Global Solutions and Blow-up Profiles for a Nonlinear Degenerate Parabolic Equation with Nonlocal Source

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Abstract: This paper deals with a degenerate parabolic equation \( v_t = \Delta v^m + av^{p_1} \|v\|^{q_1}_{\alpha_1} \) subject to homogeneous Dirichlet condition. The local existence of a nonnegative weak solution is given. The blow-up and global existence conditions of nonnegative solutions are obtained. Moreover, we establish the precise blow-up rate estimates for all the blow-up solutions.

Keywords: nonlinear degenerate parabolic equation; global existence; blow up; blow-up rate

1 Introduction

In this paper, we investigate the following parabolic equation with local source involved a product with nonlocal term:

\[
\begin{aligned}
&v_t = \Delta v^m + av^{p_1} \|v\|^{q_1}_{\alpha_1}, \quad x \in \Omega, t > 0, \\
v(x, t) = 0, \quad x \in \partial \Omega, t > 0, \\
v(x, 0) = v_0(x), \quad x \in \Omega,
\end{aligned}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N (N \geq 1) \) with smooth boundary \( \partial \Omega \), \( a > 0 \), \( m > 1 \), \( \alpha_1 \geq 1 \), \( p_1 \geq 0 \) and \( p_1 + q_1 \neq 0 \), \( \|v\|_{\alpha_1}^\alpha = \int_{\Omega} |v|^\alpha \, dx \).

Such a problem arise in the study of the flow of a fluid through a porous medium with an integral source (see [1-2] and references therein) and in the study of population dynamics (see[3]).

Porous medium equations with or without a non-local source have been studied by many authors (see [4-9] for forced reaction source and [10] for the non-local sources). The typical models are the following classical type: (i) Non-local source: \( v_t = \Delta v^m + \int_{\Omega} v^q \, dx \); (ii) Local source: \( v_t = \Delta v^m + av^{p_1} \|v\|^{q_1}_{\alpha_1} \), with the same initial-boundary conditions as problem (1.1). Although, both the problems share the same blow-up criteria and the blow-up rate, there do exist some essential differences from the two problems.

As for nonlocal problems, in [11], the authors established the critical Fujita exponent for the Cauchy problem:

\[
\begin{aligned}
&v_t = \Delta v^m + v^{r+1} \left( \int_{\mathbb{R}^N} K(x)v^q(x, \tau) \, dx \right)^{(p-1)/q}, \quad x \in \mathbb{R}^N, t > 0, \\
v(x, 0) = v_0(x),
\end{aligned}
\]

where parameters \( m, p > 1, q > 0, r \geq 0 \), the initial data \( v_0(x) \) is a bounded nonnegative function, and the kernel function \( K(x) \) is nonnegative and measurable.

In the paper [12], the authors considered the following one-dimensional problem:

\[
\begin{aligned}
&v_t = ((v^m)_x + \varepsilon v^n)_x + av \|v\|^{\alpha-1}_q, \quad x \in (0, 1), \\
v(0, t) = v(1, t) = 0, \quad t > 0, \\
v(x, 0) = v_0(x), \quad x \in [0, 1],
\end{aligned}
\]

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They proved that the solution blows up in finite time provided that \( p > \max\{1, m, n\} \) and the initial data \( u_0 \) is sufficiently large.

In [13], Deng, Duan and Xie studied the following one-dimensional problem:

\[
\begin{aligned}
v_t - (v^m)_{xx} &= a \int_{-l}^{l} v^{q} dx, & x \in (-l, l), t > 0, \\
v(-l, t) &= v(l, t) = 0, & t > 0, \\
v(x, 0) &= v_0(x), & x \in [-l, l],
\end{aligned}
\tag{1.4}
\]

with \( l, a > 0 \) and \( q > m > 1 \). They proved that if the solution \( v(x) \) of (1.4) blows up in finite time, then the blow-up set is the whole domain \([-l, l]\), and the following estimate holds (here \( T' \) denotes the blow-up time):

\[
C_1(T' - t)^{-1/(q-1)} \leq \max_{x \in [-l, l]} v(x, t) \leq C_2(T' - t)^{-1/(q-1)}.
\]

In [14], Souplet considered the following problem:

\[
\begin{aligned}
v_t - \Delta v &= ||v||_p^p, & x \in \Omega, t > 0, \\
v(x, t) &= 0, & x \in \partial \Omega, t > 0, \\
v(x, 0) &= v_0(x), & x \in \Omega,
\end{aligned}
\tag{1.5}
\]

If \( 1 < r < \infty, p > 1 \), the author showed that \( v(x) \) blows up in finite time \( T'' \) and got the blow-up profile:

\[
lim_{t \to T''} (T'' - t)^{1/(p-1)} ||v(x, t)||_\infty = [(p - 1)|\Omega|^{p/r}]^{-1/(p-1)}.
\]

When \( a_1 = q_1 \), the equation (1.1) was introduced and studied in [15]. The authors showed that the solution either exists globally or blows up in finite time. Furthermore, they also yielded the blow-up rate. But the accurate estimations of the blow-up rate and the blow-up set were not discussed in that paper.

As for local problems, in the paper [16], Chen and Xie investigated the following problem:

\[
\begin{aligned}
v_t - \Delta v^m &= av^{p+q}(x_0, t), & x \in \Omega, t > 0, \\
v(x, t) &= 0, & x \in \partial \Omega, t > 0, \\
v(x, 0) &= v_0(x), & x \in \Omega,
\end{aligned}
\tag{1.6}
\]

They showed that the solution exists globally or blows up in finite time under some conditions. In [17], Du et. al. made a further analysis for the problem (1.6) and obtained the blow-up set and the blow-up rate of the solution.

Motivated by the above works, in this paper, we discuss the local existence of nonnegative solution and profile of the blow-up solution to (1.1), and extend their results to the general equation (1.1). Now we state our results as follows.

**Theorem 1.1** If \( p_1 + q_1 < m \), then the nonnegative solution of (1.1) exists globally.

**Theorem 1.2** If \( p_1 + q_1 > m \), then the nonnegative solution of (1.1) blows up in finite time for sufficiently large initial values and exists globally for sufficiently small initial values.

**Theorem 1.3** Assume that \( p_1 + q_1 = m \).

1. If the domain is sufficiently small, then the nonnegative solution of (1.1) is global.
2. If the domain contains a sufficiently large ball, and \( v_0 \) is positive and continuous in \( \Omega \), then the nonnegative solution of (1.1) blows up in finite time.

In order to describe the blow-up properties of solutions, we need the following assumptions on the initial data \( v_0 \).

(H1) \( v_0(x) \in C^{2+\delta}(\Omega) \cap C(\bar{\Omega}), 0 < \delta < 1 \);

(H2) \( v_0(x) > 0 \) in \( \Omega \), \( v_0(x) = 0 \) on \( \partial \Omega \), where \( \eta \) is the unit outward normal vector on \( \Omega \);

(H3) \( \Delta v_0^m + av_0^{p+q} ||v_0||_{\alpha_0}^q \geq 0, x \in \Omega \) and \( \Delta v_0 + av_0^{p+q} ||v_0||_{\alpha_0}^q \geq 0 \) on \( \partial \Omega \);

(H4) There exists a constant \( \varepsilon \geq \max\{\varepsilon_0, a |\Omega|^{m/(p_1 + q_1 - 1)}\} \), such that \( \Delta v_0^m + av_0^{p+q} ||v_0||_{\alpha_0}^q - \varepsilon v_0^{p_1+q_1} \geq 0 \),

where \( \varepsilon_0 = (|\Omega|^{m/(p_1 + q_1 - 1)}(p_1 + q_1 - 1)/(p_1 + 2q_1 - 1))(p_1/(p_1 + 2q_1 - 1) + 1/(p_1 + q_1 - 1)) \).

