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The Geodesic Deformation Retract of Klein Bottle and Its Folding

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Abstract: In this paper we discuss the retraction and the deformation retract of Klein bottle into itself and onto a geodesics in Klein bottle. The retraction and the deformation retract of elastic Klein bottle into itself and onto a geodesic will be obtained by using Lagrangian equations. We will present the isometric and topological folding in each case and the relations between the deformation retracts after and before folding have been deduced. Theorems governing this connection are achived.

Keywords: elastic Klein bottle; geodesics; folding

1 Introduction

As is well known, the theory of retractions is always one of interesting topics in Euclidian and Non-Euclidian space and it has been investigated from the various viewpoints by many branches of topology and differential geometry [2,3, 4, 5,6,8,14,17].

El-Ahmady [1,2,3,4,5,6,7,8,9,10] studied the variation of the density function on chaotic spheres in chaotic space-like Minkowski space time, folding of fuzzhypertori and their retractions , limits of fuzzy retractions of fuzzy hyperspheres and their foldings, fuzzy folding of fuzzy horocycle, fuzzy Lobachevskian space and its folding, the deformation retract and topological folding of Buchdahi space, retraction of chaotic Ricci space, a calculation of geodesics in chaotic flat space and its folding, fuzzy deformation retract of fuzzy horospheres and on fuzzy spheres in fuzzy Minkowski space .

Various folding problems arising in physics and chemistry, the membrance of a plant and animal is a type of folding, also the polymers considered as a folding and the lattice folding of a manifold, also many applications are wedden by P. DI-Francesco [16].]. In this article we will approach the elastic Klein bottle. The geodesic deformation retracts of the elastic Klein bottle are discussed. The relations between folding, unfolding and retraction of the elastic Klein bottle are discussed.

2 Definitions

A subset A of a topological space X is called a retract of X if there exists a continuous map $r: \ X \longrightarrow A$such that :

(i) X is open,

(ii) $r(a) = a \ \ \forall \ a \in A$ [1,2,5,6,7,11,12,13,14,17]

2- A subset A of a topological space X is a deformation retracts of X if there exists a retraction $r : X \longrightarrow A$ and a homotopy $\varphi : X \times I \longrightarrow X$ such that:

$$\left.\begin{array}{l}\varphi(x,0) = x \\ \varphi(x,1) = r(x)\end{array}\right\} \quad x \in X$$

$\varphi(a,t) = a, \ \ a \in A, \ \ t \in [0,1]$[1,5,6,11,13,14] .

3- A map $\Im : M \longrightarrow N$, where M and N are c^{∞}–Riemannian manifolds of dimension m, n respectively is said to be an isometric folding of M into N, iff for any piecewise geodesic path $\gamma : J \longrightarrow M$, the induced path $\Im \circ \gamma : J \longrightarrow N$

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is a piecewise geodesic and of the same length as γ[4,5,9]. If $\Im$does not preserve length, then $\Im$ is a topological folding [5,6,10,11].

4- The continuous map $f : M \longrightarrow N$ is said to be an unfolding of M into N if and only if $\forall\ x, y \in M,\ d(x, y) < d(\ f(x), f(y)\ )$ [9].

5- The elastic manifold M_e is a manifold Mattached with e, e is the coefficient of elasticity, i.e. $M_e = (M, e),\ e \in [0, 1]$. If $e = 0$, then $M_0 = (M, 0) = M$, the usual manifold, and $(M, 1)$ is the complete manifold also for an elastic manifold (M, e), the distance $d(x, y)$ between any two points $x, y \in (M, e)$ is not constant [15].

