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A General Law of the Moment Convergence Rates of the Error Variance Estimator in Linear Model

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Abstract: In this paper, under some mild conditions, we show that a general law of the moment convergence rates of $\hat{\sigma}_n^2$ holds. It can describe the relations among the boundary function, weighted function, convergence rate and limit value in the study of moment convergence rates more precisely.

Keywords: moment convergence rates; linear model; error variance estimator

MSC: 60F15; 60G50

1 Introduction

As is well known, we have the famous result following, for $0 < p < 2$ and $r \geq p, \sum\limits_{n=1}^{\infty} n^{r/p-2} P(|S_n| \geq \varepsilon n^{1/p}) < \infty, \varepsilon >$ 0, if and only if $E|X|^r < \infty$ and when $r \geq 1,\ EX = 0$. For $r = 2,\ p = 1$, the sufficiency was proved by Hsu and Robbins [12], and the necessity by Erdös [5, 6]. For the case $r = p = 1$, we refer to Spitzer [20], and one can refer to Baum and Katz [1] for the general result. Note that the sums obviously tend to infinity as $\varepsilon \searrow 0$. Thus it is interesting to discuss the precise rate and limit value of $\sum_{n=1}^{\infty} \varphi(n) P(|S_n| \geq \varepsilon h(n))$ as $\varepsilon \searrow a$, $a \geq 0$, where $\varphi(x)$ and $h(x)$ are the positive functions defined on $[0, \infty)$. We call $\varphi(x)$ and $h(x)$ weighted function and boundary function respectively. The first result in this direction was due to Heyde [11], who proved that

$$\lim_{\varepsilon \searrow 0} \varepsilon^2 \sum_{n=1}^{\infty} P(|S_n| \geq \varepsilon n) = EX^2,$$

if and only if $EX = 0$ and $EX^2 < \infty$. Later, Chen [2] and Gut and Spătaru [8] both studied the precise asymptotics of the infinite sums as $\varepsilon \searrow 0$. Moreover, Gut and Spătaru [9, 10] studied the precise asymptotics of the law of the iterated logarithm and the precise asymptotics for multidimensionally indexed random variables. Lanzinger and Stadtmüller [15], Spătaru [18, 19]; and Huang and Zhang [13] obtained the precise rates in some different cases.

While, Chow [4] discussed the complete moment convergence of i.i.d random variables and got the following result:

Theorem 1 *([4]) Let $\{Y, Y_k; k \geq 1\}$ be a sequence of i.i.d random variables with $EY = 0$. Suppose that $p \geq 1, \alpha > \frac{1}{2}, p\alpha > 1, E\{|Y|^p + |Y| \log(1 + |Y|)\} < \infty$. Then for any $\varepsilon > 0$, we have*

$$\sum_{n=1}^{\infty} n^{p\alpha-2-\alpha} E\{\max_{j\leq n} |\sum_{k=1}^{j} Y_k| - \varepsilon n^{\alpha}\}_+ < \infty.$$

And then Jiang and Zhang [14] also derived the precise asymptotics in the law of the iterated logarithm for the moment convergence. Recently, Zhang [23] investigated a general case of precise asymptotics for some cases.

On the other hand, consider the following linear model

$$Y_i = X_i^{'}\beta + e_i,\ i = 1, 2, \ldots, n, \tag{1}$$

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where β is a p-dimensional unknown parametric vector, and $\{e_i,\ i \geq 1\}$ are i.i.d. random variables with zero means and finite variances σ^2. By least squares and the characteristic of linear model, the estimator of σ^2 always takes the following form:

$$\hat{\sigma}_n^2 = \frac{1}{n-\gamma}\Big\{\sum_{i=1}^{n} e_i^2 - \sum_{j=1}^{\gamma}(\sum_{i=1}^{n} a_{nji}e_i)^2\Big\}, \tag{2}$$

where $\gamma = \gamma_n$ is the rank of the design matrix $\mathbf{X}_n = (X_1, \ldots, X_n)$ satisfying $\gamma_n \leq p$ and $\{a_{nli}\}$ is a sequence of real numbers satisfying

$$\sum_{i=1}^{n} a_{nli}a_{nmi} = \delta_{lm}, \tag{3}$$

where δ_{lm} is Kronecker function. The limit properties of the error variance estimator have been widely discussed, and we refer the reader to the literature [3, 17, 22] and references therein.