**Theorem 1.4** Suppose that \( p_1 + q_1 \geq m \) and \( v_0(x) \) satisfy (H1)-(H4). If \( v(x, t) \) is the classical solution of (1.1) and blows up in finite time \( T_* \), then there exist positive constants \( C_0, C_{00} \), such that

\[
C_0(T_* - t)^{-1/(p_1 + q_1 - 1)} \leq \max_{x \in \Omega} v(x, t) \leq C_{00}(T_* - t)^{-1/(p_1 + q_1 - 1)}.
\]
Theorem 1.5 Assume that $p_1 \leq 1$ and (H1)-(H4) hold. If $v(x, t)$ is the classical solution of (1.1) and blows up in finite time $T_*$, then the following limits converge uniformly on any compact subset of $\Omega$:

1. If $p_1 < 1$, \( \lim_{t \to T_*} v(x, t)(T_* - t)^{1/(p_1 + q_1 - 1)} = (a(p_1 + q_1 - 1) |\Omega|^{q_1/\alpha_1})^{-1/(p_1 + q_1 - 1)}; \)

2. If $p_1 = 1$, \( \lim_{t \to T_*} \ln(T_* - t)^{-1} \ln v(x, t) = 1/q_1. \)

This paper is organized as follows. In Section 2, we establish the local existence and give a comparison principle. In Section 3, we concern the global existence and the blow-up phenomenon. Results relating to the uniform blow-up profile are presented in the last section.

2 Local existence and comparison principle

Let $Q_T = \Omega \times (0, T)$, $S_T = \partial \Omega \times (0, T)$ for $0 < T < \infty$. As it is now well known that degenerate equation need not posses classical solutions, we begin by giving a precise definition of a weak solution of (1.1).

Definition 2.1 A vector function $v(x, t)$ defined on $Q_T$, for some $T > 0$, is called a subsolution (or supersolution) of (1.1) on $Q_T$, if all the following hold:

1. $v(x, t) \in L^\infty(Q_T);$

2. $v(x, t) \geq 0$ for $(x, t) \in S_T$, and $v(x, 0) \leq (\geq)v_0(x)$ for almost all $x \in \Omega$;

3. \[ \int_\Omega (v(x, t)\psi(x, t) - v_0(x)\psi(x, 0))dx \leq (\geq) \int_0^t \int_\Omega (v\Delta \psi + av\psi^\mu)dxds, \] for every $t \in [0, T]$ and any $\psi$ belong to the class of test functions,

$$\Psi \equiv \{ \psi \in C(Q_T); \Delta \psi \in C(Q_T) \cap L^2(Q_T); \psi \geq 0, \psi(x, t)|_{x\in \partial \Omega} = 0 \}. $$

A weak solution of (1.1) is a vector function which is both a subsolution and a supersolution of (1.1). For every $T < \infty$, if $v(x, t)$ is a solution of (1.1), we say $v(x, t)$ is global.

Clearly, every nonnegative classical(sub-,super-)solution of (1.1) is a weak (sub-,super-)solution of (1.1) in the sense of Definition 2.1. To proving the local existence of a solution to (1.1), we first give a comparison principle without proof.

Lemma 2.1 Assume that $w(x, t) \in C^{2,1}(Q_T) \cap C(\overline{Q_T})$ and satisfy

$$\begin{align*}
\left\{ \begin{array}{ll}
\Delta w - d(x, t)\Delta w & \geq c_1(x, t)w + c_2(x, t) \int_\Omega c_3(x, t)w(x, t)dx, & (x, t) \in Q_T, \\
w(x, t) & \geq 0, & (x, t) \in S_T, \\
w(x, 0) & \geq 0, & x \in \Omega,
\end{array} \right.
\end{align*}$$

where $c_i(i = 1, 2, 3)$ are bounded functions and $c_2, c_3, d \geq 0$ in $Q_T$. Then $w(x, t) \geq 0$ on $\overline{Q_T}$.

In proving local existence for degenerate parabolic equation, there are three different approaches (see [4, p.113]). We will modify the boundary conditions rather than the differential equation. Introduce, for $k = 1, 2, \cdots$, the following regularized equations:

$$\begin{align*}
\left\{ \begin{array}{ll}
v_{kt} & = \Delta f_k(v_k) + a(g(v_k))^{p_1} ||g(v_k)||_{\alpha_1}^\beta, & x \in Q_T, \\
v_k(x, t) & = 1/k, & (x, t) \in S_T, \\
v_k(x, 0) & = v_0(x) + 1/k, & x \in \Omega,
\end{array} \right.
\end{align*}$$

where

$$f_k(v_k) = \left\{ \begin{array}{ll}
v_k^m, & v_k \geq 1/k, \\
(1/k)^m, & v_k < 1/k,
\end{array} \right. \quad g_k(v_k) = \left\{ \begin{array}{ll}
v_k, & v_k \geq 1/k, \\
1/k, & v_k < 1/k.
\end{array} \right. $$

Let $v_{0_k}$ are smooth approximation of $v_0(x)$ with $\text{supp}v_{0_k} \subset \Omega$ in (2.3). By a similar discussion as that of Theorems A.1-A.4 in [18], we can show that (2.3) has a unique classical solution $v_k \in C(\overline{\Omega} \times [0, T_k]) \cap C^{2,1}(\Omega \times (0, T_k))$ for $0 < T_k < \infty$, where $T_k$ is the maximal existence time. By a direct computation and the classical maximum principle, we have $v_k \geq 1/k$. Hence, $v_k$ satisfies

$$\begin{align*}
v_{kt} = \Delta (v_k)^m + a(v_k)^{p_1} ||v_k||_{\alpha_1}^\beta, & \quad (x, t) \in Q_{T_k},
\end{align*}$$
with the corresponding initial and boundary conditions. Clearly, passing to the limit \( i \to \infty \), it follows that \( v_k(x, t) = \lim_{i \to \infty} v^i_k(x, t) \), and \( v_k \) is a weak solution of

\[
v_{kt} = \Delta (v_k)^m + av_k^{p_1} \cdot \|v_k\|^{q_1}_{\alpha_1}, \quad (x, t) \in Q_{T(k)},
\]

(2.5)

with the corresponding initial and boundary conditions on \((0, T(k))\), where \( T(k) = \lim_{i \to \infty} T_i(k) \) is the maximal existence time. Here a weak solution of (2.5) is defined in a manner similar to that for problem (1.1), only the integral equalities for \( v \), (2.1) may be replaced with

\[
\int_{\Omega} (v(x, t)\psi(x, t) - (v_0(x) + 1/k)\psi(x, 0))dx \leq \left( \geq \right) \int_0^t \int_{\Omega} (v\psi' + v^m \Delta \psi + av^{p_1} \cdot \|v\|^{q_1}_{\alpha_1})dxds + \frac{1}{k} \int_0^t \int_{\partial \Omega} (\partial \psi/\partial n)dxds.
\]

(2.6)

Since \( v_k \geq 1/k \), applying Lemma 2.1, we have the following lemma.