3 The main results

To obtain the main results, we will introduce the following definition :

The Klein bottle K can be realized as a parametric surface in R^4. At each point of the circle of radius $a$in the xyplane there is now available a three-dimensional hyperplane in R^4 perpendicular to the circle. A smaller circle of radius $b < a$ can be rotated about a diameter at half the rate of revolution about the circle of radius a, giving a Klein bottle [13,15,18]. The parameterization is given analytically as follows:

$$\begin{aligned} x &= (a + b\sin v)\cos u, \qquad (1)\\ y &= (a + b\sin v)\sin u,\\ z &= b\cos\ v\ \cos\ u/2,\\ w &= b\cos\ v\ \ \sin u/2. \end{aligned}$$

Points in the uvplane which are identified as indicated in the figure 1 are mapped into the same points in R^4 by these equations.

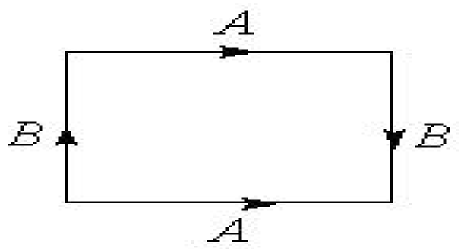


Figure 1

Theorem 1 *The geodesic of the Klein bottle are represented as hyper Klein bottle, hypersphere, hypersurface and great circle.*

Proof. In order to find a geodesic which is a subset of the Klein bottle we use Lagrangian equations

$$\frac{d}{ds}\left(\frac{\partial T}{\partial \psi_i'}\right) - \left(\frac{\partial T}{\partial \psi_i}\right) = 0, \quad i = 1, 2$$

Since $T = \frac{1}{2}\overline{ds}^2 = \frac{1}{2}\left[\ b^2 v'^2 + (\ (a + b\sin v)^2 + \frac{1}{4}b^2\cos^2 v\ )\ \ u'^2\ \right]$

Then the Lagrangian equation for Klein bottle are:

$$\frac{d}{ds}(b^2 v') - (\ (a + b\sin v)b\cos v - \frac{1}{4}b^2\sin v\cos v\ )\,u'^2 = 0 \qquad (2)$$

$$\frac{d}{ds}\ (\ (a + b\sin v)^2 + \frac{1}{4}b^2\cos^2 v\ )\,u' = 0 \qquad (3)$$

From equation (3) we get $[\ (a + b\sin v)^2 + \frac{1}{4}b^2\cos^2 v\ )\,u' = \text{constant} = \alpha$.

(I) For $\alpha = 0$ we have $[\ (a + b\sin v)^2 + \frac{1}{4}b^2\cos^2 v\ )\,u' = 0$.

Hence u' =constant, is a retraction. In the special case where $u = 0$, we obtain from equation (3.1) that the retraction r_1 is :

$$r_1 = \{\ (a + b\sin v),\ 0,\ b\cos v,\ 0\ \}.$$

which is the hyper Klein bottle $K_1 \subset K$ which is a geodesic.

If $a = 0$, hence the retraction of Klein bottle is defined by

$$x = b \sin v \cos u,$$

$$y = b \sin v \ \sin u,$$

$$z = b \cos \ v \ \cos \ u/2,$$

$$w = b \cos \ v \ \sin u/2,$$

which is the hypersphere $S_1^3 \subset K,\ x^2 + y^2 + z^2 + w^2 = b^2$ which is a geodesic.

If $b = 0$, then we obtain the following geodesic $S^1 \subset K$ given by :

$$x = a \cos u, \quad y = a \ \sin u, \quad z = w = 0,$$

where $x^2 + y^2 + z^2 + w^2 = a^2$. which is a retraction

(VI) If $a = b = 0$, hence we get the retraction of Klein bottle K is defined by :

$$x = y = z = w = 0,$$

which is the null hypersurface $K_2 \subset K$, $x^2 + y^2 + z^2 + w^2 = 0$ which is a geodesic.

Under the condition $u = 0$, equation (2) becomes $b^2 v' =$ constant, say β_2 if $\beta_2 = 0$, we have $b^2 v' = 0$. Now, if $b \neq 0$, then $v' = 0, v = \beta_3$.

