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Blow up and Blow-up Rate for the Generalized 2-component Camassa-Holm Equation

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Abstract: In this paper we investigate the local well-posedness for the generalized weakly dissipative 2-component Camassa-Holm system. We also derive a blow-up mechanism for strong solutions. In addition, we determine the exact blow-up rate of such solutions to the system.

Keywords: weak dissipative; 2-component Camassa-Holm equation; local well-posedness; blow-up; blow-up rate

1 Introduction

An interesting phenomenon in water wave channels is the appearance of waves with length much greater than the depth of the water. Various models have been proposed in understanding the long wave, shallow water problem. In the context of nonlinear elasticity, similar dynamics arise in the study of an elastic rod whose diameter is much smaller than the axial length scale. Such systems describe in a relatively simple way the competition between nonlinear and dispersive effects. The best known model is the Korteweg-de Vries (KdV) equation.

The Camassa-Holm equation

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, t > 0, x \in R \tag{1}$$

is a model for wave motion on shallow water,where $u\left(t,x\right)$ represents the fluid's free surface above a flat bottom (or equivalently,the fluid velocity at time $t \geq 0$ in the spatial x direction).

In the recent years, equation (1) has attracted the attention of a large number of researchers with two remarkable properties. The first one is the equation possesses the solutions in the form of peaked solitons or 'peakons'[1,2]. The peakon $u(t,x) = ce^{-|x-ct|}, c \neq 0$ is smooth except at its crest and the tallest among all waves of fixed energy. It is a feature observed for the traveling waves of largest amplitude which solve the governing equations for water waves[3-6]. Another remarkable property is the equation has breaking waves[1,7], that is, the solution remains bounded while its slope becomes unbounded in finite time. After wave breaking the solutions can be continued uniquely as either global conservative[8,9] or global dissipative solutions[10].The Cauchy problem of the Camassa-Holm equation has been studied extensively. It has been shown that this equation is locally well-posed[7,11-17] for initial data $u_0 \in H^s\left(R\right)$ with $s > \frac{3}{2}$. More interestingly, it has not only global strong solutions modelling permanent waves [3,7,14,16-18], but also blow-up solutions modelling wave breaking [7,14,19-21]. On the other hand, it has global weak solutions with initial data $u_0 \in H^1\left(R\right)$[8,10,18,22]. Moreover, the initial-boundary value problem for the Camassa-Holm equation on the half-line and on a finite interval were studied recently in[23]. The advantage of the Camassa-Holm equation in comparison with the KdV equation lies in the fact that the Camassa-Holm equation has peaked solitons and models wave breaking[1,7,24].

The Camassa-Holm equation also admits many integrable multicomponent generalizations. The most popular one is

$$\begin{cases} m_t - Au_x + um_x + 2u_x m + \rho\rho_x = 0, m = u - u_{xx} \\ \rho_t + (\rho u)_x = 0 \end{cases} \tag{2}$$

Notice that the CH equation can be obtained via the obvious reduction$\rho \equiv 0$and$A = 0$. System (2) was derived first in [25], where $\rho(t,x)$ is related to the free surface elevation from equilibrium (or scalar density), and $A \geq 0$ characterizes

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a linear underlying shear flow. Recently, Constantin-Ivanov [26] and Ivanov [27] established a rigorous justification of the derivation of system (1.2).Mathematical properties of the system have been also studied further in many works, for example[28-30]. The local well-posedness is improved by Gui and Liu [31] to the Besov spaces (especially in the Sobolev space $H^s(R)\times H^{s-1}(R)$with $s>\frac{3}{2}$),and they showed that the finite time blowup is determined by either the slope of the first component $u$or the slope of the second component ρ.

In this paper, we consider the following generalized weakly dissipative 2-component Camassa-Holm equation:

$$\begin{cases} u_t-u_{txx}+ku_x+3uu_x+\lambda u^{2n+1}-\beta u_{xx}=2u_xu_{xx}+uu_{xxx}+\rho\rho_x, & t>0,x\in R\\ \rho_t+(\rho u)_x=0, & t>0,x\in R \end{cases} \tag{3}$$

where $k,\lambda\geq 0$ and $\beta\geq 0$are constants,$n\geq 0$is a integer. We find that the behaviors of the equation (3) are similar to the Camassa-Holm equation in a finite interval of time, such as, the local well-posedness and the blow-up phenomena. Because of the presence of the nonlinear terms $u^{2n+1}$and the dissipative term u_{xx}, the equation (3) has not the following conservation laws: $E_1=\int_s udx$, $E_2=\int_s(u^2+u_x^2)dx$,which play an important role in the study of the Camassa-Holm equation,so it makes estimate more difficult.

The basic elementary framework is as follwos. Firstly, we give the local well-posedness of the initial value problem associated with equation (3) by applying Kato's theorem. And then, we give a blow-up mechanism for strong solutions. Finally, we determine the exact blow-up rate of such solutions to the equation(3).

