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Global Solutions and Blow-up Profiles for a Nonlinear Degenerate Parabolic Equation with Nonlocal Source

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Abstract: This paper deals with a degenerate parabolic equation $v_t = \Delta v^m + av^{p_1} \|v\|_{\alpha_1}^{q_1}$ subject to homogeneous Dirichlet condition. The local existence of a nonnegative weak solution is given. The blow-up and global existence conditions of nonnegative solutions are obtained. Moreover, we establish the precise blow-up rate estimates for all the blow-up solutions.

Keywords: nonlinear degenerate parabolic equation; global existence; blow up; blow-up rate

1 Introduction

In this paper, we investigate the following parabolic equation with local source involved a product with nonlocal term:

$$\begin{cases} v_t = \Delta v^m + av^{p_1} \|v\|_{\alpha_1}^{q_1}, & x \in \Omega, t > 0, \\ v(x,t) = 0, & x \in \partial\Omega, t > 0, \\ v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $R^N (N \geq 1)$ with smooth boundary $\partial\Omega$, $a > 0$, $m > 1$, $\alpha_1 \geq 1$, p_1 , $q_1 \geq 0$ and $p_1 + q_1 \neq 0$, $\|.\|_\alpha^\alpha = \int_\Omega |.|^\alpha dx$.

Such a problem arise in the study of the flow of a fluid through a porous medium with an integral source (see [1-2] and references therein) and in the study of population dynamics (see[3]).

Porous medium equations with or without a non-local source have been studied by many authors (see [4-9] for forced reaction source and [10] for the non-local sources). The typical models are the following classical type: (i) Non-local source:$v_\tau = \Delta v^m + \int_\Omega v^p dx$; (ii) Local source: $v_\tau = \Delta v^m + v^p$, with the same initial-boundary conditions as problem (1.1). Although, both the problems share the same blow-up criteria and the blow-up rate, there do exist some essential differences from the two problems.

As for nonlocal problems, in [11], the authors established the critical Fujita exponent for the Cauchy problem:

$$\begin{cases} v_t = \Delta v^m + v^{r+1} (\int_{R^N} K(x) v^q(x,\tau) dx)^{(p-1)/q}, x \in R^N, t > 0, \\ v(x,0) = v_0(x), \quad x \in R^N, \end{cases} \tag{1.2}$$

where parameters $m, p > 1, q > 0, r \geq 0$, the initial data $v_0(x)$ is a bounded nonnegative function, and the kernel function $K(x)$ is nonnegative and measurable.

In the paper [12], the authors considered the following one-dimensional problem:

$$\begin{cases} v_t = ((v^m)_x + \varepsilon v^n)_x + av \|v\|_q^{p-1}, & x \in (0,1), \\ v(0,t) = v(1,t) = 0, & t > 0, \\ v(x,0) = v_0(x), & x \in [0,1], \end{cases} \tag{1.3}$$

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They proved that the solution blows up in finite time provided that $p > \max\{1, m, n\}$ and the initial data u_0 is sufficiently large.

In[13], Deng, Duan and Xie studied the following one-dimensional problem:

$$\begin{cases} v_t - (v^m)_{xx} = a\int_{-l}^{l} v^q dx, & x \in (-l, l), t > 0, \\ v(-l, t) = v(l, t) = 0, & t > 0, \\ v(x, 0) = v_0(x), & x \in [-l, l], \end{cases} \tag{1.4}$$

with $l, a > 0$ and $q > m > 1$. They proved that if the solution $v(x)$ of (1.4) blows up in finite time, then the blow-up set is the whole domain $[-l, l]$, and the following estimate holds (here T' denotes the blow-up time):$C_1(T' - t)^{-1/(q-1)} \leq \max_{x\in[-l,l]} u(x, t) \leq C_2(T' - t)^{-1/(q-1)}$.

In[14], Souplet considered the following problem:

$$\begin{cases} v_t - \Delta v = \|v\|_r^p, & x \in \Omega, t > 0, \\ v(x, t) = 0, & x \in \partial\Omega, t > 0, \\ v(x, 0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.5}$$

If $1 \leq r < \infty$, $p > 1$, the author showed that $v(x)$ blows up in finite time T'' and got the blow-up profile: $\lim_{t\to T''}(T'' - t)^{1/(p-1)} v(x, t) = \lim_{t\to T''}(T'' - t)^{1/(p-1)} \|v(x, t)\|_\infty = [(p - 1\, |\Omega|^{p/r})]^{-1/(p-1)}$.

When $\alpha_1 = q_1$, the equation (1.1) was introduced and studied in [15]. The authors showed that the solution either exists globally or blows up in finite time. Furthermore, they also yielded the blow-up rate. But the accurate estimations of the blow-up rate and the blow-up set were not discussed in that paper.

As for local problems, in the paper [16], Chen and Xie investigated the following problem:

$$\begin{cases} v_t - \Delta v^m = a v^p v^q(x_0, t), & x \in \Omega, t > 0, \\ v(x, t) = 0, & x \in \partial\Omega, t > 0, \\ v(x, 0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.6}$$

They showed that the solution exists globally or blows up in finite time under some conditions. In [17], Du et. all made a further analysis for the problem (1.6) and obtained the blow-up set and the blow-up rate of the solution.

Motivated by the above works, in this paper, we discuss the local existence of nonnegative solution and profile of the blow-up solution to (1.1), and extend their results to the general equation (1.1). Now we state our results as follows.

Theorem 1.1 *If $p_1 + q_1 < m$, then the nonnegative solution of (1.1) exists globally.*

Theorem 1.2 *If $p_1 + q_1 > m$, then the nonnegative solution of (1.1) blows up in finite time for sufficiently large initial values and exists globally for sufficiently small initial values.*

Theorem 1.3 *Assume that $p_1 + q_1 = m$.*
(1) If the domain is sufficiently small, then the nonnegative solution of (1.1) is global.
(2) If the domain contains a sufficiently large ball, and v_0 is positive and continuous in Ω, then the nonnegative solution of (1.1) blows up in finite time.

In order to describe the blow-up properties of solutions, we need the following assumptions on the initial data v_0.
(H1) $v_0(x) \in C^{2+\theta}(\Omega) \cap C(\overline{\Omega}), 0 < \theta < 1$;
(H2) $v_0(x) > 0$ in Ω ,$v_0(x) = 0$, $\partial v_0/\partial\eta < 0$ on $\partial\Omega$, where η is the unit outward normal vector on Ω;
(H3) $\Delta v_0^m + a v_0^{p_1} \|v_0\|_{\alpha_1}^{q_1} \geq 0$, $x \in \Omega$ and $\Delta v_0 + a v_0^{p_1} \|v_0\|_{\alpha_1}^{q_1} = 0$ on $\partial\Omega$.
(H4) There exists a constant $\varepsilon \geq \max\{\varepsilon_0, a\, |\Omega|^{q_1/\alpha_1} m/(p_1 + q_1 - 1)\}$, such that $\Delta v_0^m + a v_0^{p_1} \|v_0\|_{\alpha_1}^{q_1} - \varepsilon v_0^{p_1+q_1} \geq 0$. where $\varepsilon_0 = (|\Omega|^{q_1/\alpha_1} a(p_1 + q_1 - 1)/(m - 1))((q_1 + m - 1)/(p_1 + 2q_1 - 1))^{(p_1+2q_1-1)/(p_1+q_1-1)}$.

