ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.12(2011) No.3,pp.331-338

Initial Value Problems for Nonlinear First-order Impulsive Integro-differential Equations with Deviating Arguments

Wenxia Wang[1] *, Junxia Wang[1], Piapiao Shi[2]
[1] Department of Mathematics, Taiyuan Normal University, Taiyuan, Shanxi 030012, P. R. China
[2] School of Mathematics, Jinzhong University, Jinzhong,Shanxi 030600, P. R. China
(Received 3 July , accepted 28 October 2011)

Abstract: By establishing the piecewise monotone iterative technique and comparison results, the existence and uniqueness of solutions for the initial value problem of first-order nonlinear impulsive integro-differential equations with deviating arguments are investigated without continuity of impulsive functions. Some known results are improved.

Keywords: Monotone iterative technique; impulsive integro-differential equation; initial value problem; deviating argument

1 Introduction and preliminaries

In this paper, by continuation of the monotone iterative technique interval by interval called piecewise monotone iterative technique we consider the initial value problems (IVP) for first-order nonlinear impulsive integro-differential equations with deviating arguments in a real Banach space E:

$$\begin{cases} x'(t) = f(t, x(t), Tx(t), x(\tau(t))),\ t \in J,\ t \neq t_k, \\ \triangle x(t_k) = I_k(x(t_k)),\ \ k = 1, 2, \cdots, m, \\ x(0) = x_0^*. \end{cases} \tag{1}$$

where $f \in C[J \times E^3, E], \tau \in C[J, J], \tau(t) \leq t, J = [0, 1], 0 < t_1 < \cdots < t_m < 1; Tx(t) = \int_0^t k(t, s)x(s)ds$ is an integral operator with kernel $k \in C[D, R_+]$, $D = \{(t, s) \in J \times J \mid t \geq s\}$ and $R_+ = [0, +\infty)$; $I_k : E \to E$ is an increasing mapping, $k = 1, 2, \cdots, m$; $x_0^* \in E$; $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ denotes the jump of $x(t)$ at $t = t_k$, where $x(t_k^+)$ and $x(t_k^-)$ represent the right and left-hand limit of $x(t)$ at $t = t_k$, respectively. Set θ be the zero element of E, $J' = J \setminus \{t_1, t_2, \cdots, t_m\}$ and $J_0 = [0, t_1], J_k = [0, t_{k+1}], J'_k = J_k \setminus \{t_1, t_2, \cdots, t_k\}, k = 1, 2, \cdots, m,\ t_{m+1} = 1$.

Impulsive differential equations, regarded as a kind of important and realistic mathematic models concerning in many evolution processes (see [1]), has attracted a great deal of attention in recent years. It is well known that the existence problem of solutions of impulsive differential equations is of fundamental importance. The monotone iterative technique coupled with the method of upper and lower solutions has been regarded as an effective method to deal with the existence of solutions for nonlinear differential equations in abstract spaces in recent years. For a small sample of such work, we refer the reader to works [2]-[9] and references therein. In those known research works, f does not contain deviating arguments and the impulse functions satisfy the continuity condition.

In this paper, by establishing the piecewise monotone iterative technique and comparison results, we obtain the existence theorems of extremal solutions and unique solution for IVP(1) without the continuity condition of impulse functions, which improve the results in [2],[3], [5]-[9].

Here we briefly recall some basic definitions and facts.

Recall that a nonempty closed convex set $P \subset E$ is a cone if it satisfies

$$x \in P, r \geq 0 \Rightarrow rx \in P \text{ and } x \in P, -x \in P \Rightarrow x = \theta.$$

* **Corresponding author**. *E-mail address*: wwxgg@126.com(W.Wang)

IJNS.2011.12.15/546

The Banach space E can be partially ordered by a cone P, that is, $x \leq y$ if and only if $y - x \in P$.

Recall that a cone P is said to be normal if there exists a positive number N, called the normal constant of P, such that $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$; and P is said to be regular if for any $\{x_n\} \subset E$ with $x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots \leq y$, there exists $x^* \in E$ such that $\|x_n - x^*\| \to 0$. If P is a regular cone then it is a normal cone.

Let $PC[J_k, E] = \{x : J_k \to E \mid x(t)$ is continuous at $t \neq t_i,$ left continuous at $t = t_i$and $x(t_i^+)$ exists, $i = 1, 2, \cdots, k\}$, then $PC[J_k, E]$ is a Banach space with norm $\|x\|_{PCk} = \sup\{\|x(t)\| \mid t \in J_k\}$. If P is a normal cone in E, then $P_k = \{x \in PC[J_k, E] \mid x(t) \geq \theta$ for $t \in J_k\}$ is a normal cone in $PC[J_k, E]$. For details on cone theory, see [10].

It is easy to see that $PC[J_0, E] = C[J_0, E]$ and $PC[J_m, E] = PC[J, E]$.

2 Some lemmas

We say $x \in PC[J, E] \cap C^1[J', E]$ is a solution of IVP(1), if it is satisfies IVP(1).

Set $k_0 = \max\{k(t, s) \mid t, s \in D\}$.