2 Local well-posedness

we consider the following generalized weakly dissipative 2-component Camassa-Holm equation:

$$\begin{cases} u_t-u_{txx}+ku_x+3uu_x+\lambda u^{2n+1}-\beta u_{xx}=2u_xu_{xx}+uu_{xxx}+\rho\rho_x, & t>0,x\in R\\ \rho_t+(\rho u)_x=0, & t>0,x\in R\\ u(0,x)=u_0(x), & x\in R\\ \rho(0,x)=\rho_0(x), & x\in R \end{cases} \tag{4}$$

Where $k,\lambda\geq 0$ and $\beta\geq 0$are constants,$n\geq 0$is a integer.

Set $y=u-u_{xx}$,then Eq (4) becomes the form of a quasi-linear evolution equation of hyperbolic type:

$$\begin{cases} y_t+uy_x+2yu_x=-ku_x-\lambda u^{2n+1}+\beta u_{xx}+\rho\rho_x, & t>0,x\in R\\ \rho_t+(\rho u)_x=0, & t>0,x\in R\\ y=u-u_{xx}, & t>0,x\in R\\ y(0,x)=y_0(x),u(0,x)=u_0(x),\rho(0,x)=\rho_0(x), & x\in R \end{cases} \tag{5}$$

Note that if $g(x)=\frac{1}{2}e^{-|x|},x\in R$,then $(1-\partial_x^2)^{-1}f=g*f$for all $f\in L^2(R)$ and $g*y=u$,where$*$denotes the spatial convolution. Using this identity, and applying the pseudo-differential operator$(1-\partial_x^2)^{-1}$to Eq (5), one can rewrite Eq (5) as a quasi-linear nonlocal evolution system of hyperbolic type:

$$\begin{cases} u_t+uu_x=-\partial_x g*(u^2+\frac{1}{2}u_x^2+ku-\frac{1}{2}\rho^2)-\lambda(g*u^{2n+1})+\beta(g*u)-\beta u,\\ \rho_t+(\rho u)_x=0,\\ u(0,x)=u_0(x),\rho(0,x)=\rho_0(x), \end{cases} \tag{6}$$

For convenience, we state here Kato's theorem in the form suitable for our purpose.Consider the abstract quasi-linear evolution equation:

$$\frac{dv}{dt}+A(v)v=f(v)\quad t\geq 0,v(0)=v_0 \tag{7}$$

Let$X$and$Y$be Hilbert spaces such that$Y$is continuously and densely embedded in$X$and let$Q:Y\to X$ be a topological isomorphism. Let $L(Y,X)$denote the space of all bounded linear operator from $Y$to$X$ $(L(X)$,if $X=Y)$. Assume that

(i)$A(y)\in L(Y,X)$for$y\in X$with

$$\|(A(y)-A(z))w\|_X\leq\mu_1\|y-z\|_X\|w\|_Y,\quad y,z,w\in Y$$

And $A(y)\in G(X,1,\beta)$,i.e.$A(y)$ is quasi-m-accretive, uniformly on bounded sets in$Y$.

(ii)$QA(y)Q^{-1}=A(y)+B(y)$,where$B(y)\in L(X)$is bounded, uniformly on bounded sets in$Y$. Moreover,

$$\|(B(y)-B(z))w\|_X\leq\mu_2\|y-z\|_Y\|w\|_X,\quad y,z\in Y\ ,w\in X$$

(iii) $f : Y \to Y$ extends to a map from$X$into$X$.$f$is bounded on bounded sets in$Y$, and

$$\begin{array}{ll} \|f(y)-f(z)\|_Y \le \mu_3 \|y-z\|_Y & y,z\in Y \\ \|f(y)-f(z)\|_X \le \mu_4 \|y-z\|_X & y,z\in Y \end{array},$$

here $\mu_1, \mu_2, \mu_3, \mu_4$depend only on max $\{\|y\|_Y, \|z\|_Y\}$.

Lemma2.1*([32])Assume that(i), (ii) and (iii) hold. Given $v_0 \in Y$, there is a $T > 0$ depending only on $\|v_0\|_Y$ and a unique solution v to Eq (7) such that*

$$v = v(\cdot, v_0) \in C([0,T);Y) \cap C^1([0,T);X).$$

Moreover, the map $v_0 \mapsto v(\cdot, v_0)$ is continuous from Y to $C([0,T);Y) \cap C^1([0,T);X)$.

Applying Kato's theorem, one can follow the similar argument as in [33] to obtain the following local well-posedness result for the system (4).