(V) If $\beta_3 = 0$, then we obtain the following geodesic $S_2^3 \subset K$ given by :

$x = a \ \cos u, \quad y = a \ \sin u, \quad z = b \ \cos \ u/2, \quad w = b \ \sin u/2$.where $x^2 + y^2 + z^2 + w^2 = a^2 + b^2$. which is a retraction.

The deformation retract of Klein bottle K is given by :

$\varphi : (K - \{\beta_i\}) \times I \longrightarrow (K - \{\beta_i\})$,where I is the closed interval $[0, 1]$ and $\{\beta_i\}$ are any antipodal points. Thus the retraction of the Klein bottle K is obtained as :

$$R : (K - \{\beta_i\}) \longrightarrow K_1,\ S_1^3,\ S^1,\ K_2,\ S_2^3.$$

The deformation retract of (3.1) into a geodesic $K_1 \subset K$ is :

$$\varphi(m, t) = ((\ (a + b \sin v) \cos u, (a + b \sin v) \sin u, b \cos v \cos u/2, b \cos v \sin u/2\ )$$
$$-\{\beta_i\}\ )\,(1 - t) + t\,(\ (a + b \sin v), 0, b \cos v, 0\ )\,,$$

where

$\varphi(m, 0) = ((\ (a + b \sin v) \cos u, (a + b \sin v) \sin u, b \cos v \cos u/2, b \cos v \sin u/2\ ) - \{\beta_i\}\ )$and

$$\varphi(m, 1) = (\ (a + b \sin v), 0, b \cos v, 0\ )\,.$$

The deformation retract of (3.1) into a geodesic $S_1^3 \subset K$ is :

$$\varphi(m, t) = ((\ (a + b \sin v) \cos u, (a + b \sin v) \sin u, b \cos v \cos u/2, b \cos v \sin u/2\ )$$
$$-\{\beta_i\}\ )\,(1 - t) + t\,(\ b \sin v \cos u, b \sin v \sin u, b \cos v \cos u/2, b \cos v \sin u/2\ )$$

The deformation retract of (3.1) into a retraction $S^1 \subset K$ is given by:

$$\varphi(m, t) = ((\ (a + b \sin v) \cos u, (a + b \sin v) \sin u, b \cos v \cos u/2, b \cos v \sin u/2\ )$$
$$-\{\beta_i\}\ )\,(1 - t) + t\,(\ a \cos u, a \sin u, 0, 0\ )$$

The deformation retract of (3.1) into a retraction $K_2 \subset K$ is :

$$\varphi(m, t) = (1 - t)\ \ ((\ (a + b \sin v) \cos u, (a + b \sin v) \sin u, b \cos v \cos u/2, b \cos v \sin u/2\ )$$
$$-\{\beta_i\}\ ) + t\,(\ 0, 0, 0, 0\ )\,.$$

The deformation retract of (3.1) into a retraction $S_2^3 \subset K$ is defined by :

$$\varphi(m,t) = (1-t)\ ((\ (a+b\sin v)\cos u, (a+b\sin v)\sin u, b\cos v\cos u/2, b\cos v\sin u/2\ )$$

$$-\{\beta_i\}\ ) + t\ (\ a\cos u, a\sin u, b\cos u/2, b\sin u/2\ )\ .$$

Now, we are going to discuss the folding $\Im$ of the Klein bottle K. Let $\Im : K \longrightarrow K$,where,

$$\Im(x_1, x_2, x_3, x_4) = (x_1, x_2, |x_3|, x_4) \tag{4}$$

be an isometric folding $\Im$ of the Klein bottle K into itself given by:

$$\Im : ((\ (a+b\sin v)\cos u, (a+b\sin v)\sin u, b\cos v\cos u/2, b\cos v\sin u/2\ ) - \{\beta_i\}\ ) \longrightarrow$$

$$((\ (a+b\sin v)\cos u, (a+b\sin v)\sin u, |b\cos v\cos u/2|, b\cos v\sin u/2\ ) - \{\beta_i\}\ )\ .$$

