maximal existence time $T < +\infty$.that is,by the assumption(10), the solution of Eq(4) does not blow up in finite time.

Conversely, the Sobolev embedding theorem $H^s(R) \hookrightarrow L^\infty(R)$(with $s > \frac{1}{2}$) implies that if (8)holds, the corresponding solution of Eq (4) blows up in finite time, which completes the proof of Theorem 10. ■

Lemma 11 *([7])Let $T > 0$ and $u \in C^1([0,T); H^2(R))$. Then for all $t \in [0,T)$, there exists at least one point $\xi(t) \in R$ with*

$$m(t) = \inf_{x \in R}(u_x(t,x)) = u_x(t,\xi(t)) \tag{16}$$

The function $m(t)$ is absolutely continuous on $(0,T)$with

$$\frac{dm(t)}{dt} = u_{tx}(t,\xi(t))\textit{a.e. on }(0,T) \tag{17}$$

Theorem 12 *Given $u_0 \in H^s, s > \frac{3}{2}$, and assume that there exists $x_0 \in R$ such that*

$$u'_{0x}(x_0) < -\beta - \sqrt{\beta^2 + 2\left(\frac{1}{4}\|u_0\|^2_{H^1} + (\sqrt{2}|k| + \frac{\sqrt{2}}{2}\beta)\|u_0\|_{H^1} + \lambda(\frac{\|u_0\|_{H^1}}{\sqrt{2}})^{2n+1}\right)}$$

Then the corresponding solution of Eq (4) blows up in finite time.

Proof. The local well-posedness theorem and a density argument implies that it suffices to prove the desired estimates for $s \geq 3$. Thus we take $s = 3$ in the proof.

Let $T > 0$ be the existence time of the solution $u(t, \cdot)$ of Eq (4) (or Eq (6)) with the initial data u_0. Differentiating Eq (6) with respect to x, we have

$$u_{tx} = -u_x^2 - uu_{xx} - \partial_x^2 g * (u^2 + \frac{1}{2}u_x^2 + ku) - \lambda\partial_x(g * u^{2n+1}) + \beta\partial_x(g * u) - \beta u_x \tag{18}$$

Note that $\partial_x^2 g * f = g * f - f$, then

$$u_{tx} = -\frac{1}{2}u_x^2 - \beta u_x - uu_{xx} + u^2 + ku - g * (u^2 + \frac{1}{2}u_x^2 + ku) - \lambda\partial_x(g * u^{2n+1}) + \beta\partial_x(g * u) \tag{19}$$

By using the inequalities $g * (u^2 + \frac{1}{2}u_x^2)(x) \geq \frac{1}{2}u^2(x)$ and

$$\|u(t,\cdot)\|^2_{L^\infty} \leq \frac{1}{2}\|u(t,\cdot)\|^2_{H^1} \leq \frac{1}{2}\|u_0\|^2_{H^1}$$

By using Young's inequality, we get for all $t \in [0, T)$

$$\|g * (ku)\|_{L^\infty} \leq \|ku\|_{L^\infty} \leq \frac{|k|}{\sqrt{2}}\|u_0\|_{H^1}$$

$$\left\|\partial_x(g * u^{2n+1})\right\|_{L^\infty} = \left\|\partial_x g * u^{2n+1}\right\|_{L^\infty} \leq \left\|u^{2n+1}\right\|_{L^\infty} \leq (\frac{\|u_0\|_{H^1}}{\sqrt{2}})^{2n+1}$$

$$\|\partial_x(g * u)\|_{L^\infty} = \|\partial_x g * u\|_{L^\infty} \leq \|u\|_{L^\infty} \leq \frac{1}{\sqrt{2}}\|u_0\|_{H^1}$$

Define

$$m(t) = \inf_{x\in R}(u_x(t, x)) = u_x(t, \xi(t)), \tag{20}$$

because we deal with a minimum, then $u_{xx}(t, \xi(t)) = 0$ for all $t \in [0, T)$. Thus, we have a.e. on for all $[0, T)$

$$\frac{dm(t)}{dt} \leq -\frac{1}{2}m^2(t) - \beta m(t) + [\frac{1}{4}\|u_0\|^2_{H^1} + (\sqrt{2}|k| + \frac{\sqrt{2}}{2}\beta)\|u_0\|_{H^1} + \lambda(\frac{\|u_0\|_{H^1}}{\sqrt{2}})^{2n+1}] \tag{21}$$