Theorem2.1. *Assume $z_0 = (u_0, \rho_0) \in H^s \times H^{s-1}, s > 2$, there exist a maximal $T = T(\lambda, n, \beta, k, \|z_0\|_{H^s\times H^{s-1}}) > 0$and a unique solution u to Eq(2.1), such that $z = (u, \rho)$ to system (4) such that*

$$z = z(\cdot, z_0) \in C([0,T);H^s \times H^{s-1}) \cap C^1([0,T);H^{s-1} \times H^{s-2})$$

and the solution depends continuously on the initial data, i.e., the mapping

$$z_0 \to z(\cdot, z_0) : H^s \times H^{s-1} \to C([0,T);H^s \times H^{s-1}) \cap C^1([0,T);H^{s-1} \times H^{s-2})$$

is continuous.

Proof. The argument is similar to the proof of Theorem 2.2(see [33]), then here we omit it. ■

3 Blow-up

Given any initial data $z_0 \in H^s \times H^{s-1}$ with $s > 2$, theorem2.1 ensures the existence of a maximal $T > 0$ and a unique solution $z = (u, \rho)$ to system (4) such that

$$z = z(\cdot, z_0) \in C([0,T);H^s \times H^{s-1}) \cap C^1([0,T);H^{s-1} \times H^{s-2}).$$

Now we consider the trajectory equation

$$\begin{cases} \frac{dq(t,x)}{dt} = u(t, q(t,x)), & t \in [0,T) \\ q(0,x) = x, & x \in R \end{cases} \tag{8}$$

We know that $q(t,\cdot) : R \to R$ is the diffeomorphism for every $t \in [0,T)$, and

$$q_x(t,x) = e^{\int_0^t u_x(\tau, q(\tau,x))d\tau} > 0, \quad \forall (t,x) \in [0,T) \times R.$$

Lemma3.1*(see[33]) Assume $z_0 = (u_0, \rho_0) \in H^s \times H^{s-1}, s \ge 2$ and let T be the maximal existence time of the solution $z = (u, \rho)$ to Eq (4) with the initial data z_0.Then we have $\rho(t, q(t,x))q_x(t,x) = \rho_0(x), \forall (t,x) \in [0,T) \times R$. Moreover, if there exists $M_1 > 0$such that $u_x(t,x) \ge -M_1$ for all $(t,x) \in [0,T) \times R$, then*

$$\|\rho(t,\cdot)\|_{L^\infty} = \|\rho(t, q(t,\cdot))\|_{L^\infty} \le e^{M_1 T} \|\rho_0(\cdot)\|_{L^\infty}, \quad \forall t \in [0,T) \tag{9}$$

Furthermore, if $\rho_0 \in L^1$, then

$$\int_R |\rho(t,x)|dx = \int_R |\rho_0(x)|dx, \quad \forall t \in [0,T). \tag{10}$$

Next, we describe the precise blow-up scenario for sufficiently regular solutions to system (4).

Theorem 3.1 *Assume $z_0 = (u_0, \rho_0) \in H^s \times H^{s-1}, s \ge \frac{5}{2}$ and let T be the maximal existence time of the solution $z = (u, \rho)$ to Eq (4) with the initial data z_0.Then the corresponding solution blows up in finite time if and only if*

$$\lim_{t\to T} \inf_{x\in R}\{u_x(t,x)\} = -\infty or \lim_{t\to T} \sup\{\|\rho_x(t,\cdot)\|_{L^\infty}\} = +\infty.$$

Proof: Let $z = (u, \rho)$be the solution to Eq(4) with initial data $z_0 \in H^s \times H^{s-1}$,$s \geq \frac{5}{2}$, and let T be the maximal existence time of the solution z,which is guaranteed by Theorem 2.1. We only need to show that Theorem 3.1 holds with $s \geq 3$. So here we assume $s = 3$ to prove Theorem 3.1.

Multiplying the first equation in (4) by u and integrating by parts, we get

$$\frac{d}{dt}\int_R (u^2 + u_x^2)dx + 2\lambda\int_R u^{2n+2}dx + 2\beta\int_R u_x^2 dx \leq -\int_R \rho^2 u_x dx \tag{11}$$

Since $2\lambda\int_R u^{2n+2}dx \geq 0$ and $2\beta\int_R u_x^2 dx \geq 0$, so

$$\frac{d}{dt}\int_R (u^2 + u_x^2)dx \leq -\int_R \rho^2 u_x dx \tag{12}$$

Differentiating the first equation in (4) with respect to x, multiplying the obtained equation by u_x, and integrating by parts, we get

$$\frac{d}{dt}\int_R (u_x^2 + u_{xx}^2)dx \leq -3\int_R u_x^3 dx - 3\int_R u_x u_{xx}^2 dx + \int_R \rho^2 u_{xxx} dx \tag{13}$$

Differentiating the first equation twice in (4) with respect to x, multiplying the obtained equation by u_{xx}, and integrating by parts, we get

$$\frac{d}{dt}\int_R (u_{xx}^2 + u_{xxx}^2)dx \leq -15\int_R u_x u_{xx}^2 dx + \frac{4n\lambda(2n+1)(2n-1)}{3}\int_R u^{2n-2}u_x^4 dx$$
$$-5\int_R u_x u_{xxx}^2 dx - 2\int_R \rho_x^2 u_{xxx} dx - 2\int_R \rho_x \rho_{xx} u_{xxx} dx \tag{14}$$

