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Dynamic Analysis of a Fractional Order Rössler System

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Abstract: Based on the qualitative theory, the existence and uniqueness of solution for the fractional order Rössler system is investigated theoretically in this paper. The stability of the corresponding equilibria is also argued similarly to the integer order counterpart. Finally, Numerical solutions, together with simulations are given to verify the correctness of our analysis.

Keywords: fractional order Rössler system; qualitative theory; numerical simulation

1 Introduction

The notion of fractional order derivative dates back three centuries [1],but its applications to physics and engineering have been attracted lots of attention recently. It has been found that in interdisciplinary fields, many systems can be described by fractional differential equations. For instance, viscoelastic systems [2], dielectric polarization [3], electrode-electrolyte polarization [4], electromagnetic waves [5], quantitative finance [6] and quantum evolution of complex systems [7].

It is known that some fractional order differential systems behave chaotically, e.g., the fractional order Chua circuit [8], the fractional order Duffing system [9], the fractional order jerk model [10], the fractional order Chen system [11], the fractional order $L\ddot{u}$ system [12], the fractional order $R\ddot{o}ssler$ system [13], the fractional order Arneodo system [14], and the fractional order Newton-Leipnik system [15].

Due to essential differences between ordinary differential equations and fractional order differential equations, most of characteristics or conclusions of the ordinary differential equations systems cannot be directly extended to the case of the fractional order differential equations systems. Therefore, the fractional order systems have been paid more attention. Recently, many efforts have been devoted to the study of chaotic dynamics of fractional order differential systems [16-18]. Many current results about fractional order chaotic systems, however, are attained only by numerical simulations. Because of low accuracy associated with some of the numerical methods or limitations of them to detect chaos [19,20], wrong results have been reported in special cases. Due to this deficiency, a logical need is observed to develop analytical methods in order to investigate chaos in fractional order systems. One of these methods which is constructed based on the stability analysis in fractional order systems [21]. In Ref.[22], some interesting results for stability analysis of fractional order time variant ordinary differential systems (ODEs) are presented by using the Hermit eigenvalues analysis. These results are directly extended based on the related results of commensurate integer order ODEs. Based on theoretical results, one can design practical control scheme for some fractional dynamical system. There are some differences between Lorenz system and $R\ddot{o}ssler$ system. The Lorenz system describes the main properties of dissipative systems, while $R\ddot{o}ssler$ system is a emanative system. It is important to study the solution of different systems. Furthermore, $R\ddot{o}ssler$ system is easy realized in experiments and practical, however, the output amplitudes of variables in the group of Lorenz model are much large, it is realized by scale narrowed in circuit, otherwise electronic components cannot work safely in circuit. Theoretically the existence of solutions of $R\ddot{o}ssler$ system is studied in this paper, so we can design and choose circuit element parameters in $R\ddot{o}ssler$ circuit to ensure the safety of working circuit.

In this paper, our aim is to investigate the existence and uniqueness about solution of the fractional order $R\ddot{o}ssler$ system, similar to the results presented in Ref. [23]. At the same time, the stability of equilibria of the system will also be analyzed. The paper is organized as follows. In Section 2, the definition of fractional order derivative and some of its basic properties are introduced. In Section 3, the stability of equilibria for the fractional order $R\ddot{o}ssler$ system are

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analyzed according to the qualitative theory. Numerical simulations to illustrate the validity of the results are presented in Section 4, and finally, conclusions in Section 5 close the paper.

2 The fractional order derivative and its characteristics

There are different definitions for fractional derivatives in [1]. Among them, the Riemann-Liouville and the Caputo definitions are more reputed than others. The simplest definition which is also the easiest one to use is the Riemann-Liouville definition. This definition is described as

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \tag{1}$$

where $\Gamma(.)$ is the gamma function and $n-1 \leq \alpha < n$. The Caputo definition of the fractional derivative, which sometimes is called smooth fractional derivative, is described as

$$\frac{d^\alpha f(t)}{dt^\alpha} = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau, & m-1 < \alpha < m, \\ \frac{d^m}{dt^m} f(t) & \alpha = m, \end{cases} \tag{2}$$

where m is the first integer larger than α. For zero initial conditions, the Laplace transform of the fractional derivative (for both definitions) is

$$L\{\frac{d^q f(t)}{dt^q}\} = s^q L\{f(t)\}. \tag{3}$$

In the rest of this paper, the notation $\frac{d^\alpha}{dt^\alpha}$ represents the Caputo fractional derivative of order α.