Lemma 1 *(Comparison result) Let P be a cone in E. Assume that $u \in PC[J, E] \cap C^1[J', E]$ satisfies*

$$\begin{cases} u'(t) \leq -M_1 u(t) - M_2 \int_0^t k(t,s)u(s)ds - M_3 u(\tau(t)),\ t \in J, t \neq t_k, \\ \triangle u(t_k) \leq \theta, \\ u(0) \leq \theta, \end{cases} \tag{2}$$

where M_1, M_2 and M_3 are nonnegative constants and satisfy one of the following inequalities

$$2M_1 + M_2 k_0 + 2M_3 \leq 2, \tag{3}$$

$$\frac{M_2 k_0 + M_3}{M_1}(e^{M_1} - 1) \leq 1, M_1 > 0. \tag{4}$$

Then $u(t) \leq \theta$ for $t \in J$.

Proof. Let $P^* = \{g \in E^* \mid g(x) \geq 0$ for $x \in P\}$, where E^* is the conjugate space of E.

Now, we prove the conclusion is true when (3) holds.

For any given $g \in P^*$, set $n(t) = g(u(t))$. Then $n \in C^1[J, R^1]$. From (2) we have

$$\begin{cases} n'(t) \leq -M_1 n(t) - M_2 \int_0^t k(t,s)n(s)ds - M_3 n(\tau(t)), t \neq t_k, t \in J, \\ \triangle n(t_k) \leq 0, \\ n(0) \leq 0. \end{cases} \tag{5}$$

We now claim that

$$n(t) \leq 0,\ \ t \in J. \tag{6}$$

Assume that there exists a $t^* \in J$ such that $n(t^*) > 0$. It is clear that $t^* > 0$. Let $\min\{n(t) \mid 0 \leq t \leq t^*\} = -n_s$. Then $n_s \geq 0$. If $n_s = 0$, then $n(t) \geq 0$ for $t \in [0, t^*]$. From (5), we get $n'(t) \leq 0,\ t \in [0, t^*]$. Thus, $n(t)$ is nonincreasing on $[0, t^*]$, which implies that $n(t^*) \leq n(0) \leq 0$. A contradiction to $n(t^*) > 0$. If $n_s > 0$, then there exists $t_* \in [0, t^*)$ such that $n(t_*) = -n_s$ or $n(t_*^+) = -n_s$. By (5) we have

$$\begin{aligned} 0 &< n(t^*) \leq n(t_*) - M_1 \int_{t_*}^{t^*} n(s)ds - M_2 k_0 \int_{t_*}^{t^*} \int_0^t n(s)dsdt - M_3 \int_{t_*}^{t^*} n(\tau(s))ds \\ &= -n_s + M_1 n_s + \frac{1}{2} M_2 k_0 m_s + M_3 n_s, \end{aligned}$$

which contradicts to (3). Hence, (6) holds, which implies that $u(t) \leq \theta$ for $t \in J$.

Next, we prove the conclusion is true when (4) holds.

Set $m(t) = e^{M_1 t} g(u(t))$. Then $m \in C^1[J, R^1]$. From (2) we have

$$\begin{cases} m'(t) \leq -M_2 \int_0^t e^{M_1(t-s)} k(t,s)m(s)ds - M_3 e^{M_1(t-\tau(t))} m(\tau(t)), t \in (t_k, t_{k+1}], \\ \triangle m(t_k) \leq 0, \\ m(0) \leq 0. \end{cases} \tag{7}$$

We now show that

$$m(t) \le 0, \ \ t \in J_k. \tag{8}$$

Assume that there exists a $t^* \in J$ such that $m(t^*) > 0$ and let $\min\{m(t) \mid 0 \le t \le t^*\} = -m_s$. Then $m_s \ge 0$. If $m_s = 0$, then $m(t) \ge 0$ for $t \in [0, t^*]$. From (7), we get $m'(t) \le 0, t \in [0, t^*]$. Thus, $m(t)$ is nonincreasing on $[0, t^*]$, which implies that $m(t^*) \le m(0) \le 0$. A contradiction to $m(t^*) > 0$. If $m_s > 0$, then there exists $t_* \in [0, t^*)$ such that $m(t_*) = -m_s$ or $m(t_*^+) = -m_s$. By (7) we have

$$\begin{aligned} 0 \quad &< \quad m(t^*) \le m(t_*) + m_s M_2 k_0 \int_{t_*}^{t^*} \int_0^t e^{M_1(t-s)} ds dt + M_3 m_s \int_{t_*}^{t^*} e^{M_1(t-\tau(t))} dt, \\ &\le \quad -m_s + m_s(M_2k_0 + M_3)\frac{e^{M_1} - 1}{M_1}, \end{aligned}$$

which contradicts to (4). Hence, (8) holds, which implies that $u(t) \le \theta$ for $t \in J$. The proof is complete. ■

Similarly, we can obtain the following comparison result.

Lemma 2 *(Comparison result) Let P be a cone in E and $k \in \{0, 1, 2, \cdots, m\}$. Assume that $u \in PC[J_k, E] \cap C^1[J_k', E]$ satisfies*

$$\begin{cases} u'(t) \le -M_1 u(t) - M_2 \int_0^t k(t,s)u(s)ds - M_3 u(\tau(t)), \ t \in (t_k, t_{k+1}], \\ u(t) = \theta, \ t \in J_{k-1}, \\ u_(t_k^+) \le \theta, \end{cases} \tag{9}$$

where M_1, M_2 and M_3 are nonnegative constants and satisfy one of the inequalities (3) and (4). Then $u(t) \le \theta$ for $t \in J_k$.