Theorem 1.4 *Suppose that $p_1 + q_1 \geq m$ and $v_0(x)$ satisfy (H1)-(H4). If $v(x, t)$ is the classical solution of (1.1) and blows up in finite time T_* , then there exist positive constants C_0, C_{00} , such that*

$$C_0(T_* - t)^{-1/(p_1+q_1-1)} \leq \max_{x\in\Omega} v(x, t) \leq C_{00}(T_* - t)^{-1/(p_1+q_1-1)}.$$

Theorem 1.5 *Assume that $p_1 \leq 1$ and (H1)-(H4) hold. If $v(x,t)$ is the classical solution of (1.1) and blows up in finite time T_*, then the following limits converge uniformly on any compact subset of Ω:*
(1) If $p_1 < 1$, $\lim_{t\to T_*} v(x,t)(T_* - t)^{1/(p_1+q_1-1)} = (a(p_1+q_1-1)\,|\Omega|^{q_1/\alpha_1})^{-1/(p_1+q_1-1)}$;
(2) If $p_1 = 1$, $\lim_{t\to T_*} |\ln(T_* - t)|^{-1} \ln v(x,t) = 1/q_1$.

This paper is organized as follows. In Section 2, we establish the local existence and give a comparison principle. In Section 3, we concern the global existence and the blow-up phenomenon. Results relating to the uniform blow-up profile are presented in the last section.

2 Local existence and comparison principle

Let $Q_T = \Omega \times (0,T)$, $S_T = \partial\Omega \times (0,T)$ for $0 < T < \infty$. As it is now well known that degenerate equation need not posses classical solutions, we begin by giving a precise definition of a weak solution of (1.1).

Definition 2.1 *A vector function $v(x,t)$ defined on $\overline{Q_T}$, for some $T > 0$, is called a subsolution (or supersolution) of (1.1) on $\overline{Q_T}$, if all the following hold:*
(1) $v(x,t) \in L^\infty(Q_T)$;
(2) $v(x,t) \leq (\geq) 0$ for $(x,t) \in S_T$, and $v(x,0) \leq (\geq) v_0(x)$ for almost all $x \in \Omega$;
(3)

$$\int_\Omega (v(x,t)\psi(x,t) - v_0(x)\psi(x,0))dx \leq (\geq) \int_0^t \int_\Omega (v\psi_s + v^m \Delta\psi + a v^{p_1} \|v\|_{\alpha_1}^{q_1} \psi) dx ds, \tag{2.1}$$

for every $t \in [0,T]$ and any ψ belong to the class of test functions,

$$\Psi \equiv \{\psi \in C(\overline{Q_T}); \psi_t, \Delta\psi \in C(Q_T) \cap L^2(Q_T); \psi \geq 0, \psi(x,t)\,|_{x\in\partial\Omega} = 0\}.$$

A weak solution of (1.1) is a vector function which is both a subsolution and a supersolution of (1.1). For every $T < \infty$, if $v(x,t)$ is a solution of (1.1) , we say $v(x,t)$ is global.

Clearly, every nonnegative classical(sub-,super-)solution of (1.1) is a weak (sub-,super-)solution of (1.1) in the sense of Definition 2.1. To proving the local existence of a solution to (1.1), we first give a comparison principle without proof.

Lemma 2.1 *Assume that $w(x,t) \in C^{2,1}(Q_T) \cap C(\overline{Q_T})$ and satisfy*

$$\begin{cases} w_t - d(x,t)\Delta w \geq c_1(x,t)w + c_2(x,t)\int_\Omega c_3(x,t)w(x,t)dx, & (x,t) \in Q_T, \\ w(x,t) \geq 0\ , & (x,t) \in S_T, \\ w(x,0) \geq 0\ , & x \in \Omega, \end{cases} \tag{2.2}$$

where $c_i (i = 1,2,3)$ are bounded functions and $c_2, c_3, d \geq 0$ in Q_T. Then $w(x,t) \geq 0$ on $\overline{Q_T}$.

In proving local existence for degenerate parabolic equation, there are three different approaches (see [4, p.113]). We will modify the boundary conditions rather than the differential equation. Introduce, for $k = 1, 2, \cdots$, the following regularized equations:

$$\begin{cases} v_{kt} = \Delta f_k(v_k) + a(g(v_k))^{p_1} \|g(v_k)\|_{\alpha_1}^{q_1}\ , & x \in Q_T, \\ v_k(x,t) = 1/k, & (x,t) \in S_T, \\ v_k(x,0) = v_0(x) + 1/k, & x \in \Omega. \end{cases} \tag{2.3}$$

where

$$f_k(v_k) = \begin{cases} v_k^m\ , & v_k \geq 1/k, \\ (1/k)^m, & v_k < 1/k, \end{cases} \qquad g_k(v_k) = \begin{cases} v_k\ , & v_k \geq 1/k, \\ 1/k, & v_k < 1/k. \end{cases}$$

Let v_{0i} are smooth approximation of $v_0(x)$ with $\mathrm{supp} v_{0i} \subset \Omega$ in (2.3). By a similar discussion as that of Theorems A.1-A.4 in [18], we can show that (2.3) has a unique classical solution $v_k^i \in C(\overline{\Omega} \times [0, T_i(k))) \cap C^{2,1}(\Omega \times (0, T_i(k)))$ for $0 < T_i(k) < \infty$, where $T_i(k)$ is the maximal existence time. By a direct computation and the classical maximum principle, we have $v_k^i \geq 1/k$. Hence, v_k^i satisfies

$$v_{kt}^i = \Delta(v_k^i)^m + a(v_k^i)^{p_1} \left\|v_k^i\right\|_{\alpha_1}^{q_1}, \quad (x,t) \in Q_{T_i(k)}, \tag{2.4}$$

with the corresponding initial and boundary conditions. Clearly, passing to the limit $i \to \infty$, it follows that $v_k(x,t) = \lim_{i\to\infty} v_k^i(x,t)$, and v_k is a weak solution of

$$v_{kt} = \Delta(v_k)^m + av_k^{p_1} \cdot \|v_k\|_{\alpha_1}^{q_1}, \quad (x,t) \in Q_{T(k)}, \tag{2.5}$$

with the corresponding initial and boundary conditions on $(0, T(k))$, where $T(k) = \lim_{i\to\infty} T_i(k)$ is the maximal existence time. Here a weak solution of (2.5) is defined in a manner similar to that for problem (1.1), only the integral equalities for v, (2.1) may be replaced with

$$\begin{aligned} \int_\Omega (v_k(x,t)\psi(x,t) - (v_0(x) + 1/k)\psi(x,0))dx \le (\ge) \int_0^t \int_\Omega (v\psi_s + v^m\Delta\psi + av^{p_1} \|v\|_{\alpha_1}^{q_1} \psi)dxds \\ + \frac{1}{k}\int_0^t \int_{\partial\Omega} (\partial\psi/\partial n)d\sigma ds. \end{aligned} \tag{2.6}$$

Since $v_k \ge 1/k$, applying Lemma 2.1, we have the following lemma.

