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An approximate analytical solution of Nonlinear Fractional Diffusion Equation

Subir Das[1], Praveen Kumar Gupta[2] *, Pradyumna Ghosh[3]
[1] Department of Applied Mathematics, Institute of Technology, Banaras Hindu University, Varanasi- 221 005, India
[2] Department of Mathematics & Statistics, Banasthali University, Banasthali - 304 022, India
[3] Department of Mechanical Engineering, Institute of Technology, Banaras Hindu University, Varanasi- 221 005, India
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Abstract: The article presents the approximate analytical solution of a nonlinear diffusion equation with fractional time derivative $\alpha(0 < \alpha < 1)$ and with the diffusion term as $u^n(n \neq 0)$. By using initial value, the explicit solutions of the equation are solved with a powerful mathematical tool like Adomian Decomposition Method. The speed of convergence of the method based on the properties of the convergent series which is successfully derived in this article shows the efficiency, simplicity and reliability of the method for solving nonlinear problem. Fast and slow diffusion for different particular cases are presented graphically.

Keywords: Partial differential equation; Nonlinear fractional diffusion equation; Brownian motion; Adomian decomposition method

1 Introduction

In 1959, Ovsiannikov [1] investigated the solution of the nonlinear diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(g(u)\frac{\partial u}{\partial x}\right), \tag{1}$$

by Symmetry method where $u(x,t)$ represents mass concentration. This type of equation appears in plasma physics, kinetic theory of gases, transport in porous medium etc. In many cases $g(u)$ is approximated as $g(u) = u^n$. Then the equation (1) is called fast diffusion equation for $-2 < n < 0$ and slow diffusion equation for $n > 0$. In the first case the spread of mass is faster than the linear case $n = 0$ and in the second case it is slower. Gandarias [2] investigated the fast diffusion equation for $n = -1$. Later, Popovych et al. [3] developed his idea and obtained a new wider classes of potential non-classical symmetries of the fast diffusion equation. Guo and Guo [4] studied the large time behaviors of the global and non-global solutions of the Cauchy problem for a fast diffusion equation with source. Recently, Fa and Lenzi [5] have used the Green function method to find the solution of the diffusion equation

$$\frac{\partial \rho(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(D(x,t)\frac{\partial \rho(x,t)}{\partial x}\right), \tag{2}$$

in a finite interval with diffusion coefficient $D(x,t) = D(t)\,|x|^{-\theta}$ and initial condition $\rho(x,0) = \rho_0(x)$ subject to absorbing boundaries. In 1986, Wyss [6] solved the fractional diffusion equation interms of Fox functions. In 2010, D'Ovidio [7] has used Mellin convolution of generalized Gamma densities for solving fractional diffusion equation. Here, the authors have made a sincere attempt to derive the following nonlinear diffusion equation with fractional time derivative

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial}{\partial x}\left(u^n\frac{\partial u}{\partial x}\right), \tag{3}$$

with the initial condition

*Corresponding author. *E-mail address*: praveen.rs.apm@itbhu.ac.in

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$$u(x,0) = f(x), \tag{4}$$

The above type of anomalous diffusion is a ubiquitous phenomenon in nature and appears in different branches of science and engineering. Most of the nonlinear problems do not a precise analytical solution; especially it is very hard to get it for the fractional nonlinear equations. So these types of equations should be solved by approximate analytical methods.

Adomian ([8 - 11]) developed a method to solve a wide class of linear and nonlinear partial differential equations (PDE), known as Adomian decomposition method (ADM). Actually ADM will be applied whenever it is appropriate to the solutions of partial differential equations of any order (integral and fractional) and the numerical solutions reveal that the method is user friendly, flexible, accurate, effective and very powerful to solve large class of differential equations. ADM is really new approach to provide an approximate analytical solution to linear and nonlinear problems. The technique provides its fast convergence of the solution.

In this article, ADM is used to solve the nonlinear diffusion equation with fractional time derivative, where the domain of the space variable is unbounded. The approximate analytical solution of probability density function $u(x,t)$ for different fractional Brownian motions and also for the standard motions are derived successfully and presented graphically. The quick convergence of the method and deduction of truncated error show the elegance of this non-classical method for solving nonlinear problems in simplistic approach.