Inspired by the mentioned above results, in this note, we show that a general law of the moment convergence rates of $\hat{\sigma}_n^2$ holds, and obtain the corresponding results for a class of weighted functions and boundary functions.

2 Main results

In this section, we will state our main results. It is natural that we will make some appropriate limitations to weighted functions and boundary functions, and these limitations will not impact the generality of main results. Now we will state our main results as follows.

Theorem 2 *Suppose $Ee_1 = 0$, $0 < \sigma^2 = Ee_1^2 < \infty$ and $Ee_1^r < \infty$, for $r > \max\{\frac{2}{s}+2, 4\}$, and let $\nu = Var(e_1^2)$. Assume that $g(x)$ is differentiable on the interval $[0, +\infty)$, which is nonnegative and strictly increasing to ∞, and differentiable function $g'(x)$ is nonnegative. Suppose $g'(x)$ is monotone. If $g'(x)$ is monotone nondecreasing, we assume that $\lim_{x\to\infty}\frac{g'(x+1)}{g'(x)} = 1$. Then, for $s > 0$, we have*

$$\lim_{\varepsilon\searrow 0}\varepsilon^{\frac{1}{s}}\sum_{n=1}^{\infty} g'(n)E\{\sqrt{n}|\hat{\sigma}_n^2 - \sigma^2| - \varepsilon\sqrt{\nu}g^s(n)\}_+ = \frac{s}{s+1}E|N|^{1+\frac{1}{s}}.$$

Here N is the standard normal random variable.

Remark 3 *To obtain this kind of moment convergence results, one way is using strong approximation methods (c.f. Jiang, et al. ([14])), but this method is not applicable here. Another way is using the Berry-Esseen's inequality (c.f. Li ([16])), and we do not take this approach either.*

3 Proof of Theorem 2

First, we introduce a key lemma, which will be used to prove our main results.

Lemma 4 *([3]). Let $\{e_i\}$ be in the linear model. Supposed that $Ee_1 = 0$, $0 < \sigma^2 = Ee_1^2 < \infty$ and $Ee_1^4 < \infty$, and set $\nu = Ee_1^4 - \sigma^4$. Then we have*

$$n^{1/2}\nu^{-1/2}(\hat{\sigma}_n^2 - \sigma^2) \xrightarrow{d} N,$$

where $\xrightarrow{d}$ and N denote convergence in distribution and the standard normal random variable, respectively.

In what follows, let $A(\varepsilon) = g^{-1}(M\varepsilon^{-\frac{1}{s}})$, for $M > 1$ and $\varepsilon > 0$, $g^{-1}(x)$ is the inverse function of $g(x)$. Without loss of generality, assume $\nu = 1$ and $a_{nli} = a_{ni}$ for the same l in the sequel, and in the sequel, C shall denote positive constants, possibly varying from place to place, $[x]$ means the largest integer $\leq x$. The proof of Theorem 2 is based on the following propositions.

Proposition 5 *For $s > 0$, we have*

$$\lim_{\varepsilon \searrow 0} \varepsilon^{\frac{1}{s}} \sum_{n=1}^{\infty} g'(n) E\{|N| - \varepsilon g^s(n)\}_+ = \frac{s}{s+1} E|N|^{1+\frac{1}{s}}.$$

Proof. Via the change of variable $y = \varepsilon g^s(x)$, for arbitrary $\delta > 0$, we have

$$\begin{aligned}
&\lim_{\varepsilon \searrow 0} \varepsilon^{\frac{1}{s}} \int_{\delta}^{\infty} g'(x) \int_{\varepsilon g^s(x)}^{\infty} P(|N| \geq t) dt dx \\
&= \lim_{\varepsilon \searrow 0} \frac{1}{s} \int_{\varepsilon g^s(\delta)}^{\infty} y^{\frac{1}{s}-1} \int_{y}^{\infty} P(|N| \geq t) dt dy \\
&= \lim_{\varepsilon \searrow 0} \frac{1}{s} \int_{\varepsilon g^s(\delta)}^{\infty} P(|N| \geq t) \int_{\varepsilon g^s(\delta)}^{t} y^{\frac{1}{s}-1} dy dt \\
&= \lim_{\varepsilon \searrow 0} \int_{\varepsilon g^s(\delta)}^{\infty} t^{\frac{1}{s}} P(|N| \geq t) dt \\
&= \frac{s}{s+1} E|N|^{1+\frac{1}{s}}. \qquad (4)
\end{aligned}$$