A fractional order nonlinear system can be defined by the following model:

$$\begin{cases} \frac{d^\alpha x}{dt^\alpha} = f(t,x), \\ x(0) = x_0, \end{cases} \tag{4}$$

where $0 < \alpha < 1$.

Theorem 1 (D. Matignon, [21]) *Assume that $D = [0, T^*] \times [x_0 - \delta, x_0 + \delta]$ with some $T^* > 0$ and some $\delta > 0$, and let the function $f : D \to \mathbb{R}$ be continuous. Furthermore, define $T = min\{T^*, (\frac{\delta\Gamma(\alpha+1)}{\|f\|_\infty})^{\frac{1}{\alpha}}\}$, then there exists a function $x : [0, T] \to \mathbb{R}$ solving the initial value problem* (4). *Notice that $\|f\|_\infty$ is the norm of function f.*

Theorem 2 (D. Matignon, [21]) *Assume that $D = [0, T^*] \times [x_0 - \delta, x_0 + \delta]$ with some $T^* > 0$ and some $\delta > 0$, and let the function $f : D \to \mathbb{R}$ be bounded on D and fulfill a Lipschitz condition with respect to the second variable, i.e.,*

$$|f(t,x) - f(t,y)| \leq L|x-y| \tag{5}$$

with some constant number $L > 0$ independent of t, x, y. Then denoting T as Theorem 1, there exists at most one function $x : [0, T] \to \mathbb{R}$ solving the initial value problem (4).

Remark 1 *The results of Theorems 1 and 2 can be easily generalized to the initial value problem of the n-dimension fractional order differential equations*

$$\begin{cases} \frac{d^\alpha X(t)}{dt^\alpha} = F(t, X(t)), \\ X(0) = X_0, \end{cases} \tag{6}$$

where $X(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T \in R^n$ and $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)^T \in R^n, 0 < \alpha_i < 1, i = 1, 2, \cdots, n$. If $\alpha_1 = \alpha_2 = \cdots = \alpha_n$, system (6) is called a commensurate order system; otherwise, system (6) indicates an incommensurate order system.

3 Stability analysis of fractional order Rössler system

3.1 Existence and uniqueness of solution

Rössler designed the Rössler attractor in 1976, and the originally theoretical equations were later found to be useful in modeling equilibrium in chemical reactions. The defining equations are:

$$\begin{cases} \dfrac{dx}{dt} = -y - z, \\ \dfrac{dy}{dt} = x + ay, \\ \dfrac{dz}{dt} = b + z(x - c), \end{cases} \tag{7}$$

where x, y, z are the state variables, and a, b, c are three system parameters. Two types chaotic attractors were found in integer $Rössler$ system: one is a single lobe chaotic attractor (spiral-type), the other is a more complicate chaotic attractor (screw-type).

Here, we consider the fractional order system denoted as system (8) below. The standard derivative is replaced by a fractional derivative as the followings:

$$\begin{cases} \dfrac{d^{\alpha_1} x}{dt^{\alpha_1}} = -y - z, \\ \dfrac{d^{\alpha_2} y}{dt^{\alpha_2}} = x + ay, \\ \dfrac{d^{\alpha_3} z}{dt^{\alpha_3}} = b + z(x - c), \end{cases} \tag{8}$$

where $0 < \alpha_i \leq 1 (i = 1, 2, 3)$ is derivative order.

Theorem 3 *The initial value problem of the fractional order $Rössler$ system* (8) *can be represented in the following form:*

$$\begin{cases} D^{\alpha} = AX(t) + x(t)BX(t) + E, t \in (0, T], \\ X(0) = X_0, \end{cases} \tag{9}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)^T, 0 < \alpha_i < 1 (i = 1, 2, 3), X(t) = (x(t), y(t), z(t)),$

$$A = \begin{pmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ 0 & 0 & -c \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix},$$

some constant $T > 0$, then it has a unique solution.