Lemma 3 *[10](Ascoli-Arzela) Assume that $U \subset C[J_0, E]$ is relatively compact if and only if U is equicontinuous and $U(t) \subset E$ is a relatively compact for each $t \in J_0$, where $U(t) = \{u(t) \in E \mid u \in U\}, t \in J_0$.*

Lemma 4 *[11]. Let $k \in \{1, 2, \cdots, m\}$. Assume that $U \subset PC[J_k, E]$ is bounded and equicontinuous on each $[0, t_1], (t_1, t_2], \cdots, (t_k, t_{k+1}]$. Then $\alpha(U) = \alpha(U(J_k)) = \sup\{\alpha(U(t)) \mid t \in J_k\}$, where α denotes Kuratowski noncompactness measure, $U(J_k) = \{u(t) \in E \mid u \in U, t \in J_k\}, U(t) = \{u(t) \in E \mid u \in U\}, t \in J_k$.*

Set $[y, z] = \{x \in PC[J, E] \Big| y(t) \le x(t) \le z(t), t \in J\}, y, z \in PC[J, E]$.

3 Main results

We call $x_0 \in PC[J_k, E] \cap C^1[J_k', E]$ is a lower solution of following IVP

$$\begin{cases} x'(t) = f(t, x(t), Tx(t), x(\tau(t))), \ \ t \in J_k, t \ne t_i, \\ \triangle x(t_i) = I_i(x(t_i)), \ \ i = 1, 2, \cdots, k, \\ x(0) \le x_0^*, \end{cases} \tag{10}$$

if it satisfies

$$\begin{cases} x_0'(t) \le f(t, x_0(t), Tx_0(t), x_0(\tau(t))), \ t \in J_k, t \ne t_i, \\ \triangle u_0(t_k) \le I_k(u_0(t_k)), \ \ i = 1, 2, \cdots, k, \\ u_0(0) \le x_0^*, \end{cases} \tag{11}$$

and $y_0 \in PC[J_k, E] \cap C^1[J_k', E]$ is an upper solution of IVP(10), if it satisfies (11) with inverse inequalities.

We will use the following conditions in this section:

(H1) $IVP(1)$ has a lower solutions $u_0 \in PC[J, E] \cap C^1[J', E]$ and a upper solutions $v_0 \in PC[J, E] \cap C^1[J', E]$ satisfying $u_0(t) \le v_0(t), t \in J$.

(H2) there exist nonnegative constants M_1, M_2 and M_3 such that

$$f(x, y, z) - f(\overline{x}, \overline{y}, \overline{z}) \ge -M_1(x - \overline{x}) - M_2(y - \overline{y}) - M_3(z - \overline{z}), \tag{12}$$

for any $t \in J, u_0(t) \le \overline{y} \le y \le v_0(t), \ Tu_0(t) \le \overline{y} \le y \le Tv_0(t), u_0(\tau(t)) \le \overline{z} \le z \le v_0(\tau(t))$.

(H3) nonnegative constants M_1, M_2 and M_3 satisfy one of the following inequalities

$$2M_1 + M_2k_0 + 2M_3 < 2, \tag{13}$$

$$\frac{M_2k_0+M_3}{M_1}(e^{M_1}-1)<1, M_1>0. \tag{14}$$

(H4) there exist nonnegative constants $\overline{M}_1, \overline{M}_2$ and $\overline{M}_3$ such that

$$f(t,x,y,z)-f(t,\overline{x},\overline{y},\overline{z})\le \overline{M}_1(x-\overline{x})+\overline{M}_2(y-\overline{y})+\overline{M}_3(z-\overline{z}) \tag{15}$$

for any $t\in J,\ u_0(t)\le \overline{y}\le y\le v_0(t),\ Tu_0(t)\le\overline{u}\le u\le Tv_0(t), u_0(\tau(t))\le\overline{z}\le z\le v_0(\tau(t))$.

Theorem 5 *Let P be a regular cone in E. Assume that the conditions (H1),(H2) and (H3) hold. Then*

(i) IVP(1) has a solutions $u^*\in PC[J,E]\cap C^1[J',E]$ in $[u_0,v_0]$, and there exists a nondecreasing sequence $\{u_n\}\in PC[J,E]\cap C^1[J',E]$ in $[u_0,v_0]$ uniformly converging to u^* ;

(ii) IVP(1) has a solutions $v^*\in PC[J,E]\cap C^1[J',E]$ in $[u_0,v_0]$, and there exists a nonincreasing sequence $\{v_n\}\in PC[J,E]\cap C^1[J',E]$ in $[u_0,v_0]$ uniformly converging to v^*.

(iii) u^* and v^* are the minimal and maximal solutions for IVP(1) in $[u_0,v_0]$, respectively.

Proof. We only prove the conclusion when (14) holds, the another case can be verified similarly.