Set

$$\mathrm{K} = \frac{1}{4}\|u_0\|^2_{H^1} + (\sqrt{2}|k| + \frac{\sqrt{2}}{2}\beta)\|u_0\|_{H^1} + \lambda(\frac{\|u_0\|_{H^1}}{\sqrt{2}})^{2n+1}$$

, then

$$\frac{dm(t)}{dt} \leq -\frac{1}{2}m^2(t) - \beta m(t) + \mathrm{K} = -\frac{1}{2}(m(t) + \beta - \sqrt{\beta^2 + 2\mathrm{K}})(m(t) + \beta + \sqrt{\beta^2 + 2\mathrm{K}}) \tag{22}$$

By the assumption $m(0) = u'_{0x}(x_0) < -\beta - \sqrt{\beta^2 + 2\mathrm{K}}$, we have

$$m^2(0) > (\beta + \sqrt{\beta^2 + 2\mathrm{K}})^2,$$

we now claim that this is true for any $t \in [0, T)$. In fact, assuming the contrary world, in view of $t_0 \in (0, T)$, such that for all $t \in [0, t_0)$, $m^2(t) > (\beta + \sqrt{\beta^2 + 2\mathrm{K}})^2$, but $m^2(t_0) = (\beta + \sqrt{\beta^2 + 2\mathrm{K}})^2$. Combining this with above inequality (22) , we have $\frac{dm(t)}{dt} < 0$ a.e. on $[0, t_0)$. Since $m(t)$ is absolutely continuous on $[0, t_0]$, an integration of this inequality would give the following inequality and we get the contraction $m(t_0) < m(0) = u'_{0x}(x_0) < -\beta - \sqrt{\beta^2 + 2\mathrm{K}}$.This proves the previous claim.

So we can solve the above inequality (22) to obtain

$$\frac{m(0)+\beta+\sqrt{\beta^2+2\mathrm{K}}}{m(0)+\beta-\sqrt{\beta^2+2\mathrm{K}}}e^{\sqrt{\beta^2+2\mathrm{K}}t}-1\leq\frac{2\sqrt{\beta^2+2\mathrm{K}}}{m(t)+\beta-\sqrt{\beta^2+2\mathrm{K}}}\leq 0$$

Since $0<\frac{m(0)+\beta+\sqrt{\beta^2+2\mathrm{K}}}{m(0)+\beta-\sqrt{\beta^2+2\mathrm{K}}}<1$, there exists

$$0<T\leq\frac{1}{\sqrt{\beta^2+2\mathrm{K}}}\ln(\frac{m(0)+\beta-\sqrt{\beta^2+2\mathrm{K}}}{m(0)+\beta+\sqrt{\beta^2+2\mathrm{K}}}),$$

Such that $\lim_{t\to T^-}m(t)=\lim_{t\to T^-}(\inf_{x\in R}u_x(t,x))=-\infty$. Therefore, the solution u does not exist globally in time. This completes the proof of Theorem 12. ■

4 Blow-up rate

In this section, we give blow-up results for the system (4) under study, attention is given to blow-up rate for the solution of Eq (4).

Theorem 13 *Let $T<\infty$ be the blow-up time of the corresponding solution of Eq (4) with $u_0\in H^s, s>\frac{3}{2}$, then we have*

$$\lim_{t\to T^-}\left(\inf_{x\in R}\{u_x(t,x)\}(T-t)\right)=-2 \tag{23}$$

Proof. Similarly, we may assume $s=3$ to prove this theorem.. Thanks to (19) (20), we have

$$\frac{dm(t)}{dt}=-\frac{1}{2}m^2(t)-\beta m(t)+u^2(t,\xi(t))+ku(t,\xi(t))$$

$$-g*(u^2+\frac{1}{2}u_x^2+ku)(t,\xi(t))-\lambda\partial_x(g*u^{2n+1})(t,\xi(t))+\beta\partial_x(g*u)(t,\xi(t)) \tag{24}$$

$$\frac{dm(t)}{dt}+\frac{1}{2}m^2(t)+\beta m(t)=u^2(t,\xi(t))+ku(t,\xi(t))$$

$$-g*(u^2+\frac{1}{2}u_x^2+ku)(t,\xi(t))-\lambda\partial_x(g*u^{2n+1})(t,\xi(t))+\beta\partial_x(g*u)(t,\xi(t)) \tag{25}$$