Lemma 2.2 *Assume that $w(x,t) \in C(\overline{\Omega} \times [0, T_i(k))) \cap C^{2,1}(\Omega \times (0, T_i(k)))$ for $0 < T_i(k) < \infty$ is a sub- (or super-) solution of (2.4). Then $w \le (\ge) v_k^i$ on $\overline{\Omega} \times [0, T_i(k))$.*

Lemma 2.3 *If $k_1 > k_2$, then $v_{k_1}^i(x,t) \le v_{k_2}^i(x,t)$ on $\overline{\Omega} \times [0, T_i(k_2))$ and $T_i(k_1) \ge T_i(k_2)$.*

Then, passing to the limit $i \to \infty$, it happens that $v_{k_1} \le v_{k_2}$ and $T(k_1) \ge T(k_2)$ if $k_1 > k_2$. Therefore, the limit $T_* = \lim_{k\to\infty} T(k)$ exists and, as well, the point-wise limit $v(x,t) = \lim_{k\to\infty} v_k(x,t)$ exists for any $(x,t) \in \overline{\Omega} \times [0, T_*)$. Furthermore, as the convergence of the sequence is monotone, passage to the limit $k \to \infty$ in the identities (2.6), the following theorem is thus established.

Theorem 2.1 *(Local existence and continuation). Given $v_0 \ge 0, v_0 \in L^\infty(\Omega)$, there is some $T_* = T_*(v_0) > 0$ such that there exists a nonnegative weak solution $v(x,t)$ of (1.1) for each $T < T_*$. Moreover, either $T_* = \infty$ or $\lim_{t\to T_*} \sup \|v(x,t)\| = \infty$.*

Lemma 2.4 *(Comparison principle). Let $\underline{v}$ and $\overline{v}$ be a nonnegative subsolution and a nonnegative supersolution of (1.1), respectively . If $\underline{v}_0 \le \overline{v_0}$ and either*

$$\int_\Omega \underline{v}^{\alpha_1} dx \ge \delta > 0, \quad \underline{v} \ge \delta, \tag{2.7}$$

or

$$\int_\Omega \overline{v}^{\alpha_1} dx \ge \delta > 0, \quad \overline{v} \ge \delta, \tag{2.8}$$

hold. Then $\underline{v} \le \overline{v}$ on Q_T.

Proof. The technique for proving comparison principle for degenerate parabolic equations is quite standard. For example, see [4, 19, 20]. Here we shall sketch the argument for the convenience of the reader.

Subtracting the first integral inequalities of (2.1) for $\underline{v}$ and $\overline{v}$, yields

$$\begin{aligned} \int_\Omega [\underline{v}(x,t) - \overline{v}(x,t)]\psi(x,t)dx \le \int_\Omega [\underline{v}_0(x) - \overline{v}_0(x)]\psi(x,0)dx \\ + \int_0^t \int_\Omega (\underline{v} - \overline{v})[\psi_s + \Phi(x,s)\Delta\psi + aH(x,s)\|\underline{v}\|_{\alpha_1}^{q_1} \psi]dxds \\ + a\int_0^t \int_\Omega \overline{v}^{p_1}\psi D(s)(\int_\Omega G(x,s)(\underline{v} - \overline{v})dx)dxds, \end{aligned}$$

where

$$\begin{aligned} \Phi(x,s) &= \int_0^1 m(\theta\overline{v} + (1-\theta)\underline{v})^{m-1}d\theta, \\ H(x,s) &= \int_0^1 p_1(\theta\overline{v} + (1-\theta)\underline{v})^{p_1-1}d\theta, \\ D(s) &= \int_0^1 (q_1/\alpha_1)[\theta\int_\Omega \overline{v}^{\alpha_1}dx + (1-\theta)\int_\Omega \underline{v}^{\alpha_1}dx]^{q_1/\alpha_1-1}d\theta, \\ G(x,s) &= \int_0^1 \alpha_1(\theta\overline{v} + (1-\theta)\underline{v})^{\alpha_1-1}d\theta, \end{aligned}$$

Since $\underline{v}$ and $\overline{v}$ are bounded Q_T, it follows from $m > 1, \alpha_1 \ge 1$ that $\Phi(x,s), G(x,s)$ are bounded nonnegative functions. Similarly, $H(x,s), D(s)$ are bounded if $p_1, q_1/\alpha_1 \ge 1$. Now, if $p_1, q_1/\alpha_1 < 1$, we have $H(x,s) \le \delta^{p_1-1}$, $D(s) \le \delta^{q_1/\alpha_1-1}$ by the assumptions (2.7) or (2.8). Thus, we can choose the appropriate test function ψ as in [4, p.118-123] to obtain

$$\begin{aligned} \int_\Omega [\underline{v}(x,t) - \overline{v}(x,t)]_+dx \le \|\psi\|_\infty \int_\Omega [\underline{v}_0(x) - \overline{v_0}(x)]_+dx + c_1\int_0^t \int_\Omega [\underline{v} - \overline{v}]_+dxds \\ + a\delta^{p_1}|\Omega| \|\psi_1\|_\infty \int_0^t D(s)\int_\Omega G(x,s)[\underline{v} - \overline{v}]_+dxds, \end{aligned} \tag{2.9}$$

where $w_+ \equiv \max\{w, 0\}$ and $c_1 > 0$ is bounded constant. Now, (2.9) combined with the Gronwall's lemma show that $\underline{v} \le \overline{v}$, since $\underline{v}_0 \le \overline{v_0}$. ■

3 Global existence and finite time blow-up

In this section, we use the super- and subsolution methods to give the proof of Theorems 1.1-1.3. According to Lemma 2.1, we only need to construct positive weak supersolutions or subsolutions bounded for any $T > 0$.

Proof of Theorem 1.1. Firstly, we construct supersolutions which are bounded for any $T > 0$. Let $\phi(x)$ be the solution of the following elliptic problem:

$$-\Delta\phi(x) = 1, x \in \Omega; \qquad \phi(x) = 0, \quad x \in \partial\Omega. \tag{3.1}$$

Letting

$$M = \max_{x\in\Omega} \phi(x) \quad and \quad \overline{v} = (K(\phi(x) + 1))^{1/m}, \tag{3.2}$$

where $K > 0$ will be fixed later. Clearly , $\overline{v}$ is bounded for any $t > 0$ and $\overline{v} \geq K^{1/m}$. Thus, we have

$$\begin{aligned} \overline{v}_t - \Delta\overline{v}^m - a\overline{v}^{p_1} \|\overline{v}\|_{\alpha_1}^{q_1} &= K - a(K(\phi(x) + 1))^{p_1/m} \left\|(K(\phi(x) + 1))^{1/m}\right\|_{\alpha_1}^{q_1} \\ &\geq K - aK^{(p_1+q_1)/m}(M + 1)^{(p_1+q_1)/m} |\Omega|^{q_1/\alpha_1}. \end{aligned} \tag{3.3}$$

Denote

$$K_1 = [a |\Omega|^{q_1/\alpha_1} (M + 1)^{(p_1+q_1)/m}]^{m/(m-p_1-q_1)}. \tag{3.4}$$

If $p_1 + q_1 < m$, then we can choose sufficiently large that $K \geq K_1$ and

$$(K(\phi(x) + 1))^{1/m} \geq v_0(x) \tag{3.5}$$

From (3.2)-(3.5) we see that $\overline{v}$ is a positive weak supersolution of (1.1). Hence $v \leq \overline{v}$ by comparison principle, which implies v exists globally.