**Lemma 2.2** Assume that \( w(x, t) \in C(\Omega \times [0, T_i(k)]) \cap C^{2,1}(\Omega \times (0, T_i(k))) \) for \( 0 < T_i(k) < \infty \) is a sub- (or super-) solution of (2.4). Then \( w \leq (\geq) v_k \) on \( \Omega \times [0, T_i(k)] \).

**Lemma 2.3** If \( k_1 > k_2 \), then \( v_{k_1}(x, t) \leq v_{k_2}(x, t) \) on \( \Omega \times [0, T_i(k_2)] \) and \( T_i(k_1) \geq T_i(k_2) \).

Then, passing to the limit \( i \to \infty \), it happens that \( v_{k_1} \leq v_{k_2} \) and \( T(k_1) \geq T(k_2) \) if \( k_1 > k_2 \). Therefore, the limit \( T_* = \lim_{k \to \infty} T(k) \) exists and, as well, the point-wise limit \( v(x, t) = \lim_{k \to \infty} v_k(x, t) \) exists for any \((x, t) \in \Omega \times [0, T_*] \)
Furthermore, as the convergence of the sequence is monotone, passage to the limit \( k \to \infty \) in the identities (2.6), the following theorem is thus established.

**Theorem 2.1** (Local existence and continuation). Given \( v_0 \geq 0, v_0 \in L^\infty(\Omega) \), there is some \( T_* = T_*(v_0) > 0 \) such that there exists a nonnegative weak solution \( v(x, t) \) of (1.1) for each \( T < T_* \). Moreover, either \( T_* = \infty \) or \( \lim_{t \to T_*} \sup \|v(x, t)\| = \infty \).

**Lemma 2.4** (Comparison principle). Let \( \underline{v} \) and \( \overline{v} \) be a nonnegative subsolution and a nonnegative supersolution of (1.1), respectively. If \( \underline{v}_0 \leq \overline{v}_0 \) and either

\[
\int_{\Omega} \underline{v}^{\alpha_1}dx \geq 0, \quad \underline{v} \leq 0,
\]

(2.7)

or

\[
\int_{\Omega} \overline{v}^{\alpha_1}dx \geq 0, \quad \overline{v} \leq 0,
\]

(2.8)

hold. Then \( \underline{v} \leq \overline{v} \) on \( Q_T \).

**Proof.** The technique for proving comparison principle for degenerate parabolic equations is quite standard. For example, see [4, 19, 20]. Here we shall sketch the argument for the convenience of the reader.

Subtracting the first integral inequalities of (2.1) for \( \underline{v} \) and \( \overline{v} \), yields

\[
\int_{\Omega} [\underline{v}(x, t) - \overline{v}(x, t)]\psi(x, t)dx \leq \int_{\Omega} [\underline{w}_0(x) - \overline{w}_0(x)]\psi(x, 0)dx + \int_0^t \int_{\Omega} [\underline{w}(x) - \overline{w}(x)]\psi(x, t)dx + \Phi(x, s)\Delta \psi + aH(x, s)\cdot \|\psi\|^{q_1}_{\alpha_1} \psi dxds
\]

\[
+ a\int_0^t \int_{\Omega} \overline{w}^{p_1}\psi D(s)G(x, s)\cdot \|\psi\|^{q_1}_{\alpha_1} \psi dxds,
\]

where

\[
\Phi(x, s) = \int_0^1 m(\theta \overline{\sigma} + (1 - \theta)\underline{\sigma})^{m-1}d\theta,
\]

\[
H(x, s) = \int_0^{p_1}(\theta \overline{\omega} + (1 - \theta)\underline{\omega})^{p_1-1}d\theta,
\]

\[
D(s) = \int_0^1 (q_1/\alpha_1)\theta \int_{\Omega} \overline{w}^{q_1}\psi dx + (1 - \theta) \int_{\Omega} \underline{w}^{q_1}\psi dx]^{q_1/\alpha_1} \psi dxds
\]

\[
+ \alpha_1(\theta \overline{\omega} + (1 - \theta)\underline{\omega})^{p_1-1}d\theta,
\]

\[
G(x, s) = \int_0^{q_1/\alpha_1} \alpha_1(\theta \overline{\omega} + (1 - \theta)\underline{\omega})^{q_1/\alpha_1-1}d\theta,
\]

Since \( \underline{v} \) and \( \overline{v} \) are bounded \( Q_T \), it follows from \( m > 1, \alpha_1 \geq 1 \) that \( \Phi(x, s), G(x, s) \) are bounded nonnegative functions. Similarly, \( H(x, s) \), \( D(s) \) are bounded if \( p_1, q_1/\alpha_1 \geq 1 \). Now, if \( p_1, q_1/\alpha_1 < 1 \), we have \( H(x, s) \leq \delta^{q_1/\alpha_1-1}_1 \), \( D(s) \leq \delta^{q_1/\alpha_1-1}_1 \) by the assumptions (2.7) or (2.8). Thus, we can choose the appropriate test function \( \psi \) as in [4, p.118-123] to obtain

\[
\int_{\Omega} [\underline{w}(x, t) - \overline{w}(x, t)]_+ dx \leq \|\psi\|_{\infty} \int_{\Omega} [\underline{w}_0(x) - \overline{w}_0(x)]_+ dx + c_1 \int_0^t \int_{\Omega} [\underline{w}(x) - \overline{w}(x)]_+ dxds
\]

\[
+ a\delta^{q_1/\alpha_1}_{\infty} \|\psi\|_{\infty} \int_0^t \int_{\Omega} G(x, s)[\underline{w}(x) - \overline{w}(x)]_+ dxds,
\]

(2.9)

where \( w_+ = \max\{w, 0\} \) and \( c_1 > 0 \) is bounded constant. Now, (2.9) combined with the Gronwall’s lemma show that \( \underline{v} \leq \overline{v} \), since \( \underline{w}_0 \leq \overline{w}_0 \).
3 Global existence and finite time blow-up

In this section, we use the super- and subsolution methods to give the proof of Theorems 1.1-1.3. According to Lemma 2.1, we only need to construct positive weak supersolutions or subsolutions bounded for any \( T > 0 \).

**Proof of Theorem 1.1.** Firstly, we construct supersolutions which are bounded for any \( T > 0 \). Let \( \phi(x) \) be the solution of the following elliptic problem:

\[-\Delta \phi(x) = 1, \quad x \in \Omega; \quad \phi(x) = 0, \quad x \in \partial \Omega. \tag{3.1}\]

Letting

\[ M = \max_{x \in \Omega} \phi(x) \quad \text{and} \quad \tau = (K(\phi(x) + 1))^{1/m}, \tag{3.2}\]

where \( K > 0 \) will be fixed later. Clearly, \( \tau \) is bounded for any \( t > 0 \) and \( \tau \geq K^{1/m} \). Thus, we have

\[ -\Delta \tau^{m} - a \tau^{p_{1}} \| \tau \|_{\alpha_{1}}^{q_{1}} = K - a(K(\phi(x) + 1))^{p_{1}/m} \| (K(\phi(x) + 1))^{1/m} \|_{\alpha_{1}}^{q_{1}} \geq K - aK^{(p_{1}+q_{1})/m}(M + 1)^{(p_{1}+q_{1})/m} |\Omega|^{q_{1}/\alpha_{1}}. \tag{3.3}\]

Denote

\[ K_{1} = a |\Omega|^{q_{1}/\alpha_{1}} (M + 1)^{(p_{1}+q_{1})/m} |\Omega|^{q_{1}/\alpha_{1}}. \tag{3.4}\]

If \( p_{1} + q_{1} < m \), then we can choose sufficiently large that \( K \geq K_{1} \) and

\[ (K(\phi(x) + 1))^{1/m} \geq v_{0}(x). \tag{3.5}\]

From (3.2)-(3.5) we see that \( \tau \) is a positive weak supersolution of (1.1). Hence \( v \leq \tau \) by comparison principle, which implies \( v \) exists globally.