Proof Let $F(X(t)) = AX(t) + x(t)BX(t) + E$, which is obviously continuous and bounded on the interval $[x_0 - \delta, x_0 + \delta]$ for any $\delta > 0$. Furthermore, one has

$$|F(X(t)) - F(X_1(t))| = |A(X(t) - X_1(t)) + x(t)BX(t) - x_1(t)BX_1(t)|$$

$$\leq (\|A\| + \|B\|(|X(t)| + |x_1(t)|))|X(t) - X_1(t)| \leq L|X(t) - X_1(t)|,$$

where $L = \|A\| + \|B\|(2|X_0| + \delta) > 0, X(t), Y(t) \in R^3$, $\|.\|$ and $|.|$ denote matrix norm and vector norm, respectively. The above inequality manifests that $F(x(t))$ satisfies a Lipschitz condition. Based on the results of Theorems 1 and 2, we can conclude that the initial value problem of fractional order $Rössler$ system has a unique solution.

3.2 The analysis of equilibria

Consider the following commensurate fractional order system:

$$\frac{d^{\alpha} x}{dt^{\alpha}} = f(x), \tag{10}$$

where $0 < \alpha < 1$ and $x \in \mathbb{R}^n$. The equilibrium points of system(10) are calculated by solving the following equation:

$$f(x) = 0. \tag{11}$$

Theorem 4 (D. Matignon, [25]) *The equilibrium points of system* (10) *are locally asymptotically stable if all eigenvalues* λ *of the Jacobian matrix* $J = \frac{\partial f}{\partial x}$ *evaluated at the equilibrium points satisfy :*

$$|arg(\lambda)| > \frac{\alpha\pi}{2}. \tag{12}$$

Besides, consider the Cauchy problem of the nonlinear autonomous fractional order system described by:

$$\frac{d^{\alpha}X(t)}{dt^{\alpha}} = F(X(t)), \tag{13}$$

with the initial value $X(0) = X_0 = (x_{10}, x_{20}, \cdots, x_{n0})^T$, where

$$F(X(t)) = \begin{pmatrix} f_1(x_1(t), x_2(t), \cdots, x_n(t)) \\ f_2(x_1(t), x_2(t), \cdots, x_n(t)) \\ \vdots \\ f_n(x_1(t), x_2(t), \cdots, x_n(t)) \end{pmatrix}.$$

$X(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T \in \mathbb{R}^n, \alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)^T$, and $0 < \alpha_i < 1 (i = 1, 2, \cdots, n)$. We have:

Theorem 5 (D. Matignon, [28]) *Let* $\hat{X} = (\hat{x_1}, \hat{x_2}, \cdots, \hat{x_n})^T$ *is the equilibrium of system* (13), *i.e.,* $\frac{d^{\alpha}\hat{X}}{dt^{\alpha}} = F(\hat{X}) = 0$, *and* $A = \frac{\partial F}{\partial X}|_{X=\hat{X}}$ *is the Jacobian matrix at the point* $\hat{X}$, *then the point* $\hat{X}$ *is asymptotically stable when* $|arg(eig(A))| > \frac{\alpha_m \pi}{2}$, *where* $\alpha_m = \max_{1 \le i \le n} \{\alpha_i\}$.

Proposition 1 *If* $c^2 = 4ab$, *the system* (8) *only one equilibrium* $S_0 = (\frac{c}{2}, -\frac{c}{2a}, \frac{c}{2a})$;*If* $c^2 > 4ab$, *it has two equilibria* $S_1 = (\frac{c+\sqrt{c^2-4ab}}{2}, -\frac{c+\sqrt{c^2-4ab}}{2a}, \frac{c+\sqrt{c^2-4ab}}{2a})$ *and* $S_2 = (\frac{c-\sqrt{c^2-4ab}}{2}, -\frac{c-\sqrt{c^2-4ab}}{2a}, \frac{c-\sqrt{c^2-4ab}}{2a})$.
Remark 2 *we let the right of system* (8) *is equal to zero, we can conclude the proposition 1.*
Proposition 2 *With respect to the system* (8), *we have*

1) *The equilibrium* S_0 *is unstable when* $Q_0 < 0$, *where* $Q_0 = 1 + \frac{c}{2a} - \frac{ac}{2}$.