Proof of Theorem 1.2. If $p_1 + q_1 > m$, we can choose $K \leq K_1$,where K_1 are defined in (3.4). Furthermore, assume that $v_0(x)$ is small enough to satisfy (3.5). It follows that $\overline{v}$ defined by (3.2) is a positive weak supersolution of (1.1). Hence, v exists globally.

Now we prove that the solution will blow up in finite time if the initial data is large. Denote by λ and $\varphi(x)$ the first eigenvalue and the corresponding eigenfunction of the following eigenvalue problem

$$-\Delta\varphi(x) = \lambda\varphi(x), \quad x \in \Omega; \quad \varphi(x) = 0, \quad x \in \partial\Omega. \tag{3.6}$$

Then $\varphi(x)$ may be normalized as $\varphi(x) > 0$ in Ω, $\max_{\Omega} \varphi(x) = 1$, and $\partial\varphi/\partial\eta < 0$ on $\partial\Omega$.

If $p_1 + q_1 > m$, then there exists $l > 0$ such that $ml > 1$. We define $\underline{v}(x, t)$ the functions as

$$\underline{v}(x, t) = [s(t)(\varphi(x) + 1)]^l, \tag{3.7}$$

and $s(t)$ is the solution of the initial problem

$$s'(t) = ks^\gamma, \quad s(0) = \delta > 0, \tag{3.8}$$

where $k, \delta > 0, \gamma > 1$ to be fixed later. Clearly, $s(t) \geq \delta$ become unbounded in finite time, and

$$\int_\Omega \underline{v}^{\alpha_1} dx \geq \delta^{\alpha_1 l} \int_\Omega (\varphi(x) + 1)^{\alpha_1 l} dx > 0. \tag{3.9}$$

A direct calculation yields

$$\begin{aligned} &\Delta\underline{v}^m + a\underline{v}^{p_1} \cdot \|\underline{v}\|_{\alpha_1}^{q_1} \\ &= s^{ml}[ml(ml - 1)(\varphi(x) + 1)^{ml-2} |\nabla\varphi|^2 + ml(\phi(x) + 1)^{ml-1}\Delta\phi(x)] + as^{(p_1+q_1)l}(\varphi(x) + 1)^{p_1 l} \left\|(\varphi(x) + 1)^l\right\|_{\alpha_1}^{q_1} \\ &\geq -\lambda mls^{ml}(\phi(x) + 1)^{ml-1}\phi(x) + as^{(p_1+q_1)l}(\varphi(x) + 1)^{p_1 l} \left\|(\varphi(x) + 1)^l\right\|_{\alpha_1}^{q_1} \\ &\geq -\lambda mls^{ml}(\phi(x) + 1)^{ml} + acs^{(p_1+q_1)l} \\ &= ls^{l-1}(\phi(x) + 1)^l[-\lambda ms^{ml-l+1}(\phi(x) + 1)^{ml-l} + (ac/l)s^{(p_1+q_1)l-l+1}(\phi(x) + 1)^{-l}] \\ &\geq ls^{l-1}(\phi(x) + 1)^l(ac/l2^l)s^{ml-l+1}(s^{(p_1+q_1-m)l} - \lambda ml2^{ml}/ac), \\ \\ &\underline{v}_t = ls^{l-1}(\phi(x) + 1)^l s'(t). \end{aligned} \tag{3.10}$$

where

$$c \equiv \left\|(\varphi(x)+1)^l\right\|_{\alpha_1}^{q_1} > 0. \tag{3.11}$$

We may choose

$$k = ac/l2^l > 0, \gamma = ml - l + 1 > 1, \delta = ((\lambda ml2^{ml}/ac)+1)^{1/(p_1+q_1-m)l} > 1. \tag{3.12}$$

Furthermore, assume that v_0 large enough to satisfy

$$\underline{v}(x,0) = [\delta(\varphi(x)+1)]^l \le v_0(x). \tag{3.13}$$

Then, it follows from (3.6)-(3.13) that $\underline{v}$ is a positive weak subsolution of (1.1), which blows up in finite time since $s(t)$ does. By comparison principle, $\underline{v} \le v$, which implies v blows up in finite time.

Proof of Theorem 1.3. (1) In view of $p_1 + q_1 = m$, we can choose $|\Omega| < [a(M+1)]^{-\alpha_1/q_1}$ and K large enough to satisfy (3.5). Then, $\overline{v}$ defined by (3.2) is a positive weak supersolution of (1.1). Hence v exists globally.

(2)Without loss of generality, we may assume that $0 \in \Omega$. Let $B_R(0)$ be a ball such that $B_R(0) \subset\subset \Omega$. In the following, we will prove that v blows up in finite time in the ball B_R. If so, v does blow up in the larger domain Ω.

Denote by $\lambda_{B_R} > 0$ and $\varphi_R(r)$ the first eigenvalue and the corresponding eigenfunction of the following eigenvalue problem:

$$-\varphi''(r) - \frac{N-1}{r}\varphi'(r) = \lambda\varphi(r), \quad r \in (0,R); \quad \varphi'(0) = 0 \ \ \varphi(R) = 0. \tag{3.14}$$

It is well known that $\varphi_R(r)$ can be normalized as $\varphi_R(r) > 0$ in B_R and $\varphi_R(0) = \max_{B_R} \varphi_R(r) = 1$. By the scaling property (let $s = r/R$) of eigenvalues and eigenfunctions , we see that $\lambda_{B_R} = R^{-2}\lambda_{B_1}$, $\varphi_R(r) = \varphi_1(r/R) = \varphi_1(s)$, where λ_{B_1} and $\varphi_1(s)$ are the first eigenvalue and the corresponding normalized eigenfunction of the eigenvalue problem in the unit ball $B_1(0)$. Moreover,

$$\max_{B_1} \varphi_1 = \varphi_1(0) = \varphi_R(0) = \max_{B_R} \varphi_R = 1. \tag{3.15}$$

Now, we define $\underline{v}(x,t)$ the functions as

$$\underline{v}(x,t) = [s(t)(\varphi_R(|x|)+1]^{l_*}, \tag{3.16}$$

where $s(t)$ is defined as in (3.8), l_* satisfies $ml_* > 1$ and $k, \delta > 0, \gamma > 1$ to be fixed later. Then, a similar calculation as that of (3.10) yields

$$\underline{v}_t - \Delta\underline{v}^m - a\underline{v}^{p_1} \cdot \|\underline{v}\|_{\alpha_1}^{q_1} \ \le l_* s^{l_*-1}(\phi(|x|)+1)^{l_*}[s'(t) - s^{ml_*-l_*+1}((ac_*/l_*2^{l_*}) - \lambda_{B_R} m2^{(m-1)l_*})]. \tag{3.17}$$