**Proof of Theorem 1.2.** If \( p_{1} + q_{1} > m \), we can choose \( K \leq K_{1} \), where \( K_{1} \) are defined in (3.4). Furthermore, assume that \( v_{0}(x) \) is small enough to satisfy (3.5). It follows that \( \tau \) defined by (3.2) is a positive weak supersolution of (1.1). Hence, \( v \) exists globally.

Now we prove that the solution will blow up in finite time if the initial data is large. Denote by \( \lambda \) and \( \varphi(x) \) the first eigenvalue and the corresponding eigenfunction of the following eigenvalue problem

\[-\Delta \varphi(x) = \lambda \varphi(x), \quad x \in \Omega; \quad \varphi(x) = 0, \quad x \in \partial \Omega. \tag{3.6}\]

Then \( \varphi(x) \) may be normalized as \( \varphi(x) > 0 \) in \( \Omega \), \( \max_{\Omega} \varphi(x) = 1 \), and \( \partial \varphi / \partial \eta < 0 \) on \( \partial \Omega \).

If \( p_{1} + q_{1} > m \), then there exists \( t > 0 \) such that \( \max_{\Omega} \varphi(x) = 1 \). We define \( \psi(x, t) \) the functions as

\[ g(x, t) = [s(t)(\varphi(x) + 1)]^{l}, \tag{3.7}\]

and \( s(t) \) is the solution of the initial problem

\[ s'(t) = ks^{\gamma}, \quad s(0) = \delta > 0, \tag{3.8}\]

where \( k, \delta > 0, \gamma > 1 \) to be fixed later. Clearly, \( s(t) \geq \delta \) become unbounded in finite time, and

\[ \int_{\Omega} \psi_{\alpha_{1}}^{\alpha_{1}} dx \geq \delta^{\alpha_{1}} \int_{\Omega} (\varphi(x) + 1)^{\alpha_{1}} dx > 0. \tag{3.9}\]

A direct calculation yields

\[ \Delta \Psi^{m} + a \Psi^{p_{1}} \| \Psi \|_{\alpha_{1}}^{q_{1}} \]

\[ \geq s^{m}[m\delta^{m-1}l(\varphi(x) + 1)^{m-1} \| \nabla \varphi \|^{2} + \delta^{m} \| \varphi(x) + 1 \|^{m-1} \Delta \varphi(x)] + as^{(p_{1}+q_{1})l}(\varphi(x) + 1)^{p_{1}l} \| (\varphi(x) + 1) \|_{\alpha_{1}}^{q_{1}} \]

\[ \geq -\lambda vs^{m}l(\varphi(x) + 1)^{m-1} \| \nabla \varphi \|^{2} + \delta^{m} \| \varphi(x) + 1 \|^{m-1} \Delta \varphi(x)] + as^{(p_{1}+q_{1})l}(\varphi(x) + 1)^{p_{1}l} \| (\varphi(x) + 1) \|_{\alpha_{1}}^{q_{1}} \]

\[ \geq -\lambda vs^{m}l(\varphi(x) + 1)^{m-1} + acs^{(p_{1}+q_{1})l}(\varphi(x) + 1)^{p_{1}l} \| (\varphi(x) + 1) \|_{\alpha_{1}}^{q_{1}} \]

\[ \geq 1s^{m-1}l(\varphi(x) + 1)^{l}(as^{2m}l^{2}l^{2} + acs^{(p_{1}+q_{1})l-1} \| \nabla \varphi \|^{2}) \]

\[ \geq 1s^{m-1}l(\varphi(x) + 1)^{l}(ac/2l^{2}l^{2} + acs^{(p_{1}+q_{1})l-1} \| \nabla \varphi \|^{2} - \lambda ml^{2m}l^{2}l^{2}), \]

\[ u_{t} = ls^{m-1}l(\varphi(x) + 1)^{l}s'(t). \tag{3.10}\]

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where
\[ c \equiv \| (\varphi(x) + 1)^{q_1} \|_{\alpha_1} > 0. \] (3.11)

We may choose
\[ k = ac/t^2 > 0, \gamma = ml - l + 1 > 1, \delta = ((\lambda ml^2/ac) + 1)^{1/(p_1 + q_1 - m)} > 1. \] (3.12)

Furthermore, assume that \( v_0 \) large enough to satisfy
\[ v(x,0) = [\delta (\varphi(x) + 1)^{\gamma}]^t \leq v_0(x). \] (3.13)

Then, it follows from (3.6)-(3.13) that \( v \) is a positive weak subsolution of (1.1), which blows up in finite time since \( s(t) \) does. By comparison principle, \( \underline{v} \leq v \), which implies \( v \) blows up in finite time.

**Proof of Theorem 1.3.** (1) In view of \( p_1 + q_1 = m \), we can choose \( |\Omega| < |a(M + 1)|^{-\alpha_1/q_1} \) and \( K \) large enough to satisfy (3.5). Then, \( \tau \) defined by (3.2) is a positive weak supersolution of (1.1). Hence \( v \) exists globally.

(2) Without loss of generality, we may assume that \( 0 \in \Omega \). Let \( B_R(0) \) be a ball such that \( B_R(0) \subset \subset \Omega \). In the following, we will prove that \( v \) blows up in finite time in the ball \( B_R \). If so, \( v \) does blow up in the larger domain \( \Omega \).

Denote by \( \lambda_{B_R} > 0 \) and \( \varphi_R(r) \) the first eigenvalue and the corresponding eigenfunction of the following eigenvalue problem:
\[ -\varphi''(r) + \frac{N-1}{r}\varphi'(r) = \lambda \varphi(r), \quad r \in (0,R); \quad \varphi'(0) = 0, \quad \varphi(R) = 0. \] (3.14)

It is well known that \( \varphi_R(r) \) can be normalized as \( \varphi_R(r) > 0 \) in \( B_R \) and \( \varphi_R(0) = \max_{\Omega} \varphi_R(r) = 1 \). By the scaling property (let \( s = r/R \) of eigenvalues and eigenfunctions, we see that \( \lambda_{B_R} = R^{-2}\lambda_{B_1}, \varphi_R(r) = \varphi_1(s) \), where \( \lambda_{B_1} \) and \( \varphi_1(s) \) are the first eigenvalue and the corresponding normalized eigenfunction of the eigenvalue problem in the unit ball \( B_1(0) \). Moreover,
\[ \max_{B_1} \varphi_1 = \varphi_1(0) = \varphi_R(0) = \max_{\Omega} \varphi_R = 1. \] (3.15)

Now, we define \( \underline{v}(x,t) \) the functions as
\[ \underline{v}(x,t) = [s(t)(\varphi_R(|x|) + 1)]^{\gamma}, \] (3.16)
where \( s(t) \) is defined as in (3.8), \( l_* \) satisfies \( ml_* > 1 \) and \( k_* \), \( \delta > 0 \), \( \gamma > 1 \) to be fixed later. Then, a similar calculation as that of (3.10) yields
\[ \underline{v}_t - \Delta_x^m - a\underline{v}^{p_1} \cdot \| g \|_{\alpha_1} \leq l_* s_*^{\gamma-1} [\Phi_1 + 1]^{t_*} [s'(t) - s^{m_1 - l_1 + 1} (ac_* |l_* 2^{l_*}) - \lambda_{B_R} m_2^{(m_1 - l_1)}]]. \] (3.17)