2) *The equilibrium* S_1 *asymptotically stable if* $P_1 > 0,\ Q_1 > 0,\ P_1Q_1 + M > 0$, *where* $P_1 = \frac{c-M}{2} - a,\ Q_1 = 1 + \frac{c+M}{2a} + \frac{M-c}{2}a,\ M = \sqrt{c^2 - 4ab}$.

3) *The equilibrium* S_2 *asymptotically stable if* $P_2 > 0,\ Q_2 > 0,\ P_2Q_2 - M > 0$, *where* $P_1 = \frac{c+M}{2} - a,\ Q_2 = 1 + \frac{c-M}{2a} - \frac{c+M}{2}a,\ M = \sqrt{c^2 - 4ab}$.

Proof 1) The Jacobian matrix of system (8) at the point S_0 is

$$\begin{pmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ \frac{c}{2a} & 0 & -\frac{c}{2a} \end{pmatrix},$$

whose corresponding characteristic polynomial is

$$f(\lambda) = \lambda^3 - (a - \frac{c}{2})\lambda^2 + (1 + \frac{c}{2a} - \frac{ac}{2})\lambda \tag{14}$$

The eigenvalues of Eq.(14) are at least one positive root when $1 + \frac{c}{2a} - \frac{ac}{2} < 0$. Therefore the equilibrium S_0 is unstable.

2)

$$\begin{pmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ \frac{c+\sqrt{c^2-4ab}}{2a} & 0 & \frac{\sqrt{c^2-4ab}-c}{2} \end{pmatrix}$$

is the Jacobian matrix of system (8) at the point S_1, whose corresponding characteristic polynomial is

$$f(\lambda) = \lambda^3 - (\frac{c - \sqrt{c^2 - 4ab}}{2} - a)\lambda^2 + (1 + \frac{c + \sqrt{c^2 - 4ab}}{2a} + \frac{\sqrt{c^2 - 4ab} - c}{2}a)\lambda - \sqrt{c^2 - 4ab}. \tag{15}$$

Let $P_1 = \frac{c-M}{2} - a, Q_1 = 1 + \frac{c+M}{2a} + \frac{M-c}{2}a, M = \sqrt{c^2 - 4ab}$, from Routh-Hurwitz criteria, if $P_1 > 0, Q_1 > 0, P_1Q_1 + M > 0$, the eigenvules of Eq.(15) are all negative, so the equilibrium S_1 is asymptotically stable.

3)

$$\begin{pmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ \frac{c-\sqrt{c^2-4ab}}{2a} & 0 & \frac{-\sqrt{c^2-4ab}-c}{2} \end{pmatrix}$$

is the Jacobian matrix of system (8) at the point S_2, whose corresponding characteristic polynomial is

$$f(\lambda) = \lambda^3 - (\frac{c+\sqrt{c^2-4ab}}{2} - a)\lambda^2 + (1 + \frac{c-\sqrt{c^2-4ab}}{2a} - \frac{\sqrt{c^2-4ab}+c}{2}a)\lambda + \sqrt{c^2-4ab}. \tag{16}$$

Let $P_2 = \frac{c+M}{2} - a, Q_2 = 1 + \frac{c-M}{2a} - \frac{M-c}{2}a, M = \sqrt{c^2-4ab}$, from Routh-Hurwitz criteria, if $P_2 > 0, Q_2 > 0, P_2Q_2 - M > 0$, the eigenvules of Eq.(16) are all negative, so the equilibrium S_2 is asymptotically stable.