Here we use $p_1 + q_1 = m$, and

$$c_* = [\int_{B_R} (\varphi_R(|x|)+1)^{\alpha_1 l_*} dx]^{q_1/\alpha_1} = R^{Nq_1/\alpha_1}[\int_{B_1} (\varphi_1 |y|)+1)dy]^{p_1/\alpha_1} \ = b_* R^{Nq_1/\alpha_1}, \tag{3.18}$$

is a constant independent of R. And then, in view of $\lambda_{B_R} = R^{-2}\lambda_{B_1}$, we assume that R, i.e., the ball $B_R(0)$, is sufficiently large that

$$\lambda_{B_R} < ac_*/ml_*2^m. \tag{3.19}$$

Hence

$$\sigma_1 = (ac_*/l_*2^{l_*}) - \lambda_{B_R} m2^{(m-1)l_*}. \tag{3.20}$$

Denote $\gamma_1 = ml_* - l_* + 1$, it follows from $m > 1$ that $\gamma_1 > 1$. Since v_0 is a positive and continuous in Ω, we can choose $\delta > 0$ small enough to satisfy

$$v_0 \ge (\delta(\varphi_R |x| + 1))^{l_*}, \tag{3.21}$$

in the ball $B_R(0)$. Finally, choose γ, k to satisfy

$$1 < \gamma < \gamma_1, 0 < k < \sigma_1 \delta^{\gamma_1 - \gamma}. \tag{3.22}$$

Thus, from (3.8), we have

$$s(t) \ge s(0) = \delta > (k/\sigma_1)^{1/(\gamma_1-\gamma)}. \tag{3.23}$$

That is

$$ks^{\gamma}(t) \le \sigma_1 s^{\gamma_1}(t). \tag{3.24}$$

It follows from (3.14)-(3.24) that $\underline{v}$ is a positive weak subsolution of (1.1) in the ball $B_R(0)$, which blows up in finite time since $s(t)$ does. Thus, v does blow up in the larger domain Ω.

4 Blow-up rate and uniform blow-up profile

Throughout this section, we will assume that $p_1 + q_1 \geq m$. To convenience, we firstly introduce a transformation to problem (1.1). Let $u(x,\tau) = v^m(x,t)$, $\tau = mt$, then (1.1) becomes

$$\begin{cases} u_\tau = u^r(\Delta u + au^p \|u\|_\alpha^q), & x \in \Omega, \tau > 0, \\ u(x,\tau) = 0\ , & x \in \partial\Omega, \tau > 0, \\ u(x,0)\ = u_0(x), & x \in \Omega, \end{cases} \tag{4.1}$$

where $0 < r = (m-1)/m < 1, p = p_1/m, \alpha = \alpha_1/m, q = q_1/m, u_0(x) = v_0^m(x)$.

Under this transformation, assumptions (H1)-(H4) become

$(\mathrm{H1})'$ $u_0(x) \in C^{2+\theta}(\Omega) \cap C(\overline{\Omega}), 0 < \theta < 1$;

$(\mathrm{H2})'$ $u_0(x) > 0$ in Ω, and $u_0(x) = 0\, , \partial u_0/\partial\eta < 0$ on $\partial\Omega$;

$(\mathrm{H3})'$ $\Delta u_0 + au_0^p \|u_0\|_\alpha^q \geq 0$ for $x \in \Omega$ and $\Delta u_0 + au_0^p \|u_0\|_\alpha^q = 0$ on $\partial\Omega$;

$(\mathrm{H4})'$ There exists a constant $\varepsilon \geq \max\{\varepsilon_1, ak^{-1} |\Omega|^{q/\alpha}\} > 0$, such that

$$\Delta u_0 + au_0^p \|u_0\|_\alpha^q - \varepsilon u_0^{p+q} \geq 0,$$

where $k = p+q+r-1$, $\varepsilon_1 = (|\Omega|^{q/\alpha} ak/r)((k-p+1)/(q+k))^{(q+k)/k}$.

To show the existence of the classical solution of (4.1), we consider the following regularized problem:

$$\begin{cases} u_{\varsigma\tau} = (u_\varsigma + \varsigma)^r(\Delta u_\varsigma + au_\varsigma^p \|u_\varsigma\|_\alpha^q), & x \in \Omega, \tau > 0, \\ u_\varsigma(x,\tau)\ = 0\ , & x \in \partial\Omega, \tau > 0, \\ u_\varsigma(x,0)\ = u_0(x), & x \in \Omega, \end{cases} \tag{4.2}$$

where $0 < \varsigma < 1$. Denote

$$u(x,\tau) = \lim_{\varsigma \to 0} u_\varsigma(x,\tau), \quad x \in \Omega, \tau > 0. \tag{4.3}$$

By the standard method (See [15,21]), we can show $u(x,\tau)$ defined by (4.3) is a unique classical solution of (4.1), provided that u_0 satisfy the hypotheses $(\mathtt{H1})'$ -$(\mathtt{H3})'$. We thus assume that the solution of (4.1) blows up in finite time T^* and set $U(\tau) = \max_{x\in\Omega} u(x,\tau)$, then $U(\tau)$ is Lipschitz continuous.

Lemma 4.1 *Suppose that $u_0(x)$ satisfy $(H1)'$ -$(H3)'$, then there exists a positive constant c_0 such that*

$$U(\tau) \geq c_0(T^* - \tau)^{-1/k}. \tag{4.4}$$

Proof. From that in (4.1), we see that $U(\tau)$ satisfies

$$U'(\tau) \leq a\,|\Omega|^{q/\alpha}\, U^{r+p+q}(\tau), \quad a.e.\ \tau \in (0, T^*). \tag{4.5}$$

Hence,

$$-(U^{1-(p+q+r)}(\tau))' \leq a\,|\Omega|^{q/\alpha}\,(p+q+r-1). \tag{4.6}$$

In view of $p_1 + q_1 \geq m$, then $p+q+r-1 = (p_1+q_1)/m + r - 1 \geq r > 0$. Integrating (4.6) over (τ, T^*), we can get

$$U(\tau) \geq (a(p+q+r-1)\,|\Omega|^{q/\alpha})^{-1/p+q+r-1}(T^* - \tau)^{-1/p+q+r-1}.$$

Setting

$$k = p+q+r-1, c_0 = (ak\,|\Omega|^{q/\alpha})^{-1/k}. \tag{4.7}$$

Then, we draw the conclusion. ∎

Lemma 4.2 *Suppose that $u_0(x)$ satisfies $(H1)'$ -$(H4)'$ and k is given by (4.7). Then there exists a constant $\varepsilon \geq \varepsilon_1$, such that*

$$u_\tau - \varepsilon u^{k+1} \geq 0, \qquad (x,\tau) \in \Omega \times (0, T^*). \tag{4.8}$$

Proof. Let $J(x,\tau) = u_\tau - \varepsilon u^{k+1}$ for $(x,\tau) \in \Omega \times (0,T^*)$, then by assumption (H3)$'$, we have

$$J(x,0) \geq 0 \quad \text{in } \Omega \quad \text{and} \lim_{x\to\partial\Omega} J(x,\tau) \geq 0. \tag{4.9}$$

A series of computations yields

$$\begin{aligned} & J_\tau - u^r J - (2r\varepsilon u^k + apu^{p+r-1}\|u\|_\alpha^q)J - aqu^{p+r}\|u\|_\alpha^{q-\alpha}\textstyle\int_\Omega u^{\alpha-1}Jdx \\ & = ru^{-1}J^2 + \varepsilon(k+1)ku^{k-1+r}|\nabla u|^2 + r\varepsilon^2 u^{2k+1} + ap\varepsilon u^{p+r+k}\|u\|_\alpha^q \\ & \quad - a\varepsilon(k+1)u^{p+r+k}\|u\|_\alpha^q + aq\varepsilon u^{p+r}\|u\|_\alpha^{q-\alpha}\|u\|_{\alpha+k}^{\alpha+k} \\ & \geq r\varepsilon^2 u^{2k+1} - a\varepsilon(k-p+1)u^{p+r+k}\|u\|_\alpha^q + aq\varepsilon u^{p+r}\|u\|_\alpha^{q-\alpha}\|u\|_{\alpha+k}^{\alpha+k} \\ & = a\varepsilon(k-p+1)u^{p+r}[(r\varepsilon/a(k-p+1))u^{2k+1-p-r} + (q/(k-p+1))\|u\|_\alpha^{q-\alpha}\|u\|_{\alpha+k}^{\alpha+k} - u^k\|u\|_\alpha^q]. \end{aligned} \tag{4.10}$$