For any $\eta\in[u_0,v_0]$, from (H3) it is easy to see that the following linear problem

$$\begin{cases} u'(t)=f(t,\eta(t),T\eta(t),\eta(\tau(t)))-M_1\big(u(t)-\eta(t)\big)-M_2\big(Tu(t)-T\eta(t)\big)\\ \qquad -M_3(u(\tau(t))-\eta(\tau(t))), t\in J, t\neq t_k,\\ \triangle u\mid_{t=t_k}=I_k(\eta(t_k)),\ \ k=1,2,\cdots,m,\\ u(0)=x_0^* \end{cases} \tag{16}$$

has a unique solution $u\in PC[J,E]\cap C^1[J',E]$ satisfying

$$\begin{aligned} u(t)=x_0^*e^{-M_1t}+\textstyle\int_0^t e^{-M_1(t-s)}\Big[f(s,\eta(s),T\eta(s),\eta(\tau(s))+M_1\eta(s)-M_2(Tu(s)-T\eta(s))\\ -M_3(u(\tau(s))-\eta(\tau(s))\Big]ds+\sum_{0<t_k<t}e^{-M_1(t-t_k)}I_k(\eta(t_k)). \end{aligned} \tag{17}$$

Set

$$\begin{aligned} u_n(t)=x_0^*e^{-M_1t}+\textstyle\int_0^t e^{-M_1(t-s)}\Big[f(s,u_{n-1}(s),Tu_{n-1}(s),u_{n-1}(\tau(s))+M_1u_{n-1}(s)\\ -M_2(Tu_n(s)-Tu_{n-1}(s))-M_3(u_n(\tau(s)-u_{n-1}(\tau(s))\Big]ds\\ +\sum_{0<t_k<t}e^{-M_1(t-t_k)}I_k(u_{n-1}(t_k)), t\in J, n=1,2,\cdots, \end{aligned} \tag{18}$$

$$\begin{aligned} v_n(t)=x_0^*e^{-M_1t}+\textstyle\int_0^t e^{-M_1(t-s)}\Big[f(s,v_{n-1}(s),Tv_{n-1}(s),v_{n-1}(\tau(s))+M_1v_{n-1}(s)\\ -M_2(Tv_n(s)-Tv_{n-1}(s))-M_3(v_n(\tau(s)-v_{n-1}(\tau(s))\Big]ds\\ +\sum_{0<t_k<t}e^{-M_1(t-t_k)}I_k(v_{n-1}(t_k)), t\in J, n=1,2,\cdots. \end{aligned} \tag{19}$$

Let $p_1=u_0-u_1, p_2=v_1-v_0$ and $p=u_1-v_1$, then, from (H1) and (H2) it is easy to check that p_1,p_2 and p satisfy (2). Thus, Lemma 2.1 implies that $u_0\le u_1\le v_1\le v_0$. By induction, we obtain that

$$u_0(t)\le u_1(t)\le\cdots\le u_n(t)\le\cdots\le v_n(t)\le\cdots\cdots v_1(t)\le v_0(t), t\in J. \tag{20}$$

To proof the conclusion (i), we firstly find a solution of IVP(1) defined on J_0. In this case, IVP(1) can be reduced to

$$\begin{cases} u'(t)=f(t,u(t),Tu(t),u(\tau(t))),\ \ t\in J_0,\\ u(0)=x_0^*. \end{cases} \tag{21}$$

Set $u_{00}=u_0|_{J_0}, u_{0n}=u_n|_{J_0}, v_{00}=v_0|_{J_0}$ and $v_{0n}=v_n|_{J_0}$, then $u_{0n},v_{0n}\in C[J_0,E]$ and (20) implies that

$$u_{00}(t)\le u_{01}(t)\le\cdots\le u_{0n}(t)\le\cdots\le v_{0n}(t)\le\cdots\le v_{01}(t)\le v_{00}(t),\ t\in J_0. \tag{22}$$

Consequently, the regularity of the cone P implies that there exist $u_0^*,v_0^*\in[u_{00},v_{00}]$ such that

$$\lim_{n\to\infty}u_{0n}(t)=u_0^*(t),\ \ \lim_{n\to\infty}v_{0n}(t)=v_0^*(t),\ t\in J_0, \tag{23}$$

and $\{u_{0n} \mid n = 1, 2, \cdots\}$ is a bounded subset in $C[J_0, E]$. Let $U_0 = \{u_{0n} \mid n = 1, 2, \cdots\}$ and $U_0(t) = \{u_{0n}(t) \mid n = 1, 2, \cdots\}, t \in J_0$. (23) implies that $U(t)$ is a relatively compact in E for each $t \in J_0$. In addition, by (H2) we have

$$\begin{aligned} & f(t, u_{00}(t), Tu_{00}(t), u_{00}(\tau(t))) + M_1u_{00}(t) + M_2Tu_{00}(t) + M_3u_{00}(\tau(t)) \\ \leq\ & f(t, u_{0n}(t), Tu_{0n}(t), u_{0n}(\tau(t))) + M_1u_{0n}(t) + M_2Tu_{0n}(t) + M_3u_{0n}(\tau(t)) \\ \leq\ & f(t, v_{00}(t), Tv_{00}(t), v_{00}(\tau(t))) + M_1v_{00}(t) + M_2Tv_{00}(t) + M_3v_{00}(\tau(t)),\ \ t \in J_0. \end{aligned} \tag{24}$$

Combining the normality of the cone P_0 gives that $\{f(t, u_{0n}(t), Tu_{0n}(t), u_{0n}(\tau(t))) + M_1u_{0n}(t) + M_2Tu_{0n}(t) + M_3u_{0n}(\tau(t))|n = 1, 2, \cdots\}$ is uniformly bounded on J_0. Moreover, $\{u'_{0n} \mid n = 1, 2, \cdots\}$ is a bounded subset in $C[J_0, E]$. It follows by virtue of the mean value theorem that U_0 is equicontinuous on J_0.