Thus, if $g'(x)$ is monotone nonincreasing, then $g'(x) \int_{\varepsilon g^s(x)}^{\infty} P(|N| \geq t) dt$ is nonincreasing. Hence

$$\begin{aligned}
\int_{2}^{\infty} g'(y) \int_{\varepsilon g^s(y)}^{\infty} P(|N| \geq t) dt dy &\leq \sum_{n=2}^{\infty} g'(n) E\{|N| - \varepsilon g^s(n)\}_+ \\
&\leq \int_{1}^{\infty} g'(y) \int_{\varepsilon g^s(y)}^{\infty} P(|N| \geq t) dt dy,
\end{aligned}$$

then, by (4), the proposition holds. If $g'(y)$ is nondecreasing, then by $\lim_{n\to\infty} \frac{g'(n+1)}{g'(n)} = 1$, for any $0 < \delta_0 < 1$, there exists $n_1 = n_1(\delta_0)$, such that $\frac{g'(n+1)}{g'(n)} < 1 + \delta$ and $\frac{g'(n)}{g'(n+1)} > 1 - \delta$ for $n \geq n_1$. Thus we have

$$\begin{aligned}
\frac{1}{1+\delta} \int_{2}^{\infty} g'(y) \int_{\varepsilon g^s(y)}^{\infty} P(|N| \geq t) dt dy &\leq \sum_{n=2}^{\infty} g'(n) E\{|N| - \varepsilon g^s(n)\}_+ \\
&\leq \frac{1}{1-\delta} \int_{1}^{\infty} g'(y) \int_{\varepsilon g^s(y)}^{\infty} P(|N| \geq t) dt dy,
\end{aligned}$$

then, by (4) and let $\delta \searrow 0$, we complete the proof of this proposition. ■

Proposition 6 *For any $M > 0$, we have*

$$\lim_{\varepsilon \searrow 0} \varepsilon^{\frac{1}{s}} \sum_{n \leq A(\varepsilon)} g'(n) |E\{\sqrt{n}|\hat{\sigma}_n^2 - \sigma^2| - \varepsilon g^s(n)\}_+ - E\{|N| - \varepsilon g^s(n)\}_+| = 0.$$

Proof. First, by the central limit theorem for the error variance estimator in linear models (cf. Chen ([3])), we have

$$\triangle_n := \sup_x |P(\sqrt{n}|\hat{\sigma}_n^2 - \sigma^2| \geq x) - P(|N| \geq x)| \to 0, \ as \ n \to \infty. \qquad (5)$$

Then, it is easy to see

$$\begin{aligned}
&\varepsilon^{\frac{1}{s}} \sum_{n \leq A(\varepsilon)} g'(n) |E\{\sqrt{n}|\hat{\sigma}_n^2 - \sigma^2| - \varepsilon g^s(n)\}_+ - E\{|N| - \varepsilon g^s(n)\}_+| \\
&= \varepsilon^{\frac{1}{s}} \sum_{n \leq A(\varepsilon)} g'(n) \left| \int_0^{\infty} P(\sqrt{n}|\hat{\sigma}_n^2 - \sigma^2| \geq x + \varepsilon g^s(n)) dx - \int_0^{\infty} P(|N| \geq x + \varepsilon g^s(n)) dx \right| \\
&\leq \varepsilon^{\frac{1}{s}} \sum_{n \leq A(\varepsilon)} g'(n) \int_0^{\infty} |P(\sqrt{n}|\hat{\sigma}_n^2 - \sigma^2| \geq x + \varepsilon g^s(n)) - P(|N| \geq x + \varepsilon g^s(n))| dx \\
&= \varepsilon^{\frac{1}{s}} \sum_{n \leq A(\varepsilon)} g'(n) (\triangle_{n1} + \triangle_{n2}),
\end{aligned}$$

where

$$\triangle_{n1} = \int_0^{\triangle_n^{-\frac{1}{4}}} |P(\sqrt{n}|\hat{\sigma}_n^2 - \sigma^2| \geq x + \varepsilon g^s(n)) - P(|N| \geq x + \varepsilon g^s(n))|dx,$$

$$\triangle_{n2} = \int_{\triangle_n^{-\frac{1}{4}}}^{\infty} |P(\sqrt{n}|\hat{\sigma}_n^2 - \sigma^2| \geq x + \varepsilon g^s(n)) - P(|N| \geq x + \varepsilon g^s(n))|dx.$$