By virtue of inequality, we have

$$\|u\|_\alpha^q = \|u\|_\alpha^{q(q-\alpha)/(q+k)}\|u\|_\alpha^{q(\alpha+k)/(q+k)} \leq \|u\|_\alpha^{q(q-\alpha)/(q+k)}\|u\|_{\alpha+k}^{q(\alpha+k)/(q+k)}|\Omega|^{kq/\alpha(q+k)}.$$

Furthermore, Young's inequality implies

$$\begin{aligned} & u^k\|u\|_\alpha^q \leq |\Omega|^{kq/\alpha(q+k)}u^k\|u\|_\alpha^{q(q-\alpha)/(q+k)}\|u\|_{\alpha+k}^{q(\alpha+k)/(q+k)} \\ & \leq |\Omega|^{kq/\alpha(q+k)}\{[k/(q+k)](\vartheta u^k)^{(q+k)/k} + [q/(q+k)](\vartheta\|u\|_\alpha^{q(q-\alpha)/(q+k)}\|u\|_{\alpha+k}^{q(\alpha+k)/(q+k)})^{(q+k)/q}\} \\ & = |\Omega|^{kq/\alpha(q+k)}[(k\vartheta^{(q+k)/k}/(q+k))u^{q+k} + (q\vartheta^{(q+k)/q}/(q+k))\|u\|_\alpha^{q-\alpha}\|u\|_{\alpha+k}^{\alpha+k}], \end{aligned} \tag{4.11}$$

where $\vartheta = ((k-p+1)/(q+k))^{q/(q+k)}|\Omega|^{kq^2/(\alpha(q+k)^2)}$. It follows from (4.10)-(4.11) and (H4)$'$that

$$J_\tau - u^r J - (2r\varepsilon u^k + apu^{p+r-1}\|u\|_\alpha^q)J - aqu^{p+r}\|u\|_\alpha^{q-\alpha}\int_\Omega u^{\alpha-1}Jdx \geq r\varepsilon(\varepsilon-\varepsilon_1)u^{2k+1} \geq 0,$$

where $\varepsilon_1 = (|\Omega|^{q/\alpha}ak/r)((k-p+1)/(q+k))^{(q+k)/k}$.
The comparison principle implies $J \geq 0$. That is, (4.8) holds. ■

From (4.8) ,we get

$$U'(\tau) \geq \varepsilon U^{k+1}(\tau), \tag{4.12}$$

Integrating (4.12) from τ to T^* , we conclude that

$$U(\tau) \leq c_{00}(T^*-\tau)^{-1/k}, \tag{4.13}$$

where $c_{00} = (\varepsilon k)^{-1/k}$ is a positive constant independent of τ. Combining (4.8) with (4.13), we have the following lemma.

Lemma 4.3 *Suppose that $u_0(x)$ satisfy $(H1)'$ -$(H3)'$. If $u(x,\tau)$ is the solution of (4.1) and blows up in finite time T^*, then there exist positive constants c_0, c_{00}, such that*

$$c_0(T^*-\tau)^{-1/k} \leq \max_{x\in\Omega} u(x,\tau) \leq c_{00}(T^*-\tau)^{-1/k}$$

where $k = p+q+r-1$.

According the transform and Lemma 4.3, we can obtain Theorem 1.4.

Lemma 4.4 *Assume $p+r \leq 1$, $u_0(x)$ satisfies $(H1)'$ -$(H3)'$ and $\Delta u_0 \leq 0$ in Ω . $u(x,\tau)$ is the classical solution of the problem (4.1). Then $\Delta u \leq 0$ in any compact subsets of $\Omega \times (0,T^*)$.*

Proof. Let $w(x,\tau) = \Delta u_\varsigma(x,\tau)$. According to (4.2), for all $\tau \in (0,T^*)$, we have

$$\begin{aligned} w_\tau &= r(u_\varsigma+\varsigma)^{r-1}\Delta u_\varsigma(\Delta u_\varsigma + au_\varsigma^p\|u_\varsigma\|_\alpha^q) + r(r-1)(u_\varsigma+\varsigma)^{r-2}|\nabla u_\varsigma|^2(\Delta u_\varsigma + au_\varsigma^p\|u_\varsigma\|_\alpha^q) \\ & \quad + 2r(u_\varsigma+\varsigma)^{r-1}\nabla u_\varsigma(\nabla w + apu_\varsigma^{p-1}\nabla u_\varsigma\|u_\varsigma\|_\alpha^q) \\ & \quad + (u_\varsigma+\varsigma)^r(\Delta w + ap(p-1)u_\varsigma^{p-2}|\nabla u_\varsigma|^2\|u_\varsigma\|_\alpha^q + apu_\varsigma^{p-1}\Delta u_\varsigma\|u_\varsigma\|_\alpha^q), \end{aligned} \tag{4.14}$$

Noticing that $u_\varsigma, u_{\varsigma\tau} \geq 0$ and $p + r \leq 1$, we have

$$\begin{aligned} & w_\tau - (u_\varsigma + \varsigma)^r \Delta w - 2r(u_\varsigma + \varsigma)^{r-1} \nabla u_\varsigma \nabla w \\ & \quad \leq r(r-1)(u_\varsigma + \varsigma)^{r-2} \left|\nabla u_\varsigma\right|^2 w + r(u_\varsigma + \varsigma)^{-1} u_{\varsigma\tau} w + ap(u_\varsigma + \varsigma)^r u_\varsigma^{p-1} \left\|u_\varsigma\right\|_\alpha^q w \\ & \quad\ + a[r(r-1) + 2rp + p(p-1)](u_\varsigma + \varsigma)^r u_\varsigma^{p-2} \left|\nabla u_\varsigma\right|^2 \left\|u_\varsigma\right\|_\alpha^q \\ & \quad \leq (r(r-1)(u_\varsigma + \varsigma)^{r-2} \left|\nabla u_\varsigma\right|^2 + r(u_\varsigma + \varsigma)^{-1} u_{\varsigma\tau} + ap(u_\varsigma + \varsigma)^r u_\varsigma^{p-1} \left\|u_\varsigma\right\|_\alpha^q) w. \end{aligned} \tag{4.15}$$

Moreover,

$$w|_{\partial\Omega} = -a u_\varsigma^p \left\|u_\varsigma\right\|_\alpha^q < 0, \qquad w(x, 0) = \Delta u_0 \leq 0.$$

By maximum principle, it follows $w(x,\tau) \leq 0$, which means $\Delta u_\varsigma \leq 0$ in $\overline{\Omega_T}$.
Therefore, $\Delta u \leq 0$ in any compact subsets of $\Omega \times (0, T^*)$. ■