By Lemma 2.3 U_0 is relatively compact in $C[J_0, E]$, which means that there exists a subsequence of $\{u_{0n}\}$ which uniformly converges to some $u_0^* \in C[J_0, E]$ on J_0. Since P is regular, we see from (22) that $\{u_{0n}(t)\}$ itself uniformly converges to $u_0^*(t)$ on J_0. Thus, for any $t \in J_0$ we have

$$\begin{aligned} & f(t, u_{0(n-1)}(t), Tu_{0(n-1)}(t), u_{0(n-1)}(\tau(t))) + M_1u_{0(n-1)}(t)) - M_2\big(Tu_{0n}(t) - Tu_{0(n-1)}(t)\big) \\ & -M_3(u_{0n}(\tau(t)) - Tu_{0(n-1)}(\tau(t))) \longrightarrow f(t, u_0^*(t), Tu_0^*(t), u_)^*(\tau(t))) + M_1u_0^*(t)\ (as\ n \to \infty). \end{aligned} \tag{25}$$

From (24) and (25), we can pass a limitation $n \to \infty$ to both sides of (18) and get

$$u_0^*(t) = x_0^*e^{-M_1t} + \int_0^t e^{-M_1(t-s)}\big[f(t, u_0^*(t), Tu_0^*(t), u_0^*(\tau(t))) + M_1u_0^*(t)\big]ds, t \in J_0. \tag{26}$$

It is evident that $u_0^* \in C[J_0, E] \cap C^1[J_0, E]$ is a solution of IVP(21) in $[u_{00}, v_{00}]$. Similarly, we can show that $\{v_{0n}\}$ uniformly converges to a solution of IVP(21) $v_0^* \in [u_{00}, v_{00}]$ on J_0. And (22) implies that

$$u_{00}(t) \leq u_{01}(t) \leq \cdots \leq u_{0n}(t) \leq \cdots \leq u_0^*(t) \leq v_0^*(t) \leq \cdots \leq v_{0n}(t) \leq \cdots \leq v_{01}(t) \leq v_{00}(t), t \in J_0. \tag{27}$$

It is easy to see that $u_0^*, v_0^* \in C[J_0, E] \cap C^1[J_0, E]$ are the minimal and maximal solutions of IVP(21) in $[u_{00}, v_{00}]$, respectively.

Second, we want to find solutions of IVP(1) defined on J. Consider the following system:

$$\begin{cases} u(t) = u_0^*(t), t \in J_0, \\ u'(t) = f(t, u(t), Tu(t), u(\tau(t))),\ t \in (t_1, t_2], \\ u(t_1^+) = x_1^*, \end{cases} \tag{28}$$

where $x_1^* = u_0^*(t_1) + I_1(u_0^*(t_1))$.

Set

$$u_{10}(t) = \begin{cases} u_0^*(t), t \in J_0, \\ u_0(t), t \in (t_1, t_2], \end{cases} \quad w_{10}(t) = \begin{cases} u_0^*(t), t \in J_0, \\ v_0(t), t \in (t_1, t_2]. \end{cases}$$

Then $u_{10}, w_{10} \in PC[J_1, E] \cap C^1[J'_1, E]$ and $u_0(t) \leq u_{10}(t) \leq w_{10}(t) \leq v_0(t)$ for $t \in [0, t_2]$.

Let

$$\eta_1(t) = \begin{cases} u_0^*(t), t \in J_0, \\ \eta(t), t \in (t_1, t_2], \end{cases}$$

then $\eta_1 \in [u_{10}, w_{10}] \subset PC[J_1, E]$. Consider the following problem

$$\begin{cases} u(t) = u_0^*(t), t \in J_0, \\ u'(t) = f(t, \eta_1(t), T\eta_1(t), \eta_1(\tau(t))) - M_1[u(t) - \eta_1(t)] - M_2(Tu(t) - T\eta_1(t)) \\ \qquad\quad -M_3[u(\tau(t)) - \eta_1(\tau(t))],\ \ t \in (t_1, t_2], \\ u(t_1) = x_1^*. \end{cases} \tag{29}$$

It is evident that (29) has a unique solution $u \in PC[J_1, E] \cap C^1[J'_1, E]$ satisfying

$$u(t) = \begin{cases} u_0^*(t),\ \ t \in J_0, \\ x_1^*e^{-M_1(t-t_1)} + \int_{t_1}^t e^{-M(t-s)}\Big[f(s, \eta_1(s), T\eta_1(s), \eta_1(\tau(s))) + M_1\eta_1(s) \\ \quad -M_2(Tu(s) - T\eta_1(s)) - M_3(u(\tau(s)) - \eta_1(\tau(s)))\Big]ds,\ \ t \in (t_1, t_2]. \end{cases} \tag{30}$$

Set

$$u_{1n}(t)=\begin{cases} u_0^*(t),\ \ t\in J_0,\\ x_1^*e^{-M_1(t-t_1)}+\int_{t_1}^t e^{-M_1(t-s)}\Big[f(s,u_{1(n-1)}(s),Tu_{1(n-1)}(s),u_{1(n-1)}(\tau(s)))\\ \quad +M_1u_{1(n-1)}(s)-M_2(Tu_{1n}(s)-Tu_{1(n-1)}(s))-M_3(u_{1n}(\tau(s))\\ \quad -u_{1(n-1)}(\tau(s)))\Big]ds,\ \ t\in(t_1,t_2], n=1,2,\cdots. \end{cases} \tag{31}$$