Multiplying the second equation in (4) by ρ and integrating by parts, we get

$$\frac{d}{dt}\int_R \rho^2 dx = -\int_R u_x \rho^2 dx \tag{15}$$

Differentiating the second equation in (4) with respect to x, multiplying the obtained equation by ρ_x, and integrating by parts, we get

$$\frac{d}{dt}\int_R \rho_x^2 dx = -3\int_R u_x \rho_x^2 dx + \int_R u_{xxx}\rho^2 dx \tag{16}$$

Differentiating the second equation twice in (4) with respect to x, multiplying the obtained equation by ρ_{xx}, and integrating by parts, we get

$$\frac{d}{dt}\int_R \rho_{xx}^2 dx = -5\int_R u_x \rho_{xx}^2 dx + \int_R u_{xxx}(3\rho_x^2 - 2\rho\rho_{xx})dx \tag{17}$$

Assume there exists $M_1 > 0, M_2 > 0$, such that $u_x(t,x) \geq -M_1$ and $\|\rho_x(t,\cdot)\|_{L^\infty} \leq M_2$.

By Lemma 3.1, we have $\|\rho(t,\cdot)\|_{L^\infty} \leq e^{M_1 T}\|\rho_0(\cdot)\|_{L^\infty}$

By the assumption and (12)(15), we have

$$\frac{d}{dt}\int_R (u^2 + u_x^2 + \rho^2)dx \leq -2\int_R u_x\rho^2 dx \leq 2M_1\int_R \rho^2 dx \leq 2M_1\int_R (u^2 + u_x^2 + \rho^2)dx$$

Then by Gronwall's inequality, we have

$$\|u\|_{H^1}^2 + \|\rho\|_{L^2}^2 \leq e^{2M_1 T}(\|u_0\|_{H^1}^2 + \|\rho_0\|_{L^2}^2) \tag{18}$$

Applying Sobolev imbedding theorem and (18), we have

$$\|u\|_{L^\infty}^2 \leq \frac{1}{2}\|u\|_{H^1}^2 \leq \frac{1}{2}e^{2M_1 T}(\|u_0\|_{H^1}^2 + \|\rho_0\|_{L^2}^2) \quad \forall t \in [0, T) \tag{19}$$

By the assumption and (12)(13)(15)(16), we have

$\frac{d}{dt}\int_R(u^2+2u_x^2+u_{xx}^2+\rho^2+\rho_x^2)dx$
$\leq -2\int_R\rho^2u_xdx-3\int_R u_x^3dx-3\int_R u_xu_{xx}^2dx-3\int_R u_x\rho_x^2dx-4\int_R\rho\rho_xu_{xx}dx$
$\leq 2M_1\int_R\rho^2dx+3M_1\int_R u_x^2dx+3M_1\int_R u_{xx}^2dx+3M_1\int_R\rho_x^2dx+2M_2\int_R(\rho^2+u_{xx}^2)dx$
$\leq (3M_1+2M_2)\int_R(u^2+u_x^2+u_{xx}^2+\rho^2+\rho_x^2)dx$

Then by Gronwall's inequality, we have

$$\|u\|_{H^2}^2+\|\rho\|_{H^1}^2\leq e^{(3M_1+2M_2)T}(\|u_0\|_{H^2}^2+\|\rho_0\|_{H^1}^2)\quad \forall t\in[0,T) \tag{20}$$

Applying Sobolev imbedding theorem and (20), we have

$$\|u_x\|_{L^\infty}^2\leq\frac{1}{2}\|u\|_{H^2}^2\leq\frac{1}{2}e^{(3M_1+2M_2)T}(\|u_0\|_{H^2}^2+\|\rho_0\|_{H^1}^2) \tag{21}$$