Denote

$$g(\tau) = a \left\|u\right\|_\alpha^q, \qquad G(\tau) = \int_0^\tau g(s) ds.$$

Lemma 4.5 *Under the conditions of Lemma 4.3, it holds that*

$$\lim_{\tau \to T^*} g(\tau) = \infty, \quad \lim_{\tau \to T^*} G(\tau) = \infty\ . \tag{4.16}$$

Proof. From Lemma 4.3, we have

$$u_\tau \leq u^{r+p} g(\tau), \qquad a.e.\ \tau \in [0, T^*). \tag{4.17}$$

Integrating (4.17) over $(0, \tau)$, we obtain

$$u^{1-p-r}(x,\tau) \leq (1-p-r) G(\tau) + u_0^{1-p-r}(x), \qquad p + r < 1, \tag{4.18}$$

$$\ln u(x,\tau) \leq G(\tau) + \ln u_0(x), \qquad p + r = 1. \tag{4.19}$$

In view of $\lim_{\tau \to T^*} u(x,\tau) = \infty$, $\lim_{\tau \to T^*} G(\tau) = \infty$. Noting that $u_\tau \geq 0$ by the assumption of the initial function, we then see that $g(\tau)$ is monotone nondecreasing. Therefore $\lim_{\tau \to T^*} g(\tau) = \infty$. ■

Lemma 4.6 *Under the conditions of Lemma 4.3. If u blows up in finite time T^*, then we have*

$$(1) \lim_{\tau \to T^*} \frac{u^{1-p-r}(x,\tau)}{(1-p-r)G(\tau)} = \lim_{\tau \to T^*} \frac{\left\|u(\cdot,\tau)\right\|_\infty^{1-p-r}}{(1-p-r)G(\tau)} = 1, \quad p+r < 1, \tag{4.20}$$

$$(2) \lim_{\tau \to T^*} \frac{\ln u(x,\tau)}{G(\tau)} = \lim_{\tau \to T^*} \frac{\left\|\ln u(\cdot,\tau)\right\|_\infty}{G(\tau)} = 1, \qquad p+r = 1. \tag{4.21}$$

uniformly on any compact subsets of Ω .

Proof. (1) For $(x,\tau) \in \Omega \times (0,T^*)$, we define

$$z_1(x,\tau) = G(\tau) - u^{1-r-p}/(1-r-p), \qquad \gamma_1(\tau) = \int_\Omega z_1(y,\tau)\varphi(y) dy,$$

where φ satisfies (3.6) and $\int_\Omega \varphi(x) dx = 1$. A direct computation shows

$$\begin{aligned} \gamma_1'(\tau) &= \textstyle\int_\Omega (g(\tau) - u^{-r-p}(y,\tau) u_\tau(y,\tau)) \varphi(y) dy = -\int_\Omega u^{-p}(y,\tau) \Delta u(y,\tau) \varphi(y) dy \\ &\leq -1/(1-p) \textstyle\int_\Omega \Delta u^{1-p}(y,\tau) \varphi(y) dy = \lambda/(1-p) \int_\Omega u^{1-p}(y,\tau) \varphi(y) dy \\ &\leq \lambda (1-r-p)^{(1-p)/(1-p-r)}/(1-p) \textstyle\int_\Omega (G(\tau) - z_1(y,\tau))^{(1-p)/(1-p-r)} \varphi(y) dy \\ &\leq C_1 (G^{(1-p)/(1-p-r)}(\tau) + \textstyle\int_\Omega (z_1^-(y,\tau))^{(1-p)/(1-p-r)} \varphi(y) dy), \end{aligned}$$

where $z_1^- = \max\{-z_1, 0\}$. From (4.18), we know that

$$\inf_\Omega z_1(y,\tau) \geq -C', \tag{4.22}$$

which means $z_1^- \leq C'$. Then

$$\gamma_1'(\tau) \leq C_2 G^{(1-p)/(1-p-r)}(\tau) + C_3. \tag{4.23}$$

Integrate (4.23) from 0 to τ,

$$\gamma_1(\tau) \leq C_4(1+\int_0^\tau G^{(1-p)/(1-p-r)}(s)ds). \tag{4.24}$$

Thus (4.22) and (4.24) implies

$$\int_\Omega |z_1(y,\tau)|\,\varphi(y)dy \leq C_5(1+\int_0^\tau G^{(1-p)/(1-p-r)}(s)ds).$$

Define $K_\rho=\{y\in\Omega: dist(y,\partial\Omega)\geq\rho\}$. Since $-\Delta z_1\leq 0$ in $\Omega\times(0,T^*)$. Using Lemma4.5 in [14], we obtain

$$\sup_{K_\rho} z_1(x,\tau) \leq \frac{C_6}{\rho^{N+1}}(1+\int_0^\tau G^{(1-p)/(1-p-r)}(s)ds). \tag{4.25}$$

It follows from (4.25) and (4.18) that

$$-\frac{C'}{G(\tau)} \leq 1-\frac{u^{1-r-p}(x,\tau)}{(1-r-p)G(\tau)} \leq \frac{C_6}{\rho^{N+1}}\frac{1+\int_0^\tau G^{(1-p)/(1-p-r)}(s)ds}{G(\tau)}, \tag{4.26}$$

for any $x\in K_\rho$ and $\tau\in(0,T^*)$. We know from Theorem 4.1 that

$$\int_0^\tau G^{(1-p)/(1-p-r)}(s)ds \leq C_7\int_0^\tau (T^*-s)^{-(1-p)/k}ds. \tag{4.27}$$

In view of (4.18), it follows that

$$G(\tau)\geq C_8u^{1-r-p}\geq C_9(T^*-\tau)^{-(1-r-p)/k}. \tag{4.28}$$

From (4.26)-(4.28), we get

$$-\frac{C'}{G(\tau)} \leq 1-\frac{u^{1-r-p}(x,\tau)}{(1-r-p)G(\tau)} \leq \frac{C_{10}}{\rho^{N+1}}\frac{1+\int_0^\tau (T^*-s)^{-(1-p)/k}ds}{(T^*-\tau)^{-(1-r-p)/k}}. \tag{4.29}$$

It is obvious that

$$\lim_{\tau\to T^*}\frac{\int_0^\tau (T^*-s)^{-(1-p)/k}ds}{(T^*-\tau)^{-(1-r-p)/k}}=0.$$

Thus

$$\lim_{\tau\to T^*}\frac{u^{1-p-r_1}(x,\tau)}{(1-p-r)G(\tau)}=\lim_{\tau\to T^*}\frac{\|u(\cdot,\tau)\|_\infty^{1-p-r}}{(1-p-r)G(\tau)}=1.$$

(2) Similarly,we define

$$z_2(x,\tau)=G(\tau)-\ln u(x,\tau),\quad \gamma_2(\tau)=\int_\Omega z_2(y,\tau)\varphi(y)dy,\quad (x,\tau)\in\Omega\times(0,T^*),$$

where φ satisfies (3.6) and $\int_\Omega\varphi(x)dx=1$. A direct computation shows

$$\begin{aligned}\gamma_2'(\tau) &= \textstyle\int_\Omega (g(\tau)-u^{-1}(y,\tau)u_\tau(y,\tau))\varphi(y)dy = -\int_\Omega u^{r-1}(y,\tau)\Delta u(y,\tau)\varphi(y)dy\\ &\leq -1/r\textstyle\int_\Omega \Delta u^r(y,\tau)\varphi(y)dy=\lambda/r\int_\Omega u^r(y,\tau)\varphi(y)dy\\ &= \lambda/r\textstyle\int_\Omega \exp\{r[G(\tau)-z_2(y,\tau)]\}\varphi(y)dy,\end{aligned}$$