The deformation retract of the folded Klein bottle $\Im(K)$ into the folded geodesic $\Im(K_1)$ is :

$$\begin{aligned}\varphi_\Im &: (((a+b\sin v)\cos u, (a+b\sin v)\sin u, |b\cos v\cos u/2|, b\cos v\sin u/2.) - \{\beta_i\}.) \times I \\ &\longrightarrow (((a+b\sin v)\cos u, (a+b\sin v)\sin u, |b\cos v\cos u/2|, b\cos v\sin u/2.) - \{\beta_i\}.).'\end{aligned}$$

with

$$\varphi_\Im(m,t) = ((\ (a+b\sin v)\cos u, (a+b\sin v)\sin u, |b\cos v\cos u/2|, b\cos v\sin u/2\ )$$

$$-\{\beta_i\}\ )\,(1-t) + t\ (\ (a+b\sin v, 0, |b\cos v|, 0\ )\ .,$$

where

$$\varphi_\Im(m,0) = ((\ (a+b\sin v)\cos u, (a+b\sin v)\sin u, |b\cos v\cos u/2|, b\cos v\sin u/2\ ) - \{\beta_i\}\ )\ ,$$

and $\varphi_\Im(m,1) = ((\ (a+b\sin v), 0, |b\cos v|, 0\ ).$

The deformation retract of the folded Klein bottle $\Im(K)$ into the folded geodesic $\Im(S_1^3)$ is given by :

$$\varphi_\Im(m,t) = (1-t)\,((\ (a+b\sin v)\cos u, (a+b\sin v)\sin u, |b\cos v\cos u/2|,$$

$$b\cos v\sin u/2\ ) - \{\beta_i\}\ ) + t(b\sin v\ \ \cos u, b\sin v\ \ \sin u, |b\cos v\cos u/2|,$$

$$b\cos v\sin u/2).$$

The deformation retract of the folded Klein bottle $\Im(K)$into the folded retraction $\Im(S_1)$ is :

$$\varphi_\Im(m,t) = (1-t)\,((\ (a+b\sin v)\cos u, (a+b\sin v)\sin u, |b\cos v\cos u/2|,$$

$$b\cos v\sin u/2\ ) - \{\beta_i\}\ ) + t(a\ \cos u, a\ \sin u, 0, 0).$$

The deformation retract of the folded Klein bottle $\Im(K)$ into the folded retraction $\Im(K_2)$ is given by :

$$\varphi_\Im(m,t) = (1-t\ )\ (\ (a+b\sin v)\cos u, (a+b\sin v)\sin u, |b\cos v\cos u/2|,$$

$$b\cos v\sin u/2\ )\quad - \{\beta_i\}\ ) + t(0,0,0,0).$$

The deformation retract of $\Im(K)$ into $\Im(S_2^3)$ is :

$$\varphi_\Im(m,t) = (1-t\ )\ (\ (a+b\sin v)\cos u, (a+b\sin v)\sin u, |b\cos v\cos u/2|,$$

$$b\cos v\sin u/2\ ) - \{\beta_i\}\ ) + t(a\cos u, a\ \sin u, |bv\cos u/2|, b\sin u/2)$$

■

Now, we are in a position to formulate the following theorems

Theorem 2 *Under the condition (3.4), the deformation retracts of the folded Klein bottle $\Im(K)$ into the folded geodesics $\Im(K_1)$,$\Im(S_1^3)$,$\Im(S_2^3)$ is different from the deformation retracts of K into K_1 or S_1^3 or S_2^3 respectively.*

Theorem 3 *The folding of $K(4)$ and any folding homeomorphic to the folding (3.4) have the same deformation retract of the Klein bottle into a retraction $S_1, K_2 \subset K$.*