By the assumption and (12)(13) (14) (15)(16) (17)(19) (21), we have

$$\begin{aligned}
&\frac{d}{dt}\int_R(u^2+2u_x^2+2u_{xx}^2+u_{xxx}^2+\rho^2+\rho_x^2+\rho_{xx}^2)dx\\
&\leq -2\int_R\rho^2u_xdx-3\int_R u_x^3dx-18\int_R u_xu_{xx}^2dx+2\int_R\rho^2u_{xxx}dx-5\int_R u_xu_{xxx}^2dx\\
&\quad -2\int_R\rho\rho_{xx}u_{xxx}dx-2\int_R\rho_x^2u_{xxx}dx-3\int_R u_x\rho_x^2dx-5\int_R u_x\rho_{xx}^2dx\\
&\quad +3\int_R\rho_x^2u_{xxx}dx-2\int_R\rho\rho_{xx}u_{xxx}dx+\frac{4\lambda n(2n-1)(2n+1)}{3}\int_R u^{2n-2}u_x^4dx\\
&\leq 2M_1\int_R\rho^2dx+3M_1\int_R u_x^2dx+18M_1\int_R u_{xx}^2dx+\|\rho\|_{L^\infty}\int_R(\rho^2+u_{xxx}^2)dx\\
&\quad +5M_1\int_R u_{xxx}^2dx+\|\rho\|_{L^\infty}\int_R(\rho_{xx}^2+u_{xxx}^2)dx+\|\rho_x\|_{L^\infty}\int_R(\rho_x^2+u_{xxx}^2)dx\\
&\quad +3M_1\int_R\rho_x^2dx+5M_1\int_R\rho_{xx}^2dx+\frac{3}{2}\|\rho_x\|_{L^\infty}\int_R(\rho_x^2+u_{xxx}^2)dx\\
&\quad +\|\rho\|_{L^\infty}\int_R(\rho_{xx}^2+u_{xxx}^2)dx+\frac{4\lambda n(2n-1)(2n+1)}{3}\|u_x\|_{L^\infty}^2\|u\|_{L^\infty}^{2n-2}\int_R u_x^2dx\\
&\leq (2M_1+\|\rho\|_{L^\infty})\int_R\rho^2dx+(3M_1+\frac{5}{2}\|\rho_x\|_{L^\infty})\int_R\rho_x^2dx+(2\|\rho\|_{L^\infty}+5M_1)\int_R\rho_{xx}^2dx\\
&\quad +(3M_1+C)\int_R u_x^2dx+18M_1\int_R u_{xx}^2dx+(3\|\rho\|_{L^\infty}+5M_1+\frac{5}{2}\|\rho_x\|_{L^\infty})\int_R u_{xxx}^2dx\\
&\leq (18M_1+\frac{5}{2}M_2+3e^{M_1T}\|\rho_0\|_{L^\infty}+C)\int_R(u^2+u_x^2+u_{xx}^2+u_{xxx}^2+\rho^2+\rho_x^2+\rho_{xx}^2)dx
\end{aligned}$$

Where

$$\|u\|_{L^\infty}^{2n-2}\cdot\|u_x\|_{L^\infty}^2\leq(\frac{1}{2})^ne^{(3M_1+2M_2)T}(\|u_0\|_{H^2}^2+\|\rho_0\|_{H^1}^2)(e^{2M_1T}(\|u_0\|_{H^1}^2+\|\rho_0\|_{L^2}^2))^{n-1}$$

Note $C=\frac{4\lambda n(2n-1)(2n+1)}{3}(\frac{1}{2})^ne^{(3M_1+2M_2)T}(\|u_0\|_{H^2}^2+\|\rho_0\|_{H^1}^2)(e^{2M_1T}(\|u_0\|_{H^1}^2+\|\rho_0\|_{L^2}^2))^{n-1}$
By Gronwall's inequality, we have

$$\|u\|_{H^3}^2+\|\rho\|_{H^2}^2\leq e^{(18M_1+\frac{5}{2}M_2+3e^{M_1T}\|\rho_0\|_{L^\infty}+C)}\cdot(\|u_0\|_{H^3}^2+\|\rho_0\|_{H^2}^2)\quad\forall t\in[0,T)$$

Which contradicts the assumption the maximal existence time $T<+\infty$, i.e. the solution $u$does not blow up in finite time. Conversely, the Sobolev embedding theorem implies: if

$\lim\limits_{t\to T}\inf\limits_{x\in R}\{u_x(t,x)\} = -\infty$or $\lim\limits_{t\to T}\sup\{\|\rho_x(t,\cdot)\|_{L^\infty}\} = +\infty$ holds, then the corresponding solution blows up in finite time, which completes the proof of Theorem 3.1.

Now we address the question of the formation of singularity for strong solutions to Eq (4). In order to establish blow-up solution, we need the following useful lemmas.

Lemma3.2 *Assume* $z_0 = (u_0,\rho_0) \in H^s \times H^{s-1}, s \geq 2$ *and let T be the maximal existence time of the solution* $z = (u,\rho)$ *to Eq (4) with the initial data* z_0*.Then for all* $t \in [0,T)$*,we have*

$$\int_R (u^2(t,x) + u_x^2(t,x) - \rho^2(t,x))dx \leq \int_R (u_0^2(x) + u_{0x}^2(x) - \rho_0^2(x))dx.$$

Moreover, if there exists $M > 0$ *such that* $\|\rho(t,\cdot)\|_{L^\infty} \leq M$ *for all* $(t,x) \in [0,T) \times R$ *and* $\rho_0 \in L^1$*, then*

$$\|u(t,\cdot)\|_{L^\infty} \leq \frac{\sqrt{2}}{2}\|u(t,\cdot)\|_{H^1} \leq \frac{\sqrt{2}}{2}(\|u_0\|_{H^1}^2 - \|\rho_0\|_{L^2}^2 + M\|\rho_0\|_{L^1})^{\frac{1}{2}}. \tag{22}$$