2 Solution of the problem by ADM

The equation (3) can be written in the form

$$D_t^\alpha u(x,t) = D_x\left((u(x,t))^n D_x u(x,t)\right), \tag{5}$$

where $D_t^\alpha \equiv \frac{\partial^\alpha}{\partial t^\alpha}$ and $D_x \equiv \frac{\partial}{\partial x}$;
Applying the operator J_t^α , the inverse of the operator D_t^α, on both sides of (5), we obtain

$$u(x,t) = u(x,0) + J_t^\alpha\left(D_x(\Phi(u))\right), \tag{6}$$

where $\Phi(u) = u^n D_x u$.

According to the decomposition method, let $u(x,t)$ can be written in a series solution form as

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{7}$$

The decomposition term $\Phi(u)$ can be decomposed as

$$\Phi(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, u_2, \ldots\ldots, u_n), \tag{8}$$

where A_i's are the Adomian polynomials ([9, 10]). For the nonlinearity, we use the following algorithm for Adomian polynomials A_i proposed by [11] and [12],

$$\begin{aligned} A_n &= \frac{1}{n!}\frac{d^n}{d\lambda^n}\Phi\left(\sum_{n=0}^{\infty}\lambda^n u_n\right)_{\lambda=0} \\ &= \frac{1}{n!}\frac{d^n}{d\lambda^n}\left(\sum_{n=0}^{\infty}\lambda^n u_n\right)^n D_x\left(\sum_{n=0}^{\infty}\lambda^n u_n\right)_{\lambda=0}, n \geq 0 \end{aligned} \tag{9}$$

From equation (9), we obtain the first few terms of the Adomian polynomials as

$$\begin{aligned}
A_0 &= u_0^n D_x u_0 \\
A_1 &= u_0^n D_x u_1 + n u_0^{n-1} u_1 D_x u_0 \\
A_2 &= u_0^n D_x u_2 + n u_0^{n-1} u_1 D_x u_1 + n u_0^{n-1} u_2 D_x u_0 + \frac{n(n-1)}{2} u_0^{n-1} u_1^2 D_x u_0 \\
A_3 &= u_0^n D_x u_3 + n u_0^{n-1} u_1 D_x u_2 + n u_0^{n-1} u_2 D_x u_1 + n u_0^{n-1} u_3 D_x u_0 + n(n-1) \\
& \quad u_0^{n-2} u_1 u_2 D_x u_0 + \frac{n(n-1)}{2} u_0^{n-2} u_1^2 D_x u_1 + \frac{n(n-1)(n-2)}{6} u_0^{n-3} u_1^3 D_x u_0
\end{aligned} \tag{10}$$

and so on.
From equations (6) - (8), we obtain

$$\sum_{n=0}^{\infty} u_n(x,t) = u(x,0) + J_t^{\alpha} D_x \left(\sum_{n=0}^{\infty} A_n(u_0, u_1, u_2, \ldots\ldots, u_n) \right), \tag{11}$$

which with the help of equation (10), gives rise to

$$\begin{aligned}
u_0 &= u(x,0) = f(x) = x, \\
u_1 &= J_t^{\alpha} D_x(A_0) = n x^{n-1} \frac{t^{\alpha}}{\Gamma(\alpha+1)}, \\
u_2 &= J_t^{\alpha} D_x(A_1) = 2n(n-1)(2n-1) x^{2n-3} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \\
u_3 &= J_t^{\alpha} D_x(A_2) \\
&= n(n-1)(3n-4) \left(6(n-1)(2n-1) + \frac{3n^2}{2} \frac{\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2} \right) x^{3n-5} \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \\
u_4 &= J_t^{\alpha} D_x(A_3) \\
&= n(4n-5)(4n-6) \Bigg[(n-1)(3n-4) \left(6(n-1)(2n-1) + \frac{3n^2}{2} \frac{\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2} \right) \\
& \quad + \frac{n^4(n-1)}{6} \frac{\Gamma(3\alpha+1)}{(\Gamma(\alpha+1))^3} + 2n^3(n-1)(2n-1) \frac{\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)} \Bigg] x^{4n-7} \frac{t^{4\alpha}}{\Gamma(4\alpha+1)}
\end{aligned}$$

Proceeding in this manner, the components $u_n, n \geq 0$ of the ADM can be completely obtained and the series solutions are thus entirely obtained.

Finally, we approximate the analytical solution $u(x,t)$ by truncated series

$$u(x,t) = lim_{N \to \infty} \Psi_N(x,t), \tag{12}$$

where $\Psi_N(x,t) = \sum_{n=0}^{N-1} u_n(x,t), \quad N \geq 1$,

The decomposition series (13) converges very rapidly in real physical problems. The rapid convergence means that only few terms are required to get the approximate analytical solution of the problem.