Here we use \( p_1 + q_1 = m \), and
\[ c_* = \int_{B_R(0)} (\varphi_R(|x|) + 1)^{\alpha_1} dx = R^{Nq_1/\alpha_1} \int_{B_1(0)} (\varphi_1 |y| + 1) dy = b_* R^{Nq_1/\alpha_1}, \] (3.18)
which is a constant independent of \( R \). And then, in view of \( \lambda_{B_R} = R^{-2}\lambda_{B_1} \), we assume that \( R \), i.e., the ball \( B_R(0) \), is sufficiently large that
\[ \lambda_{B_R} < ac_*/ml_* 2^{m}. \] (3.19)

Hence
\[ \sigma_1 = (ac_*/l_* 2^{l_*}) - \lambda_{B_R} m_2^{(m_1 - l_1)} \] (3.20)

Denote \( \gamma_1 = ml_* - l_* + 1 \), it follows from \( m > 1 \) that \( \gamma_1 > 1 \). Since \( v_0 \) is a positive and continuous in \( \Omega \), we can choose \( \delta > 0 \) small enough to satisfy
\[ v_0 \geq (\delta (\varphi_R |x| + 1))^{\gamma}, \] (3.21)
in the ball \( B_R(0) \). Finally, choose \( \gamma, k \) to satisfy
\[ 1 < \gamma < \gamma_1, 0 < k < \sigma_1\delta^{\gamma_1 - \gamma}. \] (3.22)

Thus, from (3.8), we have
\[ s(t) \geq s(0) = \delta > (k/\sigma_1)^{1/(\gamma_1 - \gamma)}. \] (3.23)

That is
\[ ks^{\gamma}(t) \leq \sigma_1 s^{\gamma'}(t). \] (3.24)

It follows from (3.14)-(3.24) that \( \underline{v} \) is a positive weak subsolution of (1.1) in the ball \( B_R(0) \), which blows up in finite time since \( s(t) \) does. Thus, \( v \) does blow up in the larger domain \( \Omega \).
4 Blow-up rate and uniform blow-up profile

Throughout this section, we will assume that \( p_1 + q_1 \geq m \). To convenience, we firstly introduce a transformation to problem (1.1). Let \( u(x, \tau) = v^m(x, \tau), \tau = mt \), then (1.1) becomes

\[
\begin{aligned}
&u_\tau = u^r(\Delta u + au^p \|u\|_\alpha^q), \quad x \in \Omega, \tau > 0, \\
u(x, \tau) = 0, \quad x \in \partial \Omega, \tau > 0, \\
u(x, 0) = u_0(x), \quad x \in \Omega,
\end{aligned}
\]

where \( 0 < r = (m - 1)/m < 1, p = p_1/m, \alpha = \alpha_1/m, q = q_1/m, u_0(x) = v^m_0(x). \)

Under this transformation, assumptions (H1)-(H4) become

\[
\begin{aligned}
(H1)' \quad u_0(x) \in C^{2+\theta}(\Omega) \cap C(\overline{\Omega}), 0 < \theta < 1; \\
(H2)' \quad u_0(x) \neq 0 \text{ in } \Omega, \quad \frac{\partial u_0}{\partial \eta} < 0 \text{ on } \partial \Omega; \\
(H3)' \quad \Delta u_0 + au^p \|u_0\|_\alpha^q \geq 0 \text{ for } x \in \Omega \text{ and } \Delta u_0 + au^p \|u_0\|_\alpha^q = 0 \text{ on } \partial \Omega; \\
(H4)' \quad \text{There exists a constant } \epsilon \geq \max\{\epsilon_1, ak^{-1} |\Omega|^{q/\alpha}\} > 0, \text{ such that}
\end{aligned}
\]

\[
\Delta u_0 + au^p \|u_0\|_\alpha^q - \epsilon u_0^{p+q} \geq 0,
\]

where \( k = p + q + r - 1, \epsilon_1 = (|\Omega|^{q/\alpha} ak/r)((k - p + 1)/(q + k))(q+k)/k. \)

To show the existence of the classical solution of (4.1), we consider the following regularized problem:

\[
\begin{aligned}
u_{\zeta}(x, \tau) = (u_\zeta + \zeta)^r(\Delta u_\zeta + au^p \|u_\zeta\|_\alpha^q), \quad x \in \Omega, \tau > 0, \\
u_{\zeta}(x, \tau) = 0, \quad x \in \partial \Omega, \tau > 0, \\
u_{\zeta}(x, 0) = u_\zeta(x), \quad x \in \Omega,
\end{aligned}
\]

where \( 0 < \zeta < 1. \) Denote

\[
u(x, \tau) = \lim_{\zeta \to 0} u_{\zeta}(x, \tau), \quad x \in \Omega, \tau > 0.
\]

By the standard method (See [15,21]), we can show \( u(x, \tau) \) defined by (4.3) is a unique classical solution of (4.1), provided that \( u_0 \) satisfy the hypotheses (H1)' -(H3)'. We thus assume that the solution of (4.1) blows up in finite time \( T^* \) and set \( U(\tau) = \max_{x \in \Omega} u(x, \tau) \), then \( U(\tau) \) is Lipschitz continuous.

**Lemma 4.1** Suppose that \( u_0(x) \) satisfy (H1)' -(H3)', then there exists a positive constant \( c_0 \) such that

\[
U(\tau) \geq c_0(T^* - \tau)^{-1/k}. \tag{4.4}
\]

**Proof.** From that in (4.1), we see that \( U(\tau) \) satisfies

\[
U'(\tau) \leq a \|\Omega\|^{q/\alpha} U^{r+p+q}(\tau), \quad a.e. \tau \in (0, T^*). \tag{4.5}
\]

Hence,

\[
-(U^{1-(p+q+r)}(\tau))' \leq a \|\Omega\|^{q/\alpha} (p + q + r - 1). \tag{4.6}
\]

In view of \( p_1 + q_1 \geq m \), then \( p + q + r - 1 = (p_1 + q_1)/m + m + r - 1 \geq r > 0 \). Integrating (4.6) over \((\tau, T^*)\), we can get

\[
U(\tau) \geq (a(p + q + r - 1) \|\Omega\|^{q/\alpha})^{-1/p+q+r-1} (T^* - \tau)^{-1/p+q+r-1}.
\]

Setting

\[
k = p + q + r - 1, c_0 = (ak \|\Omega\|^{q/\alpha})^{-1/k}. \tag{4.7}
\]

Then, we draw the conclusion. \( \blacksquare \)

**Lemma 4.2** Suppose that \( u_0(x) \) satisfies (H1)' -(H4)' and \( k \) is given by (4.7). Then there exists a constant \( \epsilon \geq \epsilon_1 \), such that

\[
u_\tau - \epsilon u^{k+1} \geq 0, \quad (x, \tau) \in \Omega \times (0, T^*). \tag{4.8}
\]
Proof. Let $J(x, \tau) = u_\tau - \varepsilon u^{k+1}$ for $(x, \tau) \in \Omega \times (0, T^*)$, then by assumption (H3)', we have

$$J(x, 0) \geq 0 \quad \text{in} \quad \Omega \quad \text{and} \quad \lim_{x \to \partial \Omega} J(x, \tau) \geq 0.$$  \hfill (4.9)