Now, we will discuss the deformation retract of elastic Klein bottle $\underline{K}$by using Lagrangian equations. Now, consider that $r_1,\ r_2$ are variable and $r_1 = a,\ r_2 = b$. Hence (3.1) becomes :

$$\begin{aligned} x &= (r_1 + r_2 \sin v) \cos u, \\ y &= (r_1 + r_2 \sin v) \sin u, \\ z &= r_2 \cos\ v\ \cos\ u/2, \\ w &= r_2 \cos\ v\ \ \sin u/2. \end{aligned} \tag{5}$$

The metric of elastic Klein bottle is given by:

$$\overline{dS}^2 = \overline{dr}_1^2 + \overline{dr}_2^2 + \overline{dv}^2 + (\ (r_1 + r_2 \sin v)^2 + \frac{1}{4} r_2^2 \cos^2 v)\ \overline{du}^2 + \sin v\ dr_1 dr_2 \quad + r_2 \cos v\ dr_1 dv. \tag{6}$$

Theorem 4 *The geodesics of the elastic Klein bottle $\underline{K}$ are represented as hyper Klein bottle $\underline{K}_3, \underline{K}_4 \subset \underline{K}$, hyper sphere $\underline{S}^3 \subset \underline{K}$ and great circles $\underline{S}_1^1, \underline{S}_2^1 \subset \underline{K}$.*

Proof. Using Lagrangian equations to obtain a geodesics and retractions of elastic Klein bottle $\underline{K}$. From equation (6) we get:

$$T = \frac{1}{2}({r_1}'^2 + {r_2}'^2 + v'^2 + (\ (r_1 + r_2 \sin v)^2 + \frac{1}{4} {r_2}^2 \cos^2 v\ )\, u'^2 + \sin v {r_1}' {r_2}' + r_2 \cos v {r_1}' v'.$$

Then, the Lagrangian equations for elastic Klein bottle$\underline{K}$ are :

$$\frac{d}{ds}(r_1' + \frac{1}{2} r_2' \sin v + \frac{1}{2} r_2 \cos v v') - (r_1 + r_2 \sin v) u'^2 = 0 \tag{7}$$

$$\frac{d}{ds}(r_2' + \frac{1}{2} r_1' \sin v) - (\ (r_1 \sin v + r_2 \sin^2 v) + \frac{1}{4} r_2 \cos^2 v\ )\, u'^2 + \frac{1}{2} r' \cos v v' = 0 \tag{8}$$

$$\begin{aligned} \frac{d}{ds}(v' + \frac{1}{2} r_2 \cos v r_1') - (\ (r_1 r_2 \cos v + r_2^2 \sin v \cos v) - \frac{1}{4} r_2^2 \sin v \cos v\ )\, u'^2 \\ + \frac{1}{2} r_1' (r_2' \cos v - r_2 \sin v v') = 0 \end{aligned} \tag{9}$$

$$\frac{d}{ds}\ (\ (r_1 + r_2 \sin v)^2 + \frac{1}{4} r_2^2 \cos^2 v\ )\, u' = 0 \tag{10}$$

Solving equation (7) implies

$$\frac{d}{ds}(r_1' + \frac{1}{2} r_2' \sin v + \frac{1}{2} r_2 \cos v v'\ ) = (r_1 + r_2 \sin v) u'^2,$$

consider the case $\frac{d}{ds}(r_1' + \frac{1}{2} r_2' \sin v + \frac{1}{2} r_2 \cos v v') = 0$, and $(r_1 + r_2 \sin v) u'^2 = 0$.

Then we are going to discuss the following cases :

(i) If $\frac{d}{ds}(r_1') = 0$, then $r_1 = ct + a$, where c and a are constant . If $c = 0$, then $r_1 = a$, which means that the deformation of the manifold is regular (the piecewise geodesic deformed into piecewise geodesic).