For $\triangle_{n1}$, by (5), we have

$$\triangle_{n1} \leq \triangle_n^{\frac{3}{4}} \to 0,\ as\ n \to \infty.$$

For $\triangle_{n2}$, by Markov's inequality, we have

$$\begin{aligned}\triangle_{n2} &\leq C\int_{\triangle_n^{-\frac{1}{4}}}^{\infty} \frac{1}{(x+\varepsilon g^s(n))^2}dx + C\int_{\triangle_n^{-\frac{1}{4}}}^{\infty} \frac{E|N|^2}{(x+\varepsilon g^s(n))^2}dx \\ &\leq C\int_{\triangle_n^{-\frac{1}{4}}}^{\infty} \frac{1}{x^2}dx \to 0,\ as\ n \to \infty.\end{aligned}$$

Denote $\triangle_n^{'} = \triangle_{n1} + \triangle_{n2}$, and by Toeplitz lemma (see P_{120} of Stout ([21])), we get that

$$\varepsilon^{\frac{1}{s}} \sum_{n \leq A(\varepsilon)} g'(n)|E\{\sqrt{n}|\hat{\sigma}_n^2 - \sigma^2| - \varepsilon g^s(n)\}_+ - E\{|N| - \varepsilon g^s(n)\}_+| \leq \varepsilon^{\frac{1}{s}} \sum_{n \leq A(\varepsilon)} g'(n)\triangle_n^{'} \to 0,\ as\ \varepsilon \searrow 0.$$

Consequently, the proof is completed. ■

Proposition 7 *For any $s > 0$, uniformly for $\varepsilon > 0$, we have*

$$\lim_{M\to\infty} \varepsilon^{\frac{1}{s}} \sum_{n > A(\varepsilon)} g'(n)E\{|N| - \varepsilon g^s(n)\}_+ = 0.$$

Proof. Note that, by the proof of proposition 5, it follows that

$$\begin{aligned}&\varepsilon^{\frac{1}{s}} \sum_{n > A(\varepsilon)} g'(n)E\{|N| - \varepsilon g^s(n)\}_+ \\ &= \varepsilon^{\frac{1}{s}} \sum_{n > A(\varepsilon)} g'(n) \int_{\varepsilon g^s(n)}^{\infty} P(|N| \geq x)dx \\ &\leq \varepsilon^{\frac{1}{s}} \int_{A(\varepsilon)}^{\infty} g'(y) \int_{\varepsilon g^s(y)}^{\infty} P(|N| \geq x)dxdy \\ &\leq C\int_{\varepsilon g^s(A(\varepsilon))}^{\infty} t^{\frac{1}{s}-1} \int_t^{\infty} P(|N| \geq x)dxdt \\ &\leq C\int_{M^s}^{\infty} P(|N| \geq x) \int_{M^s}^{x} t^{\frac{1}{s}-1}dtdx \\ &\leq C\int_{M^s}^{\infty} x^{\frac{1}{s}} P(|N| \geq x)dx \to 0,\ as\ M \to \infty.\end{aligned}$$

Thus, this proposition is proved. ■

Proposition 8 *For any $s > 0$, uniformly for $\varepsilon > 0$, we have*

$$\lim_{M\to\infty} \lim_{\varepsilon\searrow 0} \varepsilon^{\frac{1}{s}} \sum_{n > A(\varepsilon)} g'(n)E\{\sqrt{n}|\hat{\sigma}_n^2 - \sigma^2| - \varepsilon g^s(n)\}_+ = 0.$$

Proof. In view of the representation of $\hat{\sigma}_n^2$, we have

$$\sqrt{n}(\hat{\sigma}_n^2 - \sigma^2) = \frac{\sqrt{n}}{n-\gamma}\sum_{i=1}^{n}(e_i^2 - \sigma^2) + \frac{\gamma\sqrt{n}}{n-\gamma}\sigma^2 - \frac{\sqrt{n}}{n-\gamma}\sum_{j=1}^{\gamma}(\sum_{i=1}^{n} a_{nji}e_i)^2.$$