4 Numerical methods and simulations

By exploiting the Adams-Bashforth-Moulton scheme, the fractional order system (8) can be discretized as followings:

$$\begin{cases} x_{n+1} = x_0 + \frac{h^{q_1}}{\Gamma(q_1+2)}(-y^{\rho}_{n+1} - z^{\rho}_{n+1}) + \frac{h^{q_1}}{\Gamma(q_1+2)}\sum_{j=0}^{n}\beta_{1,j,n+1}(-y_j - z_j), \\ y_{n+1} = y_0 + \frac{h^{q_2}}{\Gamma(q_2+2)}(x^{\rho}_{n+1} + ay^{\rho}_{n+1}) + \frac{h^{q_2}}{\Gamma(q_2+2)}\sum_{j=0}^{n}\beta_{2,j,n+1}(x_j + ay_j), \\ z_{n+1} = z_0 + \frac{h^{q_3}}{\Gamma(q_3+2)}(b + z^{\rho}_{n+1}(x^{\rho}_{n+1} - c)) + \frac{h^{q_3}}{\Gamma(q_3+2)}\sum_{j=0}^{n}\beta_{3,j,n+1}(b + z_j(x_j - c)), \end{cases} \tag{17}$$

$$\begin{cases} x^{\rho}_{n+1} = x_0 + \frac{1}{\Gamma(q_1)}\sum_{j=0}^{n}\gamma_{1,j,n+1}(-y_j - z_j), \\ y^{\rho}_{n+1} = y_0 + \frac{1}{\Gamma(q_2)}\sum_{j=0}^{n}\gamma_{2,j,n+1}(x_j + ay_j), \\ z^{\rho}_{n+1} = z_0 + \frac{1}{\Gamma(q_3)}\sum_{j=0}^{n}\gamma_{3,j,n+1}(b + z_j(x_j - c)), \end{cases} \tag{18}$$

$$\beta_{i,j,n+1} = \begin{cases} n^{q_i+1} - (n-q_i)(n+1)^{q_i} & j = 0, \\ (n-j+2)^{q_i+1} + (n-j)^{q_i+1} - 2(n-j+1)^{q_i+1} & 1 \le j \le n, \\ 1 & j = n+1, \end{cases}$$

$$\gamma_{i,j,n+1} = \frac{h^{q_i}}{q_i}((n-j+1)^{q_i} - (n-j)^{q_i}), 0 \le j \le n, i = 1,2,3. \tag{19}$$

To verify the effectiveness of the obtained results, some numerical simulations for the fractional order $Rössler$ system have been conducted. All the differential equations are solved by using the method mentioned above. In the following simulations, let $q_1 = 0.91, q_2 = 0.95, q_3 = 0.98$ and $h = 0.01$.

We set $a = b = 1, c = 2$, i.e., $c^2 = 4ab, Q_0 < 0$. The equilibrium S_0 of system (8) is unstable. Figure 1 shows that the state $x(t)$ of system (8) is dramatically decreasing towards infinite, while the state $y(t)$ of system (8) is rapidly increasing towards infinite, where the corresponding initial state are set as $x_0 = 0.99, y_0 = -0.99, z_0 = 0.99$.

Figure 2 demonstrates that the equilibrium S_1 of system (8) is stable, where $a = 0.2, b = 5, c = 4$, i.e. $P_1 > 0, Q_1 > 0, P_1Q_1 + M > 0$. Figure 3 shows that the the equilibrium S_1 of system (8) is unstable, where $a = 1, b = 2, c = 4$, i.e., $P_1 < 0$.

Figure 4 demonstrates that the equilibrium S_2 of system (8) is stable, where $a = 0.4, b = 5, c = 4$, i.e. $P_2 > 0, Q_2 > 0, P_2Q_2 - M > 0$. Figure 5 shows that the the equilibrium S_2 of system (8) is unstable, where $a = 1, b = 2, c = 4$, i.e., $Q_2 < 0$.