Now, if $c \neq 0$, then $r_1 = ct + a$, and the piecewise geodesic deformed into non-piecewise geodesic and the deformation of the elastic manifold is not regular and (3.5) becomes:

$$\begin{aligned} x &= (\ (ct + a) + r_2 \sin v\ ) \cos u, \\ y &= (\ (ct + a) + r_2 \sin v\ ) \sin u, \\ z &= r_2 \cos\ v\ \cos\ u/2, \\ w &= r_2 \cos\ v\ \ \sin u/2, \end{aligned}$$

which is the elastic hyper Klein bottle $\underline{K}_1 \subset \underline{K}$. It is not a geodesic.

(ii) If $\frac{d}{ds}(r_2' \sin v) = 0$, then $r_2 = dt + b$, also from equation (3.5) we have :

$$\begin{aligned} x &= (\ r_1 + (dt + b) \sin v\ ) \cos u, \\ y &= (\ r_1 + (dt + b) \sin v\ ) \sin u, \end{aligned}$$

$$z = (dt + b)\cos\ v\ \cos\ u/2,$$

$$w = (dt + b)\cos\ v\ \ \sin u/2.$$

which is the elastic hyper Klein bottle $\underline{K}_2 \subset \underline{K}$ which is not a geodesic.

Now, $\sin v = 0$ can be true only for $v = \pm K\pi,\ K = 1, 2, \cdots$ and (3.5) becomes :

$$x = r_1\cos u, \quad y = r_1\sin u, \quad z = r_2\ \cos\ u/2, \quad w = r_2\ \sin u/2,$$

which is the elastic hyper sphere $\underline{S}^3 \subset \underline{K},\ x^2 + y^2 + z^2 + w^2 = r_1^2 + r_2^2$ which is a geodesic.

(iii) If $\frac{d}{ds}(r_2\cos vv') = 0$, then $r_2\cos vv' =$ constant β, if $\beta = 0$, there are two geodesics in elastic Klein bottle $\underline{K}$ given by :

$$x = r_1\cos u, \quad y = r_1\sin u, \quad z = w = \ 0 \quad ,$$

which is the elastic great circle . $\underline{S}_1^1 \subset \underline{K},\ x^2 + y^2 + z^2 + w^2 = r^2$. Also,

$$x = (r_1 + r_2)\cos u, \quad y = (r_1 + r_2)\sin u, \quad z = w = \ 0,$$

which is the elastic great circle $\underline{S}_2^1 \subset \underline{K},\ x^2 + y^2 + z^2 + w^2 = (r_1 + r_2)^2$.

(iv) If $(r_1 + r_2\sin v)u'^2 = 0$, then u =constant γ, if $\gamma = 0$. Hence the coordinate of elastic Klein bottle are :

$$x = r_1 + r_2\sin v, \qquad y = 0, \qquad z = r_2\cos v, \qquad w = 0,$$

where $x^2 + y^2 + z^2 + w^2 = (r_1^2 + 2r_1r_2\sin v + r_2^2)$, which is the elastic hyper Klein bottle $\underline{K}_3 \subset \underline{K}$, it is a geodesic . Also, if $(r_1 + r_2\sin v) = 0$, then we obtain the following geodesic $\underline{K}_4 \subset \underline{K}$ given by:

$$x = 0,\ y = 0,\ z = r_2\cos v\cos u/2,\ w = r_2\cos v\sin u/2,$$

where $x^2 + y^2 + z^2 + w^2 = r_2^2\cos^2 v$. ■

Corollary 5 *Any geodesic in elastic Klein bottle can be considered as a deformation retract in elastic Klein bottle.*

Corollary 6 *The deformation retract of the elastic Klein bottle $\underline{K}_1 \subset \underline{K}$ or $\underline{K}_2 \subset \underline{K}$ are elastic hyper Klein bottle induce a topological folding of the elastic Klein bottle .*

The deformation retract of the elastic Klein bottle $\underline{K}$ may be defined as follows :