Let $p_{11}=u_{10}-u_{11}$, then $p_{11}(t)=0$ for $t\in J_0$. From (20) we get $p_{11}(t)=u_0(t)-u_1(t)+u_1(t)-u_{11}(t)\le u_1(t)-u_{11}(t)$ for $t\in(t_1,t_2]$. Let $q(t)=u_1(t)-u_{11}(t), t\in(t_1,t_2]$. From (18),(31) and (H2) it is not difficult to find that $q(t)$ satisfies (9). By Lemma 2.2 we have $q(t)\le 0$ for $t\in(t_1,t_2]$. Moreover, we obtain that $p_{11}(t)\le 0$ for $t\in(t_1,t_2]$. Thus, we conclude that

$$u_{10}(t)\le u_{11}(t),\qquad t\in J_1. \tag{32}$$

Similar argument shows that $u_{11}(t)\le w_{10}(t)$ for $t\in J_1$. By induction, we obtain that

$$u_{10}(t)\le\cdots\le u_{1n}(t)\le\cdots\le w_{10}(t),\ t\in J_1. \tag{33}$$

Similar to the above proof, we get that the sequence $\{u_{1n}\}$ defined by (31) uniformly converges to some $u_1^*\in PC[J_1,E]\cap C^1[J_1',E]$ on $[0,t_2]$ satisfying

$$u_1^*(t)=\begin{cases} u_0^*(t),\ \ t\in[0,t_1],\\ x_1^*e^{-M_1t}+\int_{t_1}^t e^{-M_1(t-s)}\big[f(t,u_1^*(s),Tu_1^*(s),u_1^*(\tau(s)))+M_1u_1^*(s)\big]ds, t\in J_1, \end{cases}, \tag{34}$$

which is the minimal solution of IVP(28) in $[u_{10},w_{10}]$.

Going this step successively we can show that the problem

$$\begin{cases} u(t)=u_{k-1}^*(t),\ t\in J_{k-1},\\ u'(t)=f(t,u(t),Tu(t),u(\tau(t))),\ t\in(t_k,t_{k+1}],\\ u(t_k^+)=x_k^*, \end{cases} \tag{35}$$

where $x_k^*=u_{k-1}^*(t_k)+I_k(u_{k-1}^*(t_k))$, has the minimal solution $u_k^*\in PC[J_k,E]\cap C^1[J_k',E]$ in $[u_{k0},w_{k0}]$ satisfying

$$u_k^*(t)=\begin{cases} u_{k-1}^*(t),\ \ t\in J_{k-1},\\ x_k^*e^{-M_1t}+\int_{t_k}^t e^{-M_1(t-s)}\big[f(s,u_k^*(s),Tu_k^*(s),u_k^*(\tau(s)))+M_1u_k^*(s)\big]ds,\ t\in(t_k,t_{k+1}], \end{cases}, \tag{36}$$

where $u_{k0}(t)=\begin{cases} u_{k-1}^*(t), t\in J_{k-1}\\ u_0(t), t\in(t_k,t_{k+1}]\end{cases}$, $w_{k0}(t)=\begin{cases} u_{k-1}^*(t), t\in J_{k-1}\\ v_0(t), t\in(t_k,t_{k+1}]\end{cases}$ and the sequence $\{u_{kn}\}$ defined by

$$u_{kn}(t)=\begin{cases} u_{k-1}^*(t),\ \ t\in J_{k-1},\\ x_k^*e^{-M_1(t-t_k)}+\int_{t_k}^t e^{-M_1(t-s)}\Big[f(s,u_{k(n-1)}(s),Tu_{k(n-1)}(s),u_{k(n-1)}(\tau(s)))+M_1u_{k(n-1)}(s)\\ \quad -M_2(Tu_{kn}(s)-Tu_{k(n-1)}(s))-M_3(u_{kn}(\tau(s))-u_{k(n-1)}(\tau(s)))\Big]ds,\ t\in(t_k,t_{k+1}], \end{cases} \tag{37}$$

$(n=1,2,\cdots)$ uniformly converges to u_k^* on $[0,t_{k+1}]$ satisfying

$$u_{k0}(t)\le\cdots\le u_{kn}(t)\le\cdots\le u_k^*(t)\le w_{k0}(t),\ t\in J_k, k=2,3,\cdots,m.$$

Set $u^*(t)=u_m^*(t), t\in J$ and

$$\overline{u}_n(t)=\begin{cases} u_{0n}(t),\ \ t\in J_0,\\ u_{1n}(t),\ \ t\in(t_1,t_2],\\ \quad\cdots\\ u_{mn}(t),\ \ t\in(t_m,1]. \end{cases} \tag{38}$$

Then we see from (36) that $u^*\in PC[J,E]\cap C^1[J',E]$ is a solution of IVP(1) in $[u_0,v_0]$ and $\{\overline{u}_n\}\subset PC[J,E]\cap C^1[J',E]$ in $[u_0,v_0]$ is a nondecreasing sequence and uniformly converges to u^*.