3 Error analysis of the problem

Let the nonlinear function $\phi(u)$ is developable in entire series with a convergence radius equal to infinity i.e.,

$$\phi(u) = \sum_{n=0}^{\infty} \phi^n(0) \frac{u^n}{n!}, \quad |u| < \infty \tag{13}$$

Then equation (14) can be written as

$$\begin{aligned}\phi(u) &= \sum_{n=0}^{\infty}\frac{\phi^n(0)}{n!}\left[\left(\sum_{i=0}^{\infty}u_i\right)^n D_x\left(\sum_{i=0}^{\infty}u_i\right)\right]^n \\ &= \sum_{n=0}^{\infty}\frac{\phi^n(0)}{n!}\sum_{q=0}^{\infty}\gamma_{nq}(u_0,u_1,u_2,\ldots\ldots,u_q) \end{aligned} \tag{14}$$

Considering the series (7) is absolute convergent i.e., $\sum_{n=0}^{\infty}|u_n| = U < \infty$, we find (using [12,13])

$$\begin{aligned}\frac{\partial}{\partial x}\phi(u) &= \sum_{n=0}^{\infty}\frac{\phi^n(0)}{n!}\left(\sum_{q=0}^{\infty}\frac{\partial\gamma_{nq}}{\partial u_q}\frac{\partial u_q}{\partial x}\right) \\ &= \sum_{n=0}^{\infty}\frac{\phi^n(0)}{n!}\left(\sum_{q=0}^{\infty}\chi_{nq}(u_0,u_1,u_2,\ldots\ldots,u_q)\right)\end{aligned} \tag{15}$$

$$\left|\frac{\partial}{\partial x}\phi(u)\right| \le \sum_{n=0}^{\infty}\left|\frac{\phi^n(0)}{n!}\right|U^n, \tag{16}$$

where $\sum_{q=0}^{\infty}|\chi_{nq}| \le U^n$.

The error can be calculated as

$$\begin{aligned}\left|\sum_{n=0}^{\infty}u_n - \sum_{n=0}^{N-1}u_n\right| &= \left|\sum_{n=N}^{\infty}u_n\right| = \left|J_t^{\alpha}\frac{\partial}{\partial x}\sum_{n=N-1}^{\infty}A_n\right| = \left|J_t^{\alpha}\sum_{n=N-1}^{\infty}\frac{\partial\phi}{\partial x}\right| \\ &\le \sum_{n=N-1}^{\infty}\left|\frac{\phi^n(0)}{n!}\right|U^n \le \frac{M^{N-1}}{(N-1)!}k,\end{aligned} \tag{17}$$

where $|\phi^n(0)| \le k$ and $||U|| \le M$.

Here $u_n(x,t) = h(x)\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}$ which leads to an $error = \frac{M^{(N-1)\alpha}}{\Gamma((N-1)\alpha+1)}$.

4 Numerical results and discussion

In this section, the numerical results of $u(x,t)$ for different fractional Brownian motions $\alpha = 1/3, 1/2, 2/3$ are also for the standard motion $\alpha = 1$ are calculated for various values of time t with the degree of the diffusion term at $x = 1$ which are depicted through Figs. 1 - 3.

It is seen from the Figs.1-2 that the rate of increase of $u(x,t)$ with time decreases with the increase of α which conforms with the exponentially decay of regular Brownian motion. This result is in complete agreement with the results of Das [14] and Giona and Roman [15].

For positive values of the power of diffusivity coefficient, i.e., for $0 < n < 2$ anomalous diffusion has been observed where sub-diffusion occurs in the range of $0 < n < 0.6$ and super-diffusion in the range $0.8 < n < 2$. This phenomenon can be demonstrated from the normal length scale analysis. As revealed by Fig. 3, a threshold is being observed to exist in the region $0.6 < n < 0.8$ between i.e., after the occurrence of sub-diffusion and before that of super diffusion. But in the range $-2 < n < 0$ (Fig. 4) where no sub-diffusion or super-diffusion occurs, demarcation has been observed and $u(x,t)$ describes the asymptotic behavior with t.

5 Conclusion

Adomian decomposition method is a powerful method of solving nonlinear equations since it ensures exact solution as an infinite series of the functions. Since Adomian series is quickly convergent and truncated series can be calculated, as shown in the article, so it is easy to find the approximate analytical solution of the nonlinear problem with a finite number of terms of the series solution.