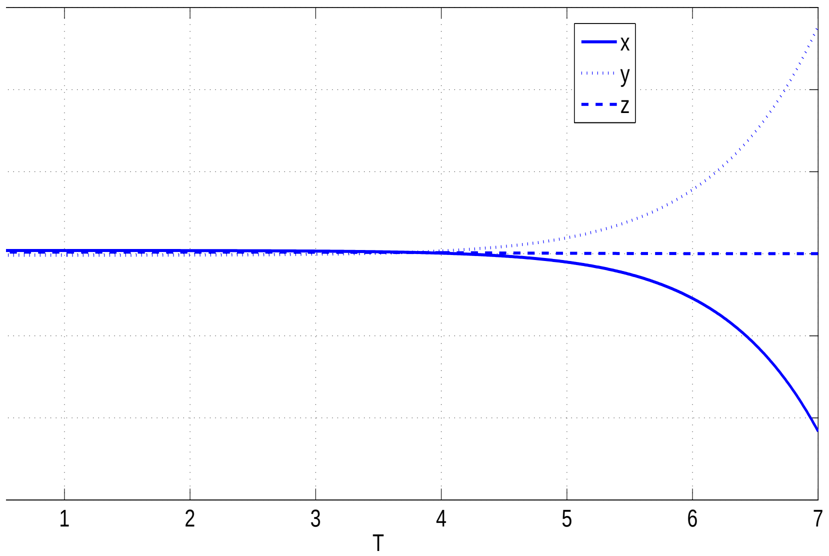

Figure 1: The equilibrium S_0 of system (8) is unstable

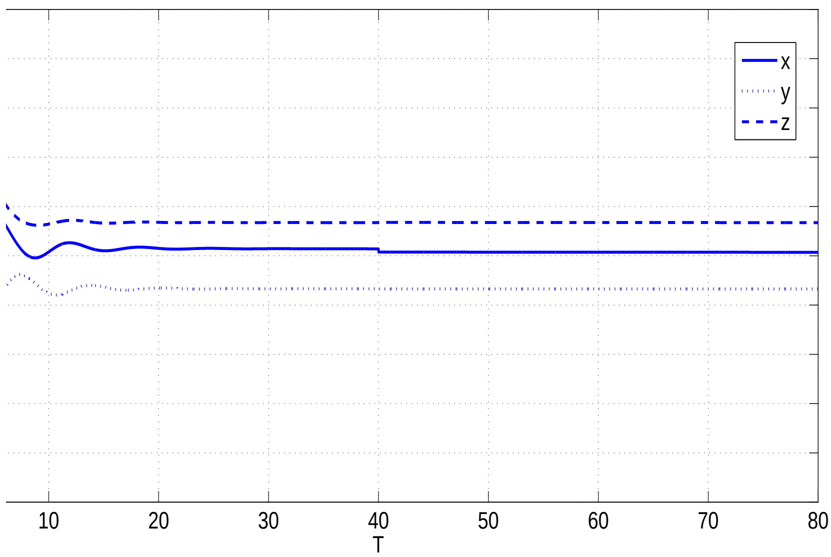

Figure 2: The equilibrium S_1 of system (8) is stable

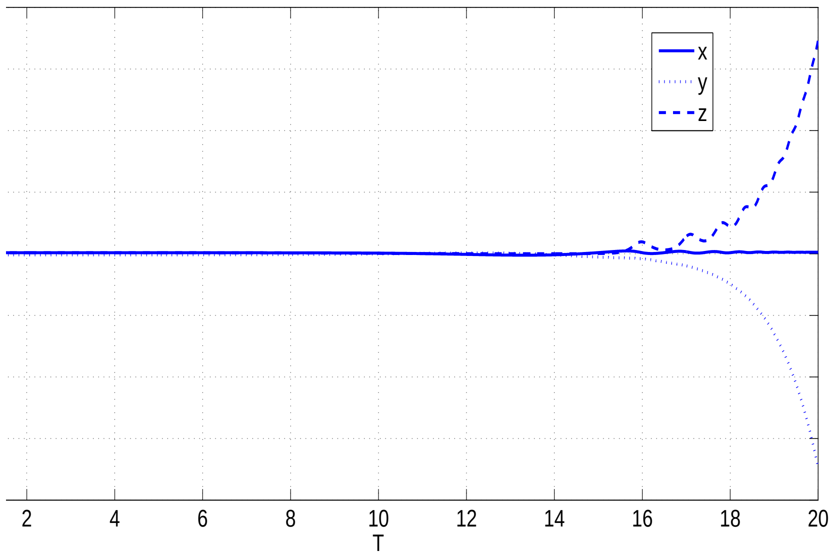

Figure 3: The equilibrium S_1 of system (8) is unstable

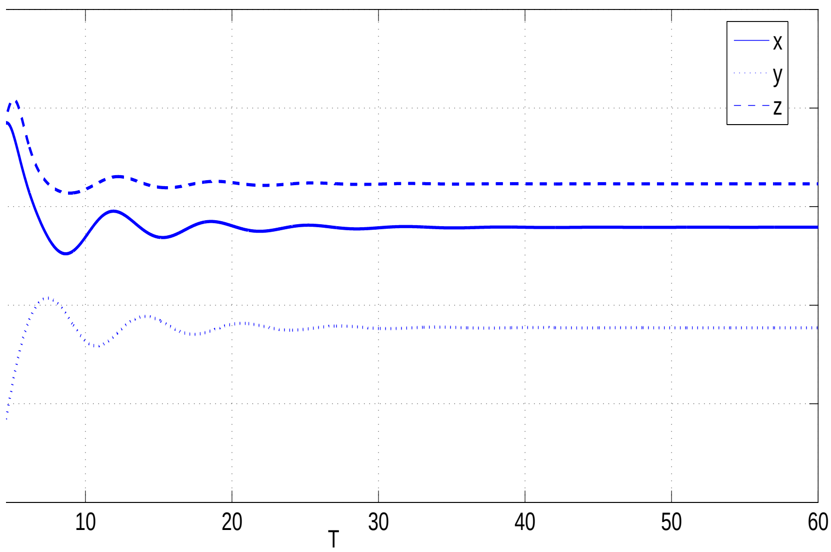

Figure 4: The equilibrium S_2 of system (8) is stable

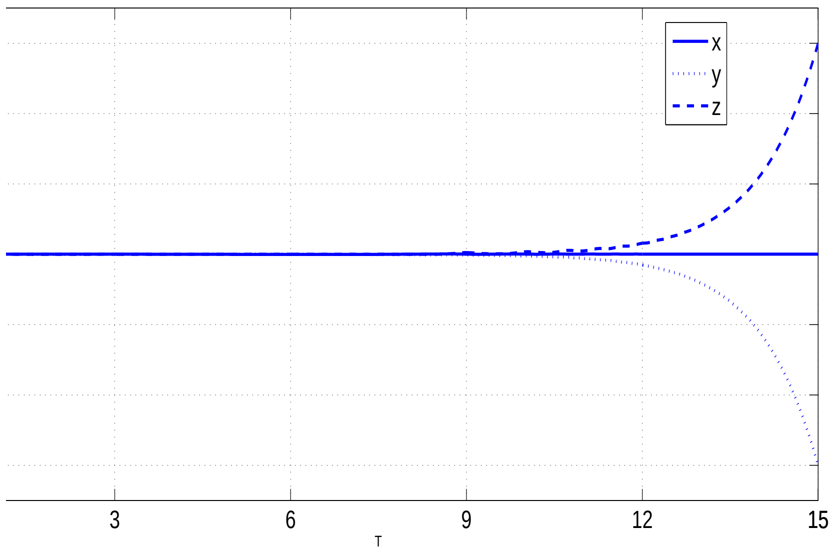

Figure 5: The equilibrium S_2 of system (8) is unstable

5 Conclusions

The dynamics for the fractional order $Rössler$ system has been extensively investigated in this paper. A rigorous proof of existence and uniqueness of solution for the fractional order $Rössler$ system has been provided. It has been shown that the fractional order $Rössler$ system has the similar equilibria and stability with the integer order counterpart. Finally, the numerical solutions and simulations are given to verify the feasibility of the results.

Acknowledgments

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References

[1] I. Podlubny. Fractional differential equations. New York, Academic Press, 1999.