From (4.19), we know that

$$\inf_\Omega z_2(y,\tau)\geq -C''. \tag{4.30}$$

Then

$$\gamma_2'(\tau)\leq C_{11}\exp[rG(\tau)]. \tag{4.31}$$

Integrate (4.31) from 0 to τ yields

$$\gamma_2(\tau)\leq C_{12}(1+\int_0^\tau \exp[rG(s)]ds). \tag{4.32}$$

Thus (4.32) and (4.30) implies

$$\int_\Omega |z_2(y,\tau)|\,\varphi(y)dy \le C_{13}(1+\int_0^\tau \exp[rG(s)]ds).$$

Define $K_\zeta = \{y \in \Omega : dist(y, \partial\Omega) \ge \zeta\}$. Since $-\Delta z_2 \le 0$ in $\Omega \times (0, T^*)$. Using Lemma4.5 in [14], we obtain

$$\sup_{K_\zeta} z_2(x,\tau) \le \frac{C_{14}}{\zeta^{N+1}}(1+\int_0^\tau \exp[rG(s)]ds). \tag{4.33}$$

It follows from (4.33) and (4.30) that

$$-\frac{C''}{G(\tau)} \le 1-\frac{\ln u(x,\tau)}{G(\tau)} \le \frac{C_{15}}{\zeta^{N+1}}\frac{1+\int_0^\tau \exp[rG(s)]ds}{G(\tau)}. \tag{4.34}$$

By Theorem 4.1, we have

$$G(\tau) = a\int_0^\tau \|u\|_\alpha^q ds \le (a\,|\Omega|^{q/\alpha}\big/(\varepsilon k))\int_0^\tau (T^*-s)^{-1}ds = \ln T^* - \ln(T^*-\tau). \tag{4.35}$$

On the other hand, we know form (4.19) and Theorem 4.1 that

$$G(\tau) \ge c\,|\ln(T^*-\tau)|\,. \tag{4.36}$$

From (4.34)-(4.36), we get

$$-\frac{C''}{G(\tau)} \le 1-\frac{\ln u(x,\tau)}{G(\tau)} \le \frac{C_{16}}{\zeta^{N+1}}\frac{1+\int_0^\tau \exp[rG(s)]ds}{G(\tau)} \le \frac{C_{17}}{\zeta^{N+1}}\frac{1+(T^*)^r\int_0^\tau (T^*-\tau)^{-r}ds}{|\ln(T^*-\tau)|}.$$

It is easy to derive

$$\lim_{\tau\to T^*}\frac{1+(T^*)^r\int_0^\tau (T^*-\tau)^{-r}ds}{|\ln(T^*-\tau)|} = 0.$$

Thus

$$\lim_{\tau\to T^*}\frac{\ln u(x,\tau)}{G(\tau)} = \lim_{\tau\to T^*}\frac{\|\ln u(\cdot,\tau)\|_\infty}{G(\tau)} = 1.$$

This completes the proof. ■

Lemma 4.7 *Assume that the conditions of Lemma 4.5 hold, then the following limits converge uniformly on any compact subset of* Ω.
(1) If $p+r<1$, $\lim\limits_{\tau\to T^*} u(x,\tau)(T^*-\tau)^{1/k} = (ak\,|\Omega|^{q/\alpha})^{-1/k}$;
(2) If $p+r=1$, $\lim\limits_{\tau\to T^*} |\ln(T^*-\tau)|^{-1}\ln u(x,\tau) = 1/q$.

Proof. Case 1: $p+r<1$. Form (4.20), we have

$$u(x,\tau) \sim ((1-p-r)G(\tau))^{1/(1-p-r)}, \quad as \quad \tau\to T^*,$$

where the notation $u \sim v$ means $\lim\limits_{\tau\to T^*} u(\tau)/v(\tau) = 1$.
Furthermore,

$$G'(\tau) = g(\tau) = a\,\|u\|_\alpha^q \sim a\,|\Omega|^{q/\alpha}\,((1-p-r)G(\tau))^{q/(1-p-r)}, \quad as \quad \tau\to T^*. \tag{4.37}$$

Integrating (4.37) over (τ, T^*) yields

$$G(\tau) \sim (1-p-r)^{-1}(ak\,|\Omega|^{q/\alpha}\,(T^*-\tau))^{-(1-p-r)/k}, \qquad as \quad \tau\to T^*. \tag{4.38}$$

So, we can get our conclusion by using (4.20) and (4.38).

Case2: $p+r=1$. In this case , for any given σ: $0<\sigma\ll 1$. By (4.21), there exists $0<\tau_0<T^*$ such that

$$(1-\sigma)G(\tau) \le \ln u(x,\tau) \le (1+\sigma)G(\tau), \quad x\in\Omega, \tau\in[\tau_0, T^*).$$

Therefore

$$a\,|\Omega|^{q/\alpha}\exp[q(1-\sigma)G(\tau)] \le G'(\tau) = a\,\|u\|_{\alpha}^{q} \le a\,|\Omega|^{q/\alpha}\exp[q(1+\sigma)G(\tau)], \quad \tau_0 \le \tau \le T^*. \tag{4.39}$$

In view of the right-hand side of the (4.39), we have

$$\exp[-q(1+\sigma)G(\tau)]dG(\tau) \le a\,|\Omega|^{q/\alpha}\,d\tau, \quad \tau_0 \le \tau \le T^*.$$

Integrating the above inequality from τ to T^* yields that

$$\exp[-q(1+\sigma)G(\tau)] \le aq(1+\sigma)\,|\Omega|^{q/\alpha}\,(T^*-\tau), \quad \tau_0 \le \tau \le T^*.$$

Namely,

$$G(\tau) \ge (-1/q(1+\sigma))\ln[aq(1+\sigma)\,|\Omega|^{q/\alpha}\,(T^*-\tau)], \quad \tau_0 \le \tau \le T^*. \tag{4.40}$$

Similar arguments to the left hand of (4.39) yield that

$$G(\tau) \le (-1/q(1-\sigma))\ln[aq(1-\sigma)\,|\Omega|^{q/\alpha}\,(T^*-\tau)], \quad \tau_0 \le \tau \le T^*. \tag{4.41}$$

Consequently, (4.40) and (4.41) guarantee that for $\tau_0 \le \tau \le T^*$,

$$(-1/q(1+\sigma))\ln[aq(1+\sigma)\,|\Omega|^{q/\alpha}\,(T^*-\tau)] \le G(\tau) \le (-1/q(1-\sigma))\ln[aq(1-\sigma)\,|\Omega|^{q/\alpha}\,(T^*-\tau)]. \tag{4.42}$$

Letting $\sigma \to 0$, we have

$$\lim_{\tau\to T^*} G(\tau)\,|\ln(T^*-\tau)|^{-1} = 1/q,$$

because of $\lim_{\tau\to T^*} G(\tau) = \infty$. Due to $\ln u(x,\tau) \sim G(\tau)$ uniformly on any compact subset of Ω, we complete our proof. ■

Combining Lemma 4.7 with the transform about $v(x,t)$, it follows that Theorem 1.5 holds.

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