Proof: Let $z = (u,\rho)$be the solution to Eq (4) with initial data $z_0 \in H^s \times H^{s-1}$,$s \geq 2$, and let T be the maximal existence time of the solution z,which is guaranteed by Theorem 2.1. Combining (12) with (15), we have $\frac{d}{dt}\int_R (u^2 + u_x^2 - \rho^2)dx \leq 0$. Then we have

$$\int_R (u^2 + u_x^2 - \rho^2)dx \leq \int_R (u_0^2 + u_{0x}^2 - \rho_0^2)dx.$$

By the above inequality, the assumption of the lemma3.2 and Lemma 3.1, we have

$$u^2(t,x) = \int_{-\infty}^x uu_x dx - \int_x^{+\infty} uu_x dx \leq \int_{-\infty}^{+\infty} |uu_x|\, dx$$

$$\leq \frac{1}{2}\int_R (u^2 + u_x^2)dx \leq \frac{1}{2}\int_R (u_0^2 + u_{0x}^2)dx - \frac{1}{2}\int_R \rho_0^2 dx + \frac{1}{2}\int_R \rho^2 dx$$

$$\leq \frac{1}{2}\int_R (u_0^2 + u_{0x}^2)dx - \frac{1}{2}\int_R \rho_0^2 dx + \frac{1}{2}M\int_R |\rho_0|\, dx$$

Which completes the proof of Lemma 3.2.

Lemma3.3*([7])* *Let* $T > 0$ *and* $u \in C^1([0,T);H^2(R))$*. Then for all* $t \in [0,T)$*, there exists at least one point* $\xi(t) \in R$ *with*

$$m(t) = \inf_{x\in R}(u_x(t,x)) = u_x(t,\xi(t)) \tag{23}$$

The function $m(t)$ *is absolutely continuous on* $(0,T)$*with*

$$\frac{dm(t)}{dt} = u_{tx}(t,\xi(t))a.e.\ on\ (0,T) \tag{24}$$

Theorem 3.2 *Assume* $z_0 = (u_0,\rho_0) \in H^s \times H^{s-1}, s \geq 2$ *and let T be the maximal existence time of the solution* $z = (u,\rho)$ *to Eq (4) with the initial data* z_0*.Assume there exists* $M > 0$ *such that* $\|\rho(t,\cdot)\|_{L^\infty} \leq M$ *for all* $t \in [0,T)$ *and* $\rho_0 \in L^1$*. If there exists some* $x_0 \in R$ *such that* $u_{0x}(x_0) < -\beta - \sqrt{\beta^2 + 2K}$*, then the solution to Eq(4) blows up in finite time in the following sense: there exists a* T*with* $0 < T \leq \frac{1}{\sqrt{\beta^2+2K}}\ln(\frac{m(0)+\beta-\sqrt{\beta^2+2K}}{m(0)+\beta+\sqrt{\beta^2+2K}})$*, such that* $\lim\limits_{t\to T}\inf\limits_{x\in R}\{u_x(t,x)\} = -\infty$*. Where*

$$K = \frac{1}{4}(\|u_0\|_{H^1}^2 - \|\rho_0\|_{L^2}^2 + 2M\|\rho_0\|_{L^1}) + (\sqrt{2}|k| + \frac{\sqrt{2}}{2}\beta)(\|u_0\|_{H^1}^2 - \|\rho_0\|_{L^2}^2 + M\|\rho_0\|_{L^1})^{\frac{1}{2}}$$

$$+\lambda(\frac{\sqrt{2}}{2}(\|u_0\|_{H^1}^2 - \|\rho_0\|_{L^2}^2 + M\|\rho_0\|_{L^1})^{\frac{1}{2}})^{2n-1}.$$

Proof: Let $z = (u,\rho)$be the solution to Eq (4) with initial data $z_0 \in H^s \times H^{s-1}$,$s \geq 2$, and let T be the maximal existence time of the solution z,which is guaranteed by Theorem 2.1.

Differentiating the first equation in Eq (6) with respect to x, and note that
$\partial_x^2 g * f = g * f - f$,we have

$$u_{tx} = -\frac{1}{2}u_x^2 - uu_{xx} + u^2 + ku - \frac{1}{2}\rho^2 - g * (u^2 + \frac{1}{2}u_x^2 + ku - \frac{1}{2}\rho^2)$$

$$-\lambda\partial_x(g * u^{2n+1}) + \beta\partial_x(g * u) - \beta u_x$$

Define $m(t) = u_x(t, \xi(t)) = \inf_{x \in R} \{u_x(t, x)\}$, then $u_{xx}(t, \xi(t)) = 0$ for all $t \in [0, T)$. Note that $g * (u^2 + \frac{1}{2}u_x^2) \geq \frac{1}{2}u^2$.We have

$$\frac{dm(t)}{dt} \leq -\frac{1}{2}m^2(t) - \beta m(t) + \frac{1}{2}u^2(t, \xi(t)) + ku(t, \xi(t))$$