[2] R. Bagley and R. Calico. Fractional order state equations for the control of viscoelastically damped structures. *J. Guid. Contr. Dyn.*, 14(1991): 304-311.

[3] H. Sun, A. Abdelwahed and B. Onaral. Linear approximation for transfer function with a pole of fractional order, *IEEE Trans. Auto. Contr.*, 29 (1984): 441-444.

[4] M. Ichise, Y. Nagayanagi and T. Kojima. An analog simulation of non-integer order transfer functions for analysis of electrode process. *J. Electroanal Chem.*, 33(1971): 253-265.

[5] O. Heaviside. Electromagnetic theory. New York, Chelsea, 1971.

[6] N. Laskin. Fractional market dynamics. *Physica A*, 2(2000): 482-492.

[7] D. Kunsezov, A. Bulagc and G. Dang. Quantum levy processes and fractional kinetics. *Phys. Rev. Lett.*, 2(1999): 1136-1139.

[8] T. T. Hartley, C. F. Lorenzo and H. K. Qammer. Chaos in a fractional order Chua's system. *IEEE. Trans. CAS-I*, 42(1995): 485-490.

[9] P. Arena, R. Caponetto, L. Fortuna and D. Porto. Chaos in a fractional order Duffing system. Proc. ECCTD, Budapest, 1997: 1259-1262.

[10] W. M. Ahmad and J. C. Sprott. Chaos in fractional-order autonomous nonlinear systems. *Chaos Solitons Fractals*, 16(2003): 339-351.

[11] J. G. Lu and G. Chen. A note on the fractional-order Chen system. *Chaos Solitons Fractals*, 27(2006): 685-688.

[12] J. G. Lu. Chaotic dynamics of the fractional-order Lu system and its synchronization. *Phys. Lett. A*, 4(2006): 305-311.

[13] C. Li and G. Chen. Chaos and hyperchaos in fractional order Rössler equations. *Physica A*, 341(2004): 55-61.

[14] J. G. Lu. Chaotic dynamics and synchronization of fractional-order Arneodo's systems. *Chaos Solitons Fractals*, 26(2005): 1125C1133.

[15] L. G. Sheu, H. K. Chen, J. H. Chen, L. M. Tam, W. C. Chen, K. T. Lin and Y. Kang. Chaos in the Newton-Leipnik system with fractional order. *Chaos Solitons Fractals*, 36(2006): 98-103.

[16] C. G. Li and G. Chen. Chaos in the fractional order Chen system and its control. *Chaos Solitons Fractals*, 22(2004): 549-554.

[17] I. Grigorenko and E. Grigorenko. Chaos dynamics of fractional Lorenz system. *Phys. Rev. Lett.*, 91(2003): 034101-034104.

[18] J. Wang and Y. Zhang. Designing synchronization schemes for chaotic fractional order unified systems. *Chaos Solitons and Fractals*, 30(2006): 1265-1272.

[19] M. S. Tavazoei and M. Haeri. A necessary condition for double scroll attractor existence in fractional order systems. *Phys. Lett. A*, 367(2007): 102-113.

[20] M. S. Tavazoei and M. Haeri. Unreliability of frequency-domain approximation in recognizing chaos in fractional-order systems. *IET Signal Process*, 1(2007): 171-181.

[21] D. Matignon. Stability results for fractional differential equations with applications to control processing. *Computational engineering in systems and application multi-conference*, 2(1996): 963-968.

[22] J. Liu, P. Dong, L. Xin and X. Li. Control of fractional order chaotic system via Hermit eigenvalue analysis. *CCDC, 2010 Chinese*: 2126 - 2131.

[23] K. Diethelm and J. Ford. Analysis of fractional differential equations. *J. Math. Anal. Appl.*, 265(2002): 229-248.

[24] R. Genesio, G. Innocenti and F. Gualdani. A global qualitative view of bifurcations and dynamics in the Rössler system. *Phys. Lett. A*, 372(2008): 1799-1809.

[25] D. Matignon. Stability results of fractional differential equations with applications to control processing. In: IMACS. IEEE-SMC, Lille, France, 1996: 963-968.