Thus, to complete the proof of this proposition, it only needs to show

$$\lim_{M\to\infty}\lim_{\varepsilon\searrow 0}\varepsilon^{\frac{1}{s}}\sum_{n>A(\varepsilon)}g'(n)E\{\frac{\left|\sum\limits_{i=1}^{n}(e_i^2-\sigma^2)\right|}{\sqrt{n}}-\varepsilon g^s(n)\}_+=0, \tag{6}$$

$$\lim_{M\to\infty}\lim_{\varepsilon\searrow 0}\varepsilon^{\frac{1}{s}}\sum_{n>A(\varepsilon)}g'(n)E\{\frac{\gamma\sigma^2}{\sqrt{n}}-\varepsilon g^s(n)\}_+=0, \tag{7}$$

and

$$\lim_{M\to\infty}\lim_{\varepsilon\searrow 0}\varepsilon^{\frac{1}{s}}\sum_{n>A(\varepsilon)}g'(n)E\{\frac{\left|\sum\limits_{i=1}^{n}a_{ni}e_i\right|^2}{\sqrt{n}}-\varepsilon g^s(n)\}_+=0. \tag{8}$$

For (6), denote $S_n=\sum\limits_{i=1}^{n}(e_i^2-\sigma^2)$. By Marcinkiewicz-Zygmund's inequality, Minkowski's inequality and the properties of $\{e_i,\ i\geq 1\}$, it follows that $E|S_n|^r\leq Cn^{\frac{r}{2}}$ for $r>2$. Then, with $q>\max\{\frac{1}{s}+1,2\}$ and Markov's inequality, it follows that

$$\begin{aligned}
&\varepsilon^{\frac{1}{s}}\sum_{n>A(\varepsilon)}g'(n)E\{\frac{|S_n|}{\sqrt{n}}-\varepsilon g^s(n)\}_+\\
&=\varepsilon^{\frac{1}{s}}\sum_{n>A(\varepsilon)}g'(n)\int_{\varepsilon g^s(n)}^{\infty}P(\frac{|S_n|}{\sqrt{n}}\geq x)dx\\
&\leq\varepsilon^{\frac{1}{s}}\sum_{n>A(\varepsilon)}g'(n)\int_{\varepsilon g^s(n)}^{\infty}\frac{E|\frac{S_n}{\sqrt{n}}|^q}{x^q}dx\\
&\leq C\varepsilon^{\frac{1}{s}}\sum_{n>A(\varepsilon)}g'(n)\int_{\varepsilon g^s(n)}^{\infty}\frac{1}{x^q}dx\\
&\leq C\varepsilon^{\frac{1}{s}-q+1}\sum_{n>A(\varepsilon)}g'(n)g^{-s(q-1)}(n)\\
&\leq C\varepsilon^{\frac{1}{s}-q+1}\int_{A(\varepsilon)}^{\infty}g'(x)g^{-s(q-1)}(x)dx\\
&\leq CM^{-s(q-1)+1}\to 0,\ as\ M\to\infty.
\end{aligned}$$

For (7), this is obvious. Since $\varepsilon g^s(n)\geq M^s\to\infty$ as $n>A(\varepsilon)$ and $M\to\infty$, and $\frac{\gamma\sigma^2}{\sqrt{n}}$ is bounded, it is obvious $E\{\frac{\gamma\sigma^2}{\sqrt{n}}-\varepsilon g^s(n)\}_+=0$ as M is large enough, and hence (7) holds true.

Now, we are in a position to cope with (8). With the mentioned above q, and Chebyshev's inequality, it follows that

$$\begin{aligned}
&\varepsilon^{\frac{1}{s}}\sum_{n>A(\varepsilon)}g'(n)E\{\frac{\left|\sum\limits_{i=1}^{n}a_{ni}e_i\right|^2}{\sqrt{n}}-\varepsilon g^s(n)\}_+\\
&=\varepsilon^{\frac{1}{s}}\sum_{n>A(\varepsilon)}g'(n)\int_{\varepsilon g^s(n)}^{\infty}P(\frac{\left|\sum\limits_{i=1}^{n}a_{ni}e_i\right|^2}{\sqrt{n}}\geq x)dx\\
&\leq C\varepsilon^{\frac{1}{s}}\sum_{n>A(\varepsilon)}g'(n)\int_{\varepsilon g^s(n)}^{\infty}\frac{E\left|\sum\limits_{i=1}^{n}a_{ni}e_i\right|^{2q}}{x^q}dx
\end{aligned}$$

$$
\begin{aligned}
&\le C\varepsilon^{\frac{1}{s}} \sum_{n>A(\varepsilon)} g'(n) \int_{\varepsilon g^s(n)}^{\infty} \frac{1}{x^q} dx \\
&\le C\varepsilon^{\frac{1}{s}-q+1} \sum_{n>A(\varepsilon)} g'(n) g^{-s(q-1)}(n) \\
&\le C\varepsilon^{\frac{1}{s}-q+1} \int_{A(\varepsilon)}^{\infty} g'(x) g^{-s(q-1)}(x) dx \\
&\le CM^{-s(q-1)+1} \to 0,\ as\ M \to \infty.
\end{aligned}
$$