$$\begin{aligned} &-g * (ku - \tfrac{1}{2}\rho^2)(t, \xi(t)) - \lambda\partial_x(g * u^{2n+1})(t, \xi(t)) + \beta\partial_x(g * u)(t, \xi(t)) \\ &\leq -\tfrac{1}{2}m^2(t) - \beta m(t) + \tfrac{1}{2}u^2(t, \xi(t)) + |ku(t, \xi(t))| \\ &+ \left|g * (ku - \tfrac{1}{2}\rho^2)(t, \xi(t))\right| + \lambda\left|\partial_x(g * u^{2n+1})(t, \xi(t))\right| + \beta\left|\partial_x(g * u)(t, \xi(t))\right| \end{aligned} \tag{25}$$

By Young's inequality and Lemma 3.2, we get for all $t \in [0, T)$,

$$\|g * u\|_{L^\infty} \leq \|g\|_{L^1} \|u\|_{L^\infty} \leq \frac{\sqrt{2}}{2}(\|u_0\|_{H^1}^2 - \|\rho_0\|_{L^2}^2 + M\|\rho_0\|_{L^1})^{\frac{1}{2}}$$

$$\left\|g * \rho^2\right\|_{L^\infty} \leq \|g\|_{L^\infty} \left\|\rho^2\right\|_{L^1} \leq \frac{1}{2}M\|\rho_0\|_{L^1}$$

$$\left\|\partial_x(g * u^{2n+1})\right\|_{L^\infty} \leq \left\|\partial_x g * u^{2n+1}\right\|_{L^\infty} \leq \left\|u^{2n+1}\right\|_{L^\infty} \leq [\frac{\sqrt{2}}{2}(\|u_0\|_{H^1}^2 - \|\rho_0\|_{L^2}^2 + M\|\rho_0\|_{L^1})^{\frac{1}{2}}]^{2n-1}$$

$$\|\partial_x(g * u)\|_{L^\infty} \leq \|u\|_{L^\infty} \leq \frac{\sqrt{2}}{2}(\|u_0\|_{H^1}^2 - \|\rho_0\|_{L^2}^2 + M\|\rho_0\|_{L^1})^{\frac{1}{2}}$$

Combining the above inequalities with (25), we have

$$\frac{dm(t)}{dt} \leq -\frac{1}{2}m^2(t) - \beta m(t) + K$$

Where $K = \frac{1}{4}(\|u_0\|_{H^1}^2 - \|\rho_0\|_{L^2}^2 + 2M\|\rho_0\|_{L^1}) + (\sqrt{2}|k| + \frac{\sqrt{2}}{2}\beta)(\|u_0\|_{H^1}^2 - \|\rho_0\|_{L^2}^2 + M\|\rho_0\|_{L^1})^{\frac{1}{2}}$

$$+\lambda(\frac{\sqrt{2}}{2}(\|u_0\|_{H^1}^2 - \|\rho_0\|_{L^2}^2 + M\|\rho_0\|_{L^1})^{\frac{1}{2}})^{2n-1}$$

From the above inequality, we have

$$\frac{dm(t)}{dt} \leq -\frac{1}{2}(m(t) + \beta - \sqrt{\beta^2 + 2\mathrm{K}})(m(t) + \beta + \sqrt{\beta^2 + 2\mathrm{K}}) \tag{26}$$

By the assumption $m(0) = u'_{0x}(x_0) < -\beta - \sqrt{\beta^2 + 2\mathrm{K}}$, we have

$$m^2(0) > (\beta + \sqrt{\beta^2 + 2\mathrm{K}})^2,$$

we now claim that this is true for any $t \in [0, T)$. In fact, assuming the contrary world, in view of $t_0 \in (0, T)$, such that for all $t \in [0, t_0)$, $m^2(t) > (\beta + \sqrt{\beta^2 + 2\mathrm{K}})^2$, but $m^2(t_0) = (\beta + \sqrt{\beta^2 + 2\mathrm{K}})^2$. Combining this with above inequality (26) , we have $\frac{dm(t)}{dt} < 0$ a.e. on $[0, t_0)$. Since $m(t)$ is absolutely continuous on $[0, t_0]$, an integration of this inequality would give the following inequality and we get the contraction $m(t_0) < m(0) = u'_{0x}(x_0) < -\beta - \sqrt{\beta^2 + 2\mathrm{K}}$.This proves the previous claim.