Another important point of the study is to present the explanation of fast and slow diffusions for both fractional Brownian motions and standard motion, which has been accomplished by the authors.

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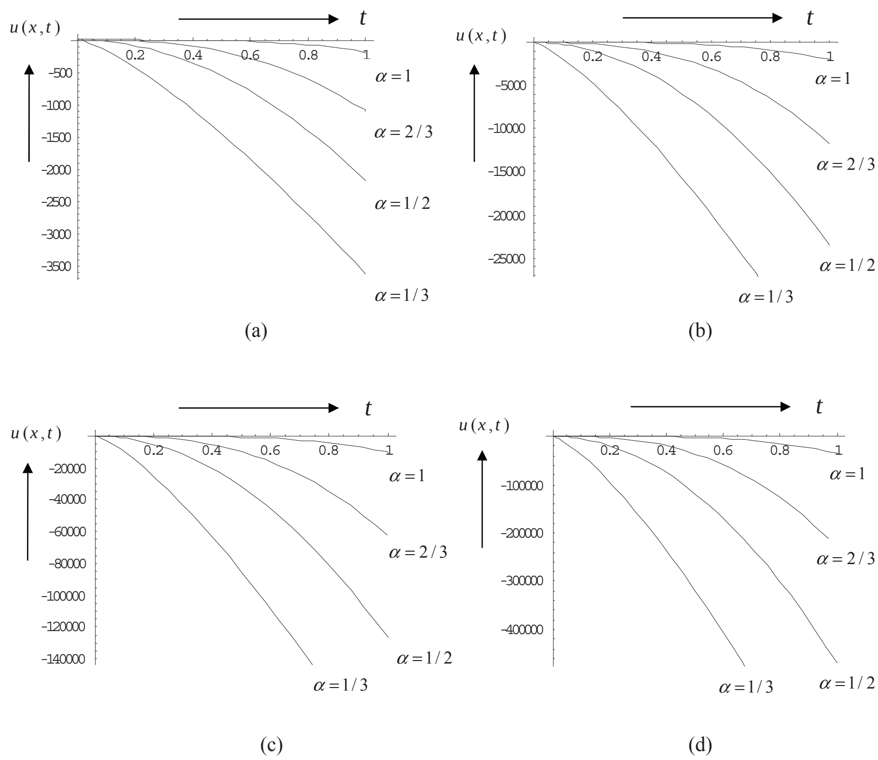

Figure 1: Plot of $u(x,t)$ vs. t at (a) $n = -\frac{1}{2}$ (b) $n = -1$ (c) $n = -\frac{3}{2}$ (d) $n = -2$ and $x = 1$ for different values of α.