Conclusion (ii). Similar to the proof of the conclusion (i) we can show that there exists $v^*=v_m^*\in PC[J,E]\cap C^1[J',E]$ satisfying

$$v_m^*(t)=\begin{cases} v_{m-1}^*(t),\ \ t\in J_{m-1},\\ x_m^*e^{-M_1t}+\int_{t_m}^t e^{-M_1(t-s)}\big[f(s,v_m^*(s),Tv_m^*(s),v_m^*(\tau(s)))+M_1v_m^*(s)\big]dst\in(t_m,1], \end{cases}$$

which is a solution of IVP(1) in $[u_0, v_0]$. Set $\overline{v}_n(t) = \begin{cases} v_{0n}(t), \ \ t \in J_0, \\ v_{1n}(t), \ \ t \in (t_1, t_2], \\ \ \ \cdots \\ v_{mn}(t), \ \ t \in (t_m, 1], \end{cases}$

where

$$v_{kn}(t) = \begin{cases} v^*_{k-1}(t), \ \ t \in J_k, \\ x^*_k e^{-M(t-t_k)} + \int_{t_k}^t e^{-M(t-s)}\Big[f(s, v_{k(n-1)}(s), Tv_{k(n-1)}(s), v_{k(n-1)}(\tau(s))) + M_1 v_{k(n-1)}(s) \\ -M_2(Tv_{kn}(s) - Tv_{k(n-1)}(s)) - M_3(v_{kn}(\tau(s)) - v_{k(n-1)}(\tau(s)))\Big]ds, \ \ t \in (t_k, t_{k+1}], \end{cases}$$

then $\{\overline{v}_n\} \subset PC[J, E] \cap C^1[J', E]$ in $[u_0, v_0]$ is a nonincreasing sequence and uniformly converges to v^*.

Conclusion (iii). Let $\xi \in PC[J, E] \cap C^1[J', E]$ be a solution for IVP(1) in $[u_0, v_0]$. Set $\xi_{k-1} = \xi|_{[0,t_k]}, k = 1, 2, \cdots, m+1$, then $\xi_m = \xi$ and $\xi_0 \in C[J_0, E] \cap C^1[J_0, E], \xi_k \in PC[J_k, E] \cap C^1[J'_k, E], k = 1, 2, \cdots, m.$

If $t \in J_0$, then ξ_0 is the solution of (21). From the proof of the conclusion (i), it is obvious that $u^*_0(t) \le \xi_1(t) \le v^*_0(t)$ for $t \in J_0$.

If $t \in J_1$, then ξ_1 is the solution of the following equation

$$\begin{cases} u(t) = \xi_0(t), t \in J_0, \\ u'(t) = f(t, u(t), Tu(t), u(\tau(t))), \ t \in (t_1, t_2], \\ u(t_1^+) = \xi_0(t_1) + I_1(\xi_0(t_1)). \end{cases}$$

This means that

$$\xi_1(t) = \begin{cases} \xi_0(t), \ \ t \in J_0, \\ \Big[\xi_0(t_1) + I_1(\xi_0(t_1))\Big]e^{-M_1(t-t_1)} + \int_{t_1}^t e^{-M_1(t-s)}\Big[f(s, \xi_1(s), T\xi_1(s), \xi_1(\tau(s))) \\ +M_1\xi_1(s)\Big]ds, \ \ t \in (t_1, t_2], \end{cases} \tag{39}$$

Similar to the proof of (32), we can show from (34) and (39) that $u^*_1(t) \le \xi_1(t) \le v^*_1(t), t \in J_1$. Hence, it is successively shown that $u^*_k(t) \le \xi_k(t) \le v^*_k(t), t \in J_k$. Consequently, $u^*(t) \le \xi(t) \le v^*(t), t \in J$. The proof is complete. ■

We further consider the uniqueness of solution for IVP(1).

Theorem 6 *Let P be a normal cone with the normal constant N in E. Assume that (H1), (H2), (H3) and (H4) hold. Then IVP(1) possesses a unique solution $u^* \in PC[J, E] \cap C^1[J', E]$ in $[u_0, v_0]$, provided that*

$$N[2(M_1 + \overline{M}_1) + (M_2 + \overline{M}_2)k_2 + 2(M_3 + \overline{M}_3)] < 2. \tag{40}$$

Proof. To begin with, we prove that IVP(21) has a solution. By virtue of the proof of Theorem 3.1, we only need to prove that $\{u_{0n}(t)\}$ defined by Theorem 3.1 uniformly converges to some $u^*_0(t) \in C[J_0, E] \cap C^1[J_0, E]$ on J_0. From (H4) we have

$$\big\|u_{0(n+1)} - u_{0n}\big\|_{PC0} \le (NM^*)^n \big\|u_{01} - u_{00}\big\|_{PC0},$$

where $M^* = (M_1 + \overline{M}_1) + \frac{1}{2}(M_2 + \overline{M}_2)k_2 + (M_3 + \overline{M}_3)$. Consequently, for any $p = 1, 2, \cdots$ we have

$$\big\|u_{0(n+p)} - u_{0n}\big\|_{PC0} \le (NM^*)^n \Big[1 + NM^* + \cdots + (NM^*)^{(p-1)}\Big] \big\|u_{01} - u_{00}\big\|_{PC0}.$$

In combination with (40), this implies that $\{u_{0n}\}$ is a Cauchy sequence in $C[J_0, E]$ and there exists $u^*_0 \in C[J_0, E]$ such that $\{u_{0n}\}$ uniformly converges to u^*_0 on J_0 and satisfies (26). By the proof of Theorem 3.1, $u^*_0 \in C[J_0, E] \cap C^1[J_0, E]$ is a solution of IVP(21) in $[u_{00}, v_{00}]$.