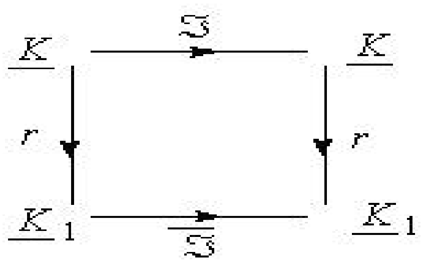
$$\varphi : (\underline{K} - \{\beta_i\}) \times I \longrightarrow (\underline{K} - \{\beta_i\}),$$

Also, the retraction of the elastic Klein bottle is defined as follows: $R : (\underline{K} - \{\beta_i\}) \longrightarrow \underline{K}_1$ or $\underline{K}_2$ or $\underline{S}^3$ or $\underline{S}_1^1$ or $\underline{S}_2^1$ or $\underline{K}_3$ or $\underline{K}_4$.

Now, under condition (3.4) and by conjugation we can obtain the same results as obtained above and we conclude that

Theorem 7 *The deformation retracts of the elastic Klein bottle is either a geodesic or not and its folding may be the deformation retracts or not.*

Corollary 8 *Let $\Im : \underline{K} \longrightarrow \underline{K}$ be a topological folding of elastic Klein bottle into itself and $r : \underline{K} \longrightarrow \underline{K}_1$ be a retraction. Then the following diagram is commutative*



$$r \circ \Im = \bar{\Im} \circ r$$

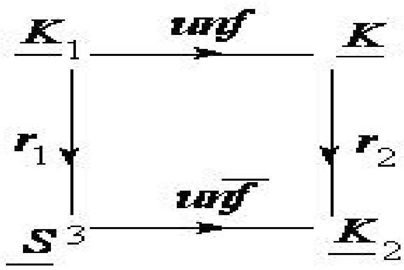
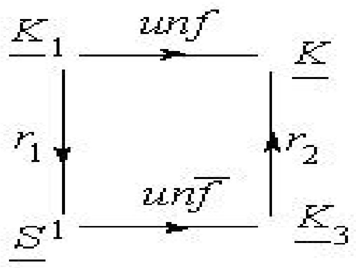


Figure 2

Theorem 9 *Let $\underline{K}$ be an elastic Klein bottle and $unf : \underline{K}_1 \longrightarrow \underline{K}$ then*
$un\,\bar{f} \circ r_1 = r_2 \circ un\,f$ *if* $\dim\,(\underline{K}) = \dim\ r(\underline{K})$,
$un\ \bar{f} \circ r_1 \not\models r_2 \circ unf$ *if* $\dim\,(\underline{K}) \not\models \dim\ r(K)$.

Proof. Let $r_1 : \underline{K}_1 \longrightarrow \underline{S}^3$, $r_2 : \underline{K} \longrightarrow \underline{K}_2$be a retraction of the elastic Klein bottle, where $\underline{S}^3$ and $\underline{K}_2$ are the elastic hyper sphere and elastic hyper Klein bottle such that $\dim\,(\underline{K}_1) = \dim\,(\underline{S}^3), \dim\,(\underline{K}) = \dim\,(\underline{K}_2)$ then the following diagram is commutative and $un\,\bar{f} \circ r_1 = r_2 \circ unf$.

Now, consider the retractions $r_1 : \underline{K}_1 \longrightarrow \underline{S}^1,\ r_2 :\ \underline{K} \longrightarrow \underline{K}_3$ where $\underline{S}^1$ and $\underline{K}_3$ are the great circle and hyper Klein bottle such that $\dim\,(\underline{K}_1) \not\models \dim\,(\underline{S}^1)$ and $\dim\,(\underline{K}) \not\models \dim\,(\underline{K}_3)$. Then the following diagram is not commutative



$$un\bar{f} \circ r_1 \not\models r_2 \circ unf$$

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