Here, we need the following fact

$$E\left|\sum_{i=1}^{n} a_{ni} e_i\right|^{2q} \le C.$$

In fact, in view of Marcinkiewicz-Zygmund's inequality, Minkowski's inequality and (3), we have that

$$E\left|\sum_{i=1}^{n} a_{ni} e_i\right|^{2q} \le CE\left(\sum_{i=1}^{n} a_{ni}^2 e_i^2\right)^{q} \le C\left(\sum_{i=1}^{n} a_{ni}^2 (E|e_i|^{2q})^{1/q}\right)^{q} \le C.$$

Until now, this proposition holds. ■

The proof of Theorem 2 now follows from the propositions.

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References

[1] L.E. Baum, M. Katz. Convergence rates in the law of large numbers. Trans. Amer. Math. Soc., 120(1965):108-123.
[2] R. Chen. A remark on the tail probability of a distribution. J. Multiv. Anal., 8(1978):328-333.
[3] X. Chen. Invariance principles for the estimation of the error variance. Chinese Sci. Bull., 25(1980):81-85.
[4] Y. S. Chow. On the rate of moment convergence of sample sums and extremes. Bull. Inst. Math. Acad. Sinica., 16(1988):177-201.
[5] P. Erdös. On a theorem of Hsu and Robbins. Ann. Math. Statist., 20(1949):286-291.
[6] P. Erdös. Remark on my paper "On a theorem of Hsu and Robbins". Ann. Math. Statist., 21(1950):138.
[7] K. Fu, W. Liu, L. Zhang. Precise asymptotics for the first moment of the error variance estimator in linear models. App. Math. Lett., 21(2008):641-647.
[8] A. Gut, A. Spătaru. Precise asymptotics in the Baum-Katz and Davis law of large numbers. J. Math. Anal. Appl. , 248(2000):233-246.
[9] A. Gut, A. Spătaru. Precise asymptotics in the law of the iterated logarithm. Ann. Probab., 28(2000):1870-1883.
[10] A. Gut, A. Spătaru. Precise asymptotics in some strong limit theorems for multidimensionally indexed random variables. J. Multivariate. Anal., 86(2003):398-422.
[11] C. C. Heyde. A supplement to the strong law of large numbers. J. Appl. Probab., 12(1975): 173-175.
[12] P. L. Hsu, H. Robbins. Complete convergence and the strong law of large numbers. Proc. Nat. Acad. Sci. USA., 33 (1947):25-31.
[13] W. Huang, L. Zhang. Precise asymptotics in the law of the logarithm in the Hilbert space. J. Math. Anal. Appl., 304(2005):734-758.
[14] Y. Jiang, L. Zhang. Precise asymptotics in the law of iterated logarithm for the moment of i.i.d. random variables. Acta. Math. Sinica., 22(2006):781-792.
[15] H. Lanzinger, U. Stadtmüller. Refined Baum-Katz Laws for weighted sums of iid random variables. Statist. Probab. Lett., 69(2004):357-368.
[16] Y. Li. Precise asmptotics in complete moment convergence of moving-average processes. Statist. Probab. Lett., 76(2006):1305-1315.
[17] C. Lu, J. Qiu. Precise asymptotics in the estimation of the error variance in linear models. Chinese Ann. Math. Ser. A., 26 (1) (2005):83-92.

[18] A. Spătaru. Exact asymptotics in log log laws for random fields. J. Theoret. Probab., 17(2004):943-965.

[19] A. Spătaru. Precise asymptotics for a series of T.L. Lai. Proc. Amer. Math. Soc., 132(2004):3387-3395.

[20] F. Spitzer. A combinatorial lemma and its applications to probability theory. Trans. Amer. Math. Soc., 82(1956):323-339.

[21] W.F. Stout. Almost Sure Convergence. Academic, New York. 1995.

[22] H. Yu. Complete convergence for the estimation of the error variance in linear models. Appl. Math. J. Chinese Univ.,3(1988):28-40.

[23] Y. Zhang. Some results on limit theory for sequences of random variables. Doctorate dissertation of Jilin University. 2009.