A series of computations yields

$$J_\tau - u^\tau J - \left(2r u^\tau k + apu^{p+r-1} \|u\|_Q^q \right) J = -aqu^{q-r} \|u\|_Q^{q-\alpha} \int_\Omega u^{\alpha-1} J \, dx$$

$$= r u^{-1} J^2 + \varepsilon (k+1) k u^{k-1+r} \|\nabla u\|^2 + r \varepsilon^2 u^{2k+1} + apu^{p+r+k} \|u\|_Q^q$$

$$- \varepsilon (k+1) u^{p+r+k} \|u\|_Q^q + apu^{p+r} \|u\|_Q^{q-\alpha} \|u\|_Q^{\alpha+k}$$

$$\geq r \varepsilon^2 u^{2k+1} - \varepsilon (k-p) u^{p+r+k} \|u\|_Q^q + apu^{p+r} \|u\|_Q^{q-\alpha} \|u\|_Q^{\alpha+k}$$

$$= \varepsilon (k-p+1) pu^{p+r} [r \varepsilon/a(k-p+1)] u^{2k+1-p+r} + (q/(k-p+1)) \|u\|_Q^{q-\alpha} \|u\|_Q^{\alpha+k} - u_k \|u\|_Q^q.$$  \hfill (4.10)

By virtue of inequality, we have

$$\|u\|_Q^{q} = \|u\|_Q^{\alpha(q-\alpha)/(q+k)} \|u\|_Q^{\alpha/(q+k)} \leq \|u\|_Q^{\alpha(q-\alpha)/(q+k)} \|u\|_Q^{\alpha/(q+k)} \Omega^{kq/(q+k)}.$$  \hfill (4.11)

Furthermore, Young’s inequality implies

$$u_k \|u\|_Q^q \leq \Omega^{kq/(q+k)} \|u_k\|_Q^{\alpha(q-\alpha)/(q+k)} \|u\|_Q^{\alpha/(q+k)}$$

$$\leq \Omega^{kq/(q+k)} \|u_k\|_Q^{\alpha(q-\alpha)/(q+k)} \|u\|_Q^{\alpha/(q+k)} \|u\|_Q^{\alpha/(q+k)} \|\nabla u\|^2 + apu^{p+r+k} \|u\|_Q^{q-\alpha} \|u\|_Q^{\alpha+k} \|u\|_Q^{\alpha+k}.$$  \hfill (4.12)

$$\|u\|_Q^{q} = \|u\|_Q^{\alpha(q-\alpha)/(q+k)} \|u\|_Q^{\alpha/(q+k)} \Omega^{kq/(q+k)}.$$

where $\vartheta = ((k-p+1)/(q+k))^{q/(q+k)} \Omega^{kq/(q+k)^2}.$ It follows from (4.10)-(4.11) and (H4)' that

$$J_\tau - u^\tau J - \left(2r u^\tau k + apu^{p+r-1} \|u\|_Q^q \right) J = -aqu^{q-r} \|u\|_Q^{q-\alpha} \int_\Omega u^{\alpha-1} J \, dx \geq r \varepsilon (\varepsilon - \varepsilon^1) u^{2k+1} \geq 0,$$

where $\varepsilon^1 = ((\Omega^{\alpha q}/akr)((k-p+1)/(q+k))^{(q+k)/k}.$

The comparison principle implies $J \geq 0.$ That is, (4.8) holds. \hfill \blacksquare

From (4.8), we get

$$U'(\tau) \geq \varepsilon U^{k+1}(\tau).$$  \hfill (4.13)

Integrating (4.12) from $\tau$ to $T^*$, we conclude that

$$U(\tau) \leq c_{00}(T^* - \tau)^{-1/k},$$  \hfill (4.14)

where $c_{00} = (\varepsilon k)^{-1/k}$ is a positive constant independent of $\tau.$ Combining (4.8) with (4.13), we have the following lemma.

**Lemma 4.3** Suppose that $u_0(x)$ satisfy (H1)' - (H3)'. If $u(x, \tau)$ is the solution of (4.1) and blows up in finite time $T^*$, then there exist positive constants $c_0, c_{00},$ such that

$$c_0(T^* - \tau)^{-1/k} \leq \max_{x \in \Omega}(x, \tau) \leq c_{00}(T^* - \tau)^{-1/k}$$

where $k = p + q + r - 1.$

According the transform and Lemma 4.3, we can obtain Theorem 1.4.

**Lemma 4.4** Assume $p + r \leq 1,$ $u_0(x)$ satisfies (H1)' - (H3)' and $\Delta u_0 \leq 0$ in $\Omega$. $u(x, \tau)$ is the classical solution of the problem (4.1). Then $\Delta u \leq 0$ in any compact subsets of $\Omega \times (0, T^*).$

Proof. Let $w(x, \tau) = \Delta w(x, \tau).$ According to (4.2), for all $\tau \in (0, T^*),$ we have

$$w_\tau = r(u + \varepsilon)^{r-1} \Delta u_\tau (\Delta u + au^{\rho} \|u\|_Q^\rho) + r(r - 1)(u + \varepsilon)^{r-2} \|\nabla u\|^2 (\Delta u + au^{\rho} \|u\|_Q^\rho)$$

$$+ 2r(u + \varepsilon)^{r-1} \Delta u_\tau (\nabla u + au^{\rho-1} \nabla u \|u\|_Q^{\rho-1})$$

$$+ (u + \varepsilon)^{r-1} (\Delta u + ap - 1) u^{p-2} \|\nabla u\|^2 (\|u\|_Q^\rho + au^{p-1} \Delta u \|u\|_Q^\rho).$$  \hfill (4.15)
Noticing that $u_\ast, u_{r\tau} \geq 0$ and $p + r \leq 1$, we have
\begin{align}
w_{r} & = (u_\ast + \zeta)^r \Delta w - 2r(u_\ast + \zeta)^{r-1}\nabla u_\ast \nabla w \\
& \leq r(r-1)(u_\ast + \zeta)^{r-2} |\nabla u_\ast|^2 w + r(u_\ast + \zeta)^{-1} u_{r\tau} w + a_p(u_\ast + \zeta)^r u_\ast^{r-1} \|u_\ast\|_\alpha^r w \quad (4.15) \\
& + a|r(r-1) + 2p + p(p-1)||u_\ast + \zeta|^{r-2} |\nabla u_\ast|^2 \|u_\ast\|_\alpha^r \\
& \leq (r(r-1)(u_\ast + \zeta)^{r-2} |\nabla u_\ast|^2 w + r(u_\ast + \zeta)^{-1} u_{r\tau} w + a_p(u_\ast + \zeta)^r u_\ast^{r-1} \|u_\ast\|_\alpha^r)w.
\end{align}
Moreover,
\begin{align*}
w|_{\partial \Omega} = -au_\ast^p \|u_\ast\|_\alpha^p < 0, \quad w(x, 0) = \Delta u_0 \leq 0.
\end{align*}
By maximum principle, it follows $w(x, \tau) \leq 0$, which means $\Delta u_\ast \leq 0$ in $\overline{\Omega_T}$.
Therefore, $\Delta u \leq 0$ in any compact subsets of $\Omega \times (0, T^*)$. ■
Denote
\[ g(\tau) = a \|u\|^p_\alpha, \quad G(\tau) = \int_0^\tau g(s)ds. \]