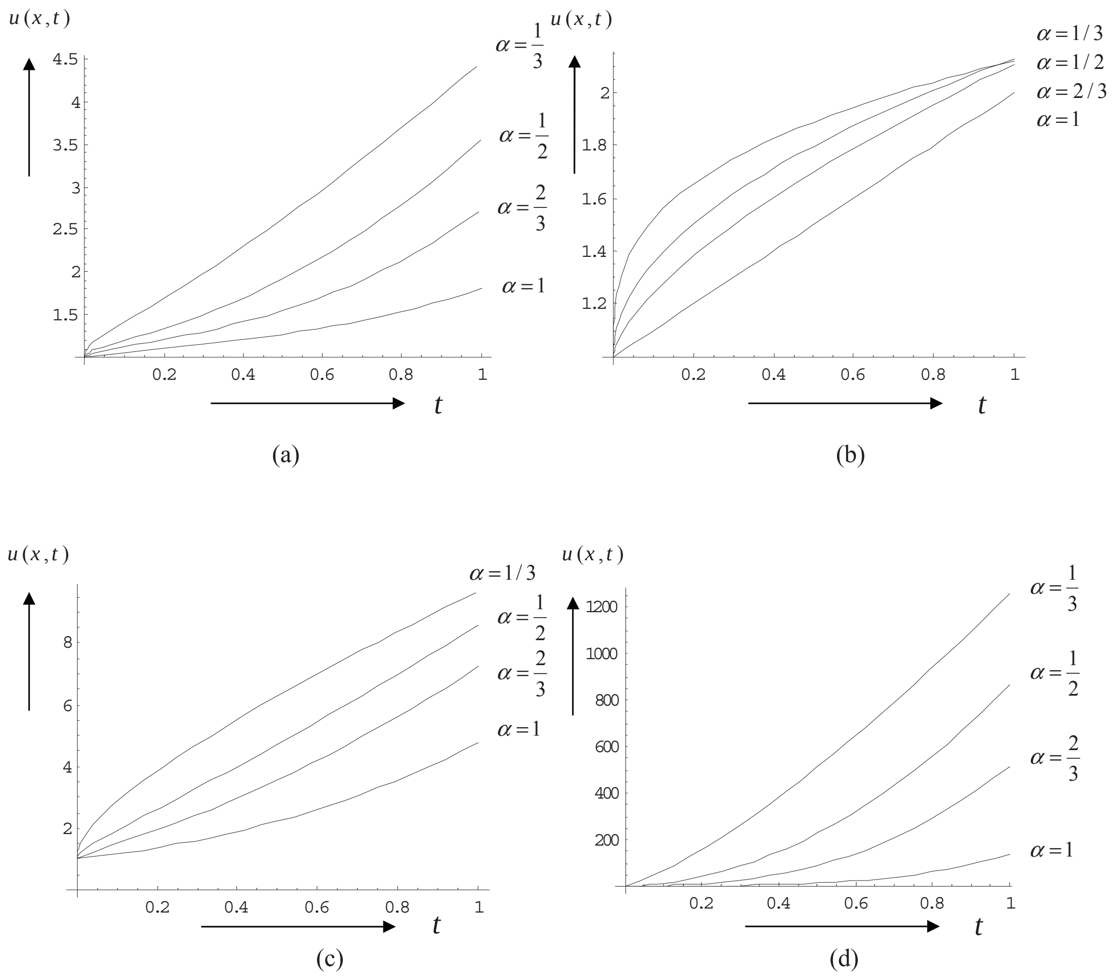

Figure 2: Plot of $u(x,t)$ vs. t at (a) $n=\frac{1}{2}$ (b) $n=1$ (c) $n=\frac{3}{2}$ (d) $n=2$ and $x=1$ for different values of α.

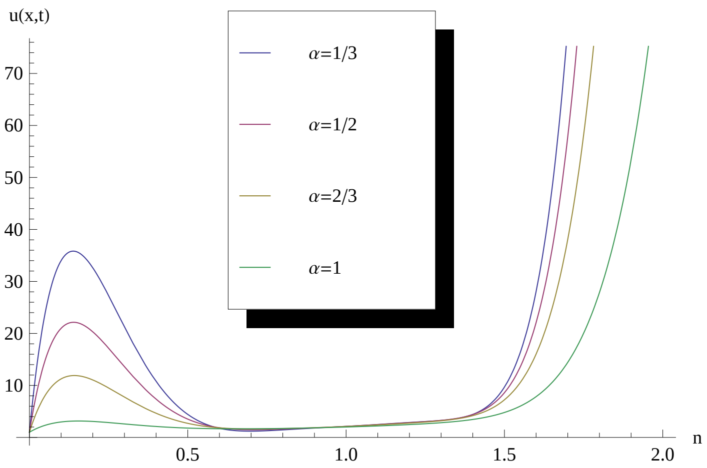

Figure 3: Plot of $u(x,t)$ vs. positive values of n for various values of α at $x = 1$ and $t = 1$.

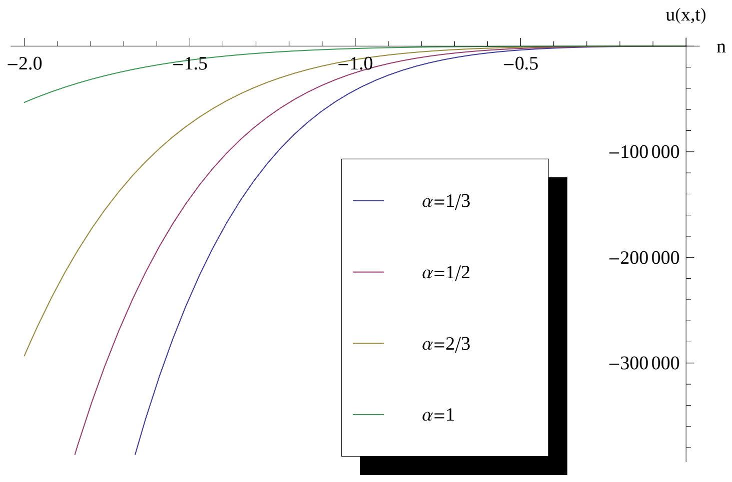

Figure 4: Plot of $u(x,t)$ vs. negative values of n for various values of α at $x = 1$ and $t = 1$.