So we can solve the above inequality (26) to obtain

$$\frac{m(0) + \beta + \sqrt{\beta^2 + 2\mathrm{K}}}{m(0) + \beta - \sqrt{\beta^2 + 2\mathrm{K}}}e^{\sqrt{\beta^2 + 2\mathrm{K}}t} - 1 \leq \frac{2\sqrt{\beta^2 + 2\mathrm{K}}}{m(t) + \beta - \sqrt{\beta^2 + 2\mathrm{K}}} \leq 0$$

Since $0 < \frac{m(0)+\beta+\sqrt{\beta^2+2\mathrm{K}}}{m(0)+\beta-\sqrt{\beta^2+2\mathrm{K}}} < 1$, there exists

$$0 < T \le \frac{1}{\sqrt{\beta^2+2\mathrm{K}}}\ln(\frac{m(0)+\beta-\sqrt{\beta^2+2\mathrm{K}}}{m(0)+\beta+\sqrt{\beta^2+2\mathrm{K}}}),$$

Such that $\lim_{t\to T^-} m(t) = \lim_{t\to T^-}(\inf_{x\in R} u_x(t,x)) = -\infty$. Therefore, the solution u does not exist globally in time. This completes the proof of Theorem3.2.

4 Blow-up rate

Having established blow up results for system (4) under study, attention is given to blow-up rate for solutions to Eq (4).

Theorem4.1.*Let $T < \infty$ be the blow-up time of the corresponding solution of Eq (4) with initial data $z_0 = (u_0, \rho_0) \in H^s \times H^{s-1}$ with $s \ge 2$ satisfying the assumptions of Theorem 3.2 .*

Then we have

$$\lim_{t\to T^-}\left(\inf_{x\in R}\{u_x(t,x)\}(T-t)\right) = -2. \tag{27}$$

Proof: Similarly as in the proof of Theorem 3.2, we still consider $\frac{dm(t)}{dt} \le -\frac{1}{2}m^2(t) - \beta m(t) + K$,where

$$K = \frac{1}{4}(\|u_0\|^2_{H^1} - \|\rho_0\|^2_{L^2} + 2M\|\rho_0\|_{L^1}) + (\sqrt{2}|k| + \frac{\sqrt{2}}{2}\beta)(\|u_0\|^2_{H^1} - \|\rho_0\|^2_{L^2} + M\|\rho_0\|_{L^1})^{\frac{1}{2}}$$

$$+\lambda(\frac{\sqrt{2}}{2}(\|u_0\|^2_{H^1} - \|\rho_0\|^2_{L^2} + M\|\rho_0\|_{L^1})^{\frac{1}{2}})^{2n-1}$$

Therefore, $-K \le \frac{dm(t)}{dt} + \frac{1}{2}m^2(t) + \beta m(t) \le K$ a.e. on $(0,T)$

Hence,

$$-K - \frac{1}{2}\beta^2 \le \frac{dm(t)}{dt} + \frac{1}{2}(m(t)+\beta)^2 \le K + \frac{1}{2}\beta^2 a.e. on(0,T) \tag{28}$$

Choose now $\varepsilon \in (0, \frac{1}{2})$, since $\lim_{t\to T^-}(m(t)+\beta) = -\infty$, there is some $t_0 \in (0,T)$ with $m(t_0)+\beta < 0$, and $(m(t_0)+\beta)^2 > \frac{1}{\varepsilon}(K + \frac{1}{2}\beta^2)$. Since m is locally Lipschitz, it follows that m is absolutely continuous. We deduce that m is decreasing on $[t_0, T)$, and

$$(m(t)+\beta)^2 > \frac{1}{\varepsilon}(K + \frac{1}{2}\beta^2) \quad t \in [t_0, T) \tag{29}$$

From (28) and (29), we get

$$-\frac{1}{2}(m(t)+\beta)^2 - K - \frac{1}{2}\beta^2 \le \frac{dm(t)}{dt} \le K + \frac{1}{2}\beta^2 - \frac{1}{2}(m(t)+\beta)^2 \tag{30}$$

Notice that m is locally Lipschitz and less than $m(t_0) < -\beta$ on $[t_0, T)$, we know that $\frac{1}{m}$ is locally Lipschitz on (t_0, T). From the inequality (30), we obtain

$$\frac{1}{2} - \varepsilon \le \frac{d}{dt}(\frac{1}{m(t)+\beta}) \le \frac{1}{2} + \varepsilon a.e. on(t_0, T) \tag{31}$$

Integrating the above inequality on (t, T) with $t \in [t_0, T)$ and noticing that $\lim_{t\to T^-} m(t) = -\infty$, we get $(\frac{1}{2} - \varepsilon)(T-t) \le -\frac{1}{m(t)+\beta} \le (\frac{1}{2} + \varepsilon)(T-t) \quad t \in (t_0, T)$

Since $\varepsilon \in (0, \frac{1}{2})$ is arbitrary, in view of the definition of $m(t)$, then

$$\lim_{t\to T^-}\{m(t)(T-t) + \beta(T-t)\} = -2$$

That is $\lim_{t\to T^-} m(t)(T-t) = -2$.

Remark: Although the occurrence of blow-up of strong solutions to Eq(4) is affected by the dissipative parameter. However, the blow-up rate of strong solutions to Eq (4) is not affected by the weak dissipative term.

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