**Lemma 4.5** Under the conditions of Lemma 4.3, it holds that
\begin{align}
\lim_{\tau \to T^*} g(\tau) = \infty, \quad \lim_{\tau \to T^*} G(\tau) = \infty.
\end{align}
**Proof.** From Lemma 4.3, we have
\begin{align}
u_{r\tau} \leq u^{r+p}g(\tau), \quad a.e. \tau \in [0, T^*).
\end{align}
Integrating (4.17) over $(0, \tau)$, we obtain
\begin{align}
u^{1-p-r}(x, \tau) \leq (1 - p - r)G(\tau) + u_0^{1-p-r}(x), \quad p + r < 1, \\
\ln u(x, \tau) \leq G(\tau) + \ln u_0(x), \quad p + r = 1.
\end{align}
In view of $\lim_{\tau \to T^*} u(x, \tau) = \infty$, $\lim_{\tau \to T^*} G(\tau) = \infty$. Noting that $u_{r\tau} \geq 0$ by the assumption of the initial function, we then see that $g(\tau)$ is monotone nondecreasing. Therefore $\lim_{\tau \to T^*} g(\tau) = \infty$. ■

**Lemma 4.6** Under the conditions of Lemma 4.3. If $u$ blows up in finite time $T^*$, then we have
\begin{enumerate}
\item \( \lim_{\tau \to T^*} \frac{u^{1-p-r}(x, \tau)}{(1 - p - r)G(\tau)} = \lim_{\tau \to T^*} \frac{\|u(x, \tau)\|^{1-p-r}_{\infty}}{G(\tau)} = 1, \quad p + r < 1, \)
\item \( \lim_{\tau \to T^*} \frac{\ln u(x, \tau)}{G(\tau)} = \lim_{\tau \to T^*} \frac{\|\ln u(\cdot, \tau)\|_{\infty}}{G(\tau)} = 1, \quad p + r = 1, \)
\end{enumerate}
uniformly on any compact subsets of $\Omega$.
**Proof.** (1) For $(x, \tau) \in \Omega \times (0, T^*)$, we define
\[ z_1(x, \tau) = G(\tau) - u^{1-r-p}/(1 - r - p), \quad \gamma_1(\tau) = \int_\Omega z_1(y, \tau)\varphi(y)dy, \]
where $\varphi$ satisfies (3.6) and $\int_\Omega \varphi(x)dx = 1$. A direct computation shows
\[ \gamma_1'(\tau) = \int_\Omega (g(\tau) - u^{1-r-p}(y, \tau)u_{r\tau}(y, \tau))\varphi(y)dy = - \int_\Omega u^{1-p}(y, \tau)\Delta u(y, \tau)\varphi(y)dy \\
\leq -1/(1 - p) \int_\Omega \Delta u^{1-p}(y, \tau)\varphi(y)dy = \lambda/(1 - p) \int_\Omega u^{1-p}(y, \tau)\varphi(y)dy \\
\leq \lambda(1 - r - p)/(1 - p - r)/(1 - p) \int_\Omega (G(\tau) - z_1(y, \tau))^{(1-p)/(1-p-r)}\varphi(y)dy \\
\leq C_1(G^{(1-p)/(1-p-r)}(\tau) + \int_\Omega (z_1(y, \tau))^{(1-p)/(1-p-r)}\varphi(y)dy), \]
where $z_1^- = \max\{-z_1, 0\}$. From (4.18), we know that
\begin{align}
\inf_{\Omega} z_1(y, \tau) \geq -C', \quad (4.22)
\end{align}
which means $z_1^- \leq C'$. Then
\begin{align}
\gamma_1'(\tau) \leq C_2G^{(1-p)/(1-p-r)}(\tau) + C_3.
\end{align}

Integrate (4.23) from 0 to \(\tau\),
\[
\gamma_1(\tau) \leq C_4(1 + \int_0^\tau G^{(1-p)/(1-p-r)}(s)ds).
\tag{4.24}
\]

Thus (4.22) and (4.24) implies
\[
\int_\Omega |\gamma_2(y,\tau)|\varphi(y)dy \leq C_5(1 + \int_0^\tau G^{(1-p)/(1-p-r)}(s)ds).
\]

Define \(K_\rho = \{y \in \Omega : \text{dist}(y, \partial\Omega) \geq \rho\}\). Since \(-\Delta z_1 \leq 0\) in \(\Omega \times (0, T^*)\). Using Lemma 4.5 in [14], we obtain
\[
\sup_{K_\rho} z_1(x, \tau) \leq C_6 \rho^{n+1}(1 + \int_0^\tau G^{(1-p)/(1-p-r)}(s)ds).
\tag{4.25}
\]

It follows from (4.25) and (4.18) that
\[
-\frac{C'}{G(\tau)} \leq 1 - \frac{u^{1-r-p}(x, \tau)}{(1-r-p)G(\tau)} \leq \frac{C_6 \rho^{n+1}(1 + \int_0^\tau G^{(1-p)/(1-p-r)}(s)ds)}{G(\tau)},
\tag{4.26}
\]
for any \(x \in K_\rho\) and \(\tau \in (0, T^*)\). We know from Theorem 4.1 that
\[
\int_0^\tau G^{(1-p)/(1-p-r)}(s)ds \leq C_7 \int_0^\tau (T^* - s)^{-(1-p)/k}ds.
\tag{4.27}
\]

In view of (4.18), it follows that
\[
G(\tau) \geq C_8 u^{1-r-p} \geq C_9 (T^* - \tau)^{-(1-r-p)/k}.
\tag{4.28}
\]

From (4.26)-(4.28), we get
\[
-\frac{C'}{G(\tau)} \leq 1 - \frac{u^{1-r-p}(x, \tau)}{(1-r-p)G(\tau)} \leq \frac{C_{10} 1 + \int_0^\tau (T^* - s)^{-(1-p)/k}ds}{(T^* - \tau)^{-(1-r-p)/k}}.
\tag{4.29}
\]

It is obvious that
\[
\lim_{\tau \to T^*} \int_0^\tau (T^* - s)^{-(1-p)/k}ds = 0.
\]

Thus
\[
\lim_{\tau \to T^*} \frac{u^{1-p-r}(x, \tau)}{(1-p-r)G(\tau)} = \lim_{\tau \to T^*} \frac{\|u(\cdot, \tau)\|^{1-p-r}_\infty}{(1-p-r)G(\tau)} = 1.
\]

(2) Similarly, we define
\[
z_2(x, \tau) = G(\tau) - \ln u(x, \tau), \quad \gamma_2(\tau) = \int_\Omega z_2(y, \tau)\varphi(y)dy, \quad (x, \tau) \in \Omega \times (0, T^*),
\]
where \(\varphi\) satisfies (3.6) and \(\int_\Omega \varphi(x)dx = 1\). A direct computation shows
\[
\gamma_2'(\tau) = \int_\Omega (g(\tau) - u^{-1}(y, \tau)u_{\tau}(y, \tau))\varphi(y)dy = -\int_\Omega u^{-1}(y, \tau)\Delta u(y, \tau)\varphi(y)dy
\leq -\frac{1}{r} \int_\Omega \Delta u''(y, \tau)\varphi(y)dy = \lambda/r \int_\Omega u''(y, \tau)\varphi(y)dy = \lambda/r \int_\Omega \exp[rG(\tau) - z_2(y, \tau)]\varphi(y)dy.
\]

From (4.19), we know that
\[
\inf_{\Omega} z_2(y, \tau) \geq -C''.
\tag{4.30}
\]