Next, we prove that IVP(21) has only one solution $u^* \in C[J_0, E] \cap C^1[J_0, E]$ in $[u_{00}, v_{00}]$. Let $w^*_0 \in C[J_0, E] \cap C^1[J_0, E]$ is a solution of IVP(21) in $[u_{00}, v_{00}]$. It is evident that w^*_0 satisfies (26) and $u^*_0 \le w^*_0$. From (H4), we have

$$\begin{aligned} \theta \ \ \le \ \ & w^*_0(t) - u^*_0(t) \le \int_0^t \big[(M_1 + \overline{M}_1)(w^*_0(r) - u^*(r)) + \overline{M}_2 k_0 \int_0^s (w^*_0(\tau) - u^*(\tau))d\tau \\ & +\overline{M}_3(w^*_0(\tau(s)) - u^*(\tau(s))\big]ds. \end{aligned}$$

Thus, we obtain that $\|w^*_0 - u^*_0\| \le N\big[(M_1 + \overline{M}_1) + \frac{1}{2}\overline{M}_2 k_0 + \overline{M}_3\big]\|w^*_0 - u^*\|$, which implies that $w^*_0(t) = u^*_0(t), t \in J_0$.

Consider the system (28). Similarly the sequence $\{u_{1n}\} \subset PC[J_1, E]$ defined by (31) is a Cauchy sequence in $PC[J_1, E]$ which uniformly converges to $u_1^*(t)$ on $[0, t_2]$ and $u_1^* \in PC[J_1, E] \cap C^1[J_1', E]$ is a unique solution of IVP(28) in $[u_{10}, w_{10}] = [u_{10}, v_{10}]$. Hence, we can successively show that the system (35) has a unique solution $u_k^* \in PC[J_k, E] \cap C^1[J_k', E]$ in $[u_{k0}, w_{k0}]$, and the sequence $\{u_{kn}\}$ defined by (37) uniformly converges to u_k^* on $[0, t_{k+1}]$, $k = 2, 3, \cdots, m$. Let $u^*(t) = u_m^*(t), t \in J$. Then u^* is a unique solution for IVP(1) in $[u_0, v_0]$. The proof is complete. ■

Remark 7 *Theorem 3.1 and Theorem 3.2 obtained in this paper need not the continuity of impulsive functions I_k. Which can not be done by the classical monotone iterative method in [5], [6]-[9]. In addition, we extend the monotoneity condition used in [7]-[9]. Hence, Our results improve and extend the results in [5] and [7]-[8], even if f does not contain the deviating arguments.*

Remark 8 *If there is no any impulse in IVP(1), Theorem 3.1 is the main result in [3]. If there is neither an impulse nor an integral operator T in IVP(1), Theorem 3.1 is the main result in [2].*

Acknowledgments

This research was partially supported by the NNSF of China (10571078), the Science Foundation of Shanxi Province of China (2006011013) and the University Natural Science Research Develop Foundation of Shanxi Province of China (20111021).

References

[1] V. Lakshmikantham, D.D. Bainov and P.S. Simeonov, Theory of Impulsive Differential Equation. World Scientific, Singapore.1989.

[2] S.Du and V.Lakshmikantham, Monotone iterative technique for differential equation in a Banach space. *J. Math. Anal. Appl.* 89(1982):454-459.

[3] D. Guo, Initial value problems for integro-differential equations of Volterra type in a Banach space, *Appl. Math. Stochastic and Anal.* 143(1994):13-23.

[4] L.Liu, Iterative method for solutions and coupled quasi-solution for nonlinear integro-differential equations of Mixed type in Banach space, *Nonlinear Anal.* 42(2000):583-598.

[5] D. Guo and X. Liu, Extremal solutions of nonlinear impulsive integro-differential equation in Banach space, *J. Anal. Math. Anal.* 177(1993):538-553.

[6] X. Liu and D.J. Guo, Initial value problems for first-order impulsive integro-differential equation in Banach space, *Comm. Appl. Nonlinear Anal* 2(1995):65-83.

[7] S. Xie and Z. Yang, Solvability of nonlinear impulsive Volterra integral equations and integro-differential equations in Banch spaces, *Acta Mathematica Sinica*, 46(2003):445-452.

[8] Y.X. Li and Z. Liu, Monotone iterative technique for addressing impulsive Integro-differential Equation in Banach space, *Nonlinear Anal.* 66(2007):83-92.

[9] S.Zhang and F.Wang, On initial value problems for first order impulsive differential systems, *Appl. Math. Mech.* 19(4)(1998):303-308.

[10] D. Guo, Monotone iterative method in nonlinear analysis. Shandong Scientific And Technical Publishers,Jinan. 2000.

[11] D.Guo, Existence of solutions of boundary value problems for nonlinear second order impulsive differential equations in a Banach space, *J. Math. Anal. Appl.* 181(1994):233-249.