Applying the Sobolev embedding inequality, together with Young's inequality , we deduce that

$$\left|u^2(t,\xi(t))+ku(t,\xi(t))-g*(u^2+\frac{1}{2}u_x^2+ku)(t,\xi(t))-\lambda\partial_x(g*u^{2n+1})(t,\xi(t))+\beta\partial_x(g*u)(t,\xi(t))\right|$$

$$\leq\left\|u^2(t,\xi(t))\right\|_{L^\infty}+|k|\left\|u(t,\xi(t))\right\|_{L^\infty}+\left\|g*(ku)(t,\xi(t))\right\|_{L^\infty}$$

$$+\lambda\left\|\partial_x(g*u^{2n+1})(t,\xi(t))\right\|_{L^\infty}+\beta\left\|\partial_x(g*u)(t,\xi(t))\right\|_{L^\infty}$$

$$\leq\frac{1}{2}\left\|u_0\right\|_{H^1}^2+\sqrt{2}\,|k|\left\|u_0\right\|_{H^1}+\lambda(\frac{\left\|u_0\right\|_{H^1}}{\sqrt{2}})^{2n+1}+\frac{\sqrt{2}}{2}\beta\left\|u_0\right\|_{H^1} \tag{26}$$

Set

$$\mathrm{M}=\frac{1}{2}\left\|u_0\right\|_{H^1}^2+\sqrt{2}\,|k|\left\|u_0\right\|_{H^1}+\lambda(\frac{\left\|u_0\right\|_{H^1}}{\sqrt{2}})^{2n+1}+\frac{\sqrt{2}}{2}\beta\left\|u_0\right\|_{H^1}$$

Therefore,

$$-\mathrm{M}\leq\frac{dm(t)}{dt}+\frac{1}{2}m^2(t)+\beta m(t)\leq\mathrm{M}a.e.on(0,T)$$

Hence,

$$-\mathrm{M}-\frac{1}{2}\beta^2\leq\frac{dm(t)}{dt}+\frac{1}{2}(m(t)+\beta)^2\leq\mathrm{M}+\frac{1}{2}\beta^2 a.e.on(0,T) \tag{27}$$

Choose now $\varepsilon\in(0,\frac{1}{2})$, since $\lim_{t\to T^-}(m(t)+\beta)=-\infty$, there is some $t_0\in(0,T)$ with $m(t_0)+\beta<0$, and $(m(t_0)+\beta)^2>\frac{1}{\varepsilon}(\mathrm{M}+\frac{1}{2}\beta^2)$. Since m is locally Lipschitz, it follows that m is absolutely continuous. We deduce that m is decreasing on $[t_0,T)$, and

$$(m(t)+\beta)^2>\frac{1}{\varepsilon}(\mathrm{M}+\frac{1}{2}\beta^2)\quad t\in[t_0,T) \tag{28}$$

Using (27) and (28), we get

$$-\frac{1}{2}(m(t)+\beta)^2-\mathrm{M}-\frac{1}{2}\beta^2\leq\frac{dm(t)}{dt}\leq\mathrm{M}+\frac{1}{2}\beta^2-\frac{1}{2}(m(t)+\beta)^2 \tag{29}$$

Notice that m is locally Lipschitz and less than $m(t_0)<-\beta$ on $[t_0,T)$, we know that $\frac{1}{m}$ is locally Lipschitz on (t_0,T). From the inequality (29), we obtain

$$\frac{1}{2}-\varepsilon\leq\frac{d}{dt}(\frac{1}{m(t)+\beta})\leq\frac{1}{2}+\varepsilon a.e.on(t_0,T) \tag{30}$$

Integrating the above inequality on (t,T) with $t\in[t_0,T)$ and noticing that $\lim_{t\to T^-}m(t)=-\infty$, we get

$$(\frac{1}{2}-\varepsilon)(T-t)\leq-\frac{1}{m(t)+\beta}\leq(\frac{1}{2}+\varepsilon)(T-t)\quad t\in(t_0,T)$$

Since $\varepsilon\in(0,\frac{1}{2})$ is arbitrary, in view of the definition of $m(t)$, then

$$\lim_{t\to T^-}\{m(t)(T-t)+\beta(T-t)\}=-2$$

That is

$$\lim_{t\to T^-}m(t)(T-t)=-2$$

■

Remark 14 *:Although the occurrence of blow-up of strong solutions to Eq(4) is affected by the dissipative parameter. However, the blow-up rate of strong solutions to Eq (4) is not affected by the weak dissipative term.*

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