Then
\[
\gamma_2'(\tau) \leq C_{11} \exp[rG(\tau)].
\tag{4.31}
\]

Integrate (4.31) from 0 to \(\tau\) yields
\[
\gamma_2(\tau) \leq C_{12} (1 + \int_0^\tau \exp[rG(s)]ds).
\tag{4.32}
\]
Thus (4.32) and (4.30) implies
\[
\int_{\Omega} |z_2(y, \tau)| \varphi(y) dy \leq C_{13}(1 + \int_0^{\tau} \exp[rG(s)] ds).
\]
Define \( K_\zeta = \{ y \in \Omega : \text{dist}(y, \partial \Omega) \geq \zeta \} \). Since \(-\Delta z_2 \leq 0 \) in \( \Omega \times (0, T^*) \). Using Lemma 4.5 in [14], we obtain
\[
\sup_{K_\zeta} z_2(x, \tau) \leq C_{14} \frac{1 + \int_0^{\tau} \exp[rG(s)] ds}{\zeta^{N+1}}.
\]
(4.33)
It follows from (4.33) and (4.30) that
\[
- \frac{C''}{G(\tau)} \leq 1 - \frac{\ln u(x, \tau)}{G(\tau)} \leq \frac{\ln u(x, \tau)}{G(\tau)} \leq \frac{1 + \int_0^{\tau} \exp[rG(s)] ds}{G(\tau)} \leq \frac{1 + \int_0^{\tau} (T^* - s)^{-r} ds}{\ln(T^* - \tau)}.
\]
(4.34)
By Theorem 4.1, we have
\[
G(\tau) = a \int_0^{\tau} \left\| u(\eta, \tau) \right\|^q d\eta \leq (\Lambda^{q/\alpha}) \left( \frac{\zeta}{(4\eta)^{\alpha}} \right) \int_0^{\tau} (T^* - s)^{-1} ds = \ln T^* - \ln(T^* - \tau).
\]
(4.35)
On the other hand, we know form (4.19) and Theorem 4.1 that
\[
G(\tau) \geq e |\ln(T^* - \tau)|.
\]
(4.36)
From (4.34)-(4.36), we get
\[
- \frac{C''}{G(\tau)} \leq 1 - \frac{\ln u(x, \tau)}{G(\tau)} \leq \frac{\ln u(x, \tau)}{G(\tau)} \leq \frac{1 + \int_0^{\tau} \exp[rG(s)] ds}{G(\tau)} \leq \frac{1 + (T^*)^r \int_0^{\tau} (T^* - \tau)^{-r} ds}{\ln(T^* - \tau)}.
\]
It is easy to derive
\[
\lim_{\tau \to T^*} \frac{1 + (T^*)^r \int_0^{\tau} (T^* - \tau)^{-r} ds}{\ln(T^* - \tau)} = 0.
\]
Thus
\[
\lim_{\tau \to T^*} \frac{\ln u(x, \tau)}{G(\tau)} = \lim_{\tau \to T^*} \frac{\left\| u(\cdot, \tau) \right\|_\infty}{G(\tau)} = 1.
\]
This completes the proof. ■

**Lemma 4.7** Assume that the conditions of Lemma 4.5 hold, then the following limits converge uniformly on any compact subset of \( \Omega \).
(1) If \( p + r < 1 \), \( \lim_{\tau \to T^*} u(x, \tau)(T^* - \tau)^{-1/k} = (a \Omega^{q/\alpha})^{-1/k} \);
(2) If \( p + r = 1 \), \( \lim_{\tau \to T^*} |\ln(T^* - \tau)|^{-1} \ln u(x, \tau) = 1/q. \)

**Proof.** Case 1: \( p + r < 1 \). Form (4.20), we have
\[
u(x, \tau) \sim ((1 - p + r)G(\tau))^{1/(1-p-r)}, \text{ as } \tau \to T^*,
\]
where the notation \( u \sim v \) means \( \lim_{\tau \to T^*} u(\tau)/v(\tau) = 1. \)
Furthermore,
\[
G'(\tau) = g(\tau) = a \left\| u \right\|_{\alpha}^q \sim a \alpha \left\| u \right\|_{\alpha}^{q/\alpha} ((1 - p + r)G(\tau))^{g/(1-p-r)}, \text{ as } \tau \to T^*.
\]
(4.37)
Integrating (4.37) over \((\tau, T^*) \) yields
\[
G(\tau) \sim (1 - p + r)^{-1} (a \alpha \left\| u \right\|_{\alpha}^{q/\alpha} (T^* - \tau))^{-(1-p-r)/k}, \text{ as } \tau \to T^*.
\]
(4.38)
So, we can get our conclusion by using (4.20) and (4.38).
Case 2: \( p + r = 1 \). In this case, for any given \( \sigma: 0 < \sigma < 1 \). By (4.21), there exists \( 0 < \tau_0 < T^* \) such that
\[
(1 - \sigma)G(\tau) \leq \ln u(x, \tau) \leq (1 + \sigma)G(\tau), \text{ for } x \in \Omega, \tau \in [\tau_0, T^*].
\]

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Therefore
\[ a |\Omega|^{q/\alpha} \exp[q(1 - \sigma)] G(\tau) \leq G'(\tau) = a \|u\|_\alpha^q \leq a |\Omega|^{q/\alpha} \exp[q(1 + \sigma)] G(\tau), \quad \tau_0 \leq \tau \leq T^*. \quad (4.39) \]

In view of the right-hand side of the (4.39), we have
\[ \exp[-q(1 + \sigma)] dG(\tau) \leq a |\Omega|^{q/\alpha} d\tau, \quad \tau_0 \leq \tau \leq T^*. \]

Integrating the above inequality from $\tau$ to $T^*$ yields that
\[ \exp[-q(1 + \sigma)] G(\tau) \leq aq(1 + \sigma) |\Omega|^{q/\alpha} (T^* - \tau), \quad \tau_0 \leq \tau \leq T^*. \]

Namely,
\[ G(\tau) \geq (-1/q(1 + \sigma)) \ln[aq(1 + \sigma) |\Omega|^{q/\alpha} (T^* - \tau)], \quad \tau_0 \leq \tau \leq T^*. \quad (4.40) \]

Similar arguments to the left hand of (4.39) yield that
\[ G(\tau) \leq (-1/q(1 - \sigma)) \ln[aq(1 - \sigma) |\Omega|^{q/\alpha} (T^* - \tau)], \quad \tau_0 \leq \tau \leq T^*. \quad (4.41) \]

Consequently, (4.40) and (4.41) guarantee that for $\tau_0 \leq \tau \leq T^*$,
\[ (-1/q(1 + \sigma)) \ln[aq(1 + \sigma) |\Omega|^{q/\alpha} (T^* - \tau)] \leq G(\tau) \leq (-1/q(1 - \sigma)) \ln[aq(1 - \sigma) |\Omega|^{q/\alpha} (T^* - \tau)]. \quad (4.42) \]

Letting $\sigma \to 0$, we have
\[ \lim_{\tau \to T^*} G(\tau) [\ln(T^* - \tau)]^{-1} = 1/q, \]

because of $\lim_{r \to T^*} G(\tau) = \infty$. Due to $\ln u(x, \tau) \sim G(\tau)$ uniformly on any compact subset of $\Omega$, we complete our proof. \[ \square \]

Combining Lemma 4.7 with the transform about $v(x, t)$, it follows that Theorem 1.5 holds.

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**References**


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