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Homotopy Analysis and Homotopy Padé Methods for (2+1)-dimensional Boiti-Leon-Pempinelli System

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Abstract: In this paper, analytic solutions of (2+1)-dimensional Boiti-Leon-Pempinelli (BLP) system are obtained by the Homotopy analysis and the Homotopy Padé methods. The obtained approximation by using Homotopy method contains an auxiliary parameter which is a simple way to control and adjust the convergence region and rate of solution series. The approximation solutions by $[m, m]$ Homotopy Padé technique is often independent of auxiliary parameter $\hbar$ and this technique accelerate the convergence of the related series.

Keywords: Homotopy analysis method; Homotopy Padé technique; (2+1)-dimensional Boiti-Leon-Pempinelli system.

1 Introduction

Solving nonlinear problems is inherently difficult even by means of numerical methods, and the stronger the nonlinearity, the more intractable solutions becomes.Very a few nonlinear problems have closed-form solutions. In most cases, approximate techniques are used to give asymptotic results or series solution of a given nonlinear equation. Perturbation techniques [1]-[5] are widely applied to solve nonlinear equations in science and engineering.However, it is a pity that perturbation techniques are in principle based on small/large physical parameters (perturbation quantity) and thus perturbation approximations often break down as the nonlinearity becomes strong. To avoid this restrictions, some non-perturbative techniques, such as Lyapunov artificial small parameter method [6], Adomian decomposition method [7, 8] and the d-expansion method [9, 10], are developed. Although these non-perturbative techniques seem to have nothing to do with small/large physical parameters, they however cannot guarantee the convergence of solution series, and thus are still valid only for problems with weak nonlinearity, too.

Homotopy analysis method(HAM), first proposed by Liao in his Ph.D dissertation [11], is an elegant method which has proved its effectiveness and efficiency in solving many types of nonlinear equations [12]-[14]. Liao in his book [15] proved that HAM is a generalization of some previously used techniques such as the δ-expansion method, artificial small parameter method [16] and Adomian decomposition method. Moreover, unlike previous analytic techniques, the HAM provides a convenient way to adjust and control the region and rate of convergence [17]. There exist some techniques to accelerate the convergence of a given series. Among them, the so-called Padé method is widely applied [12, 15].

In this paper we apply Homotopy analysis and Homotopy Padé methods for BLP system [18]:

$$\begin{aligned} &u_{ty} = (u^2 - u_x)_{xy} + 2v_{xxx}, \\ &v_t = v_{xx} + 2uv_x. \end{aligned} \tag{1}$$

A considerable research work has been invested in [18]-[24] to study the BLP system (1). The integrability of this system was studied in [18] by using the sine-Gordon and the sinh-Gordon equations. Other works have been conducted by using other methods such as Jacobi elliptic methods, balance methods and others [19]-[24].

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2 Homotopy Analysis Method

For convenience of the readers, we will first present a brief description of the standard HAM. To achieve our goal, let us assume the nonlinear system of differential equations be in the form of

$$N_j[u_1(x,t), u_2(x,t), \cdots, u_m(x,t)] = 0, \quad j = 1 \cdots n, \tag{2}$$

where N_j are nonlinear operators, t is an independent variable, $u_i(t)$ are unknown functions. By means of generalizing the traditional homotopy method, Liao construct the zeroth-order deformation equation as follows

$$\begin{aligned}&(1-q)L_j[\phi_i(x,t,q) - u_{i,0}(x,t)]\\ &= q\hbar H(t) N_j[\phi_1(x,t,q), \phi_2(x,t,q), \cdots, \phi_m(x,t,q)],\\ &\quad i = 1 \cdots m; \qquad\qquad j = 1, \cdots n,\end{aligned} \tag{3}$$

where $q \in [0,1]$ is an embedding parameter, L_j are linear operators, $u_{i,0}(x,t)$ are initial guesses of $u_i(x,t)$, $\phi_i(x,t;q)$ are unknown functions, $\hbar$ and $H(x,t)$ are auxiliary parameter and auxiliary function respectively. It is important to note that, one has great freedom to choose auxiliary objects such as $\hbar$ and L_j in HAM; This freedom plays an important role in establishing the keystone of validity and flexibility of HAM as shown in this paper. Obviously, when $q = 0$ and $q = 1$, both

$$\phi_i(x,t,0) = u_{i,0}(x,t) \quad \text{and} \quad \phi_i(x,t,1) = u_i(x,t), \quad i = 1 \cdots m, \tag{4}$$

hold. Thus as q increases from 0 to 1, the solutions of $\phi_i(x,t;q)$ change from the initial guesses $u_{i,0}(x,t)$ to the solutions $u_i(x,t)$. Expanding $\phi_i(x,t;q)$ in Taylor series with respect to q, one has

$$\phi_i(x,t,q) = u_{i,0}(x,t) + \sum_{k=1}^{+\infty} u_{i,k}(x,t) q^k, \quad i = 1 \cdots m, \tag{5}$$

where

$$u_{i,k}(x,t) \quad = \quad \frac{1}{k!}\frac{\partial^k \phi_i(x,t,q)}{\partial q^k}\Big|_{q=0}, \quad i = 1 \cdots m. \tag{6}$$

If the auxiliary linear operators, the initial guesses, the auxiliary parameter $\hbar$, and the auxiliary function are so properly chosen, then the series (5) converges at $q = 1$, then one has

$$\phi_i(x,t,1) = u_{i,0}(x,t) + \sum_{k=1}^{+\infty} u_{i,k}(x,t), \quad i = 1 \cdots m, \tag{7}$$

which must be one of the solutions of the original nonlinear equations, as proved by Liao. Define the vectors

$$\vec{u}_{i,n}(t) = \{u_{i,0}(x,t), u_{i,1}(x,t), \cdots, u_{i,n}(x,t)\}, \quad i = 1 \cdots m. \tag{8}$$

Differentiating (3), k times with respect to the embedding parameter q and then setting $q = 0$ and finally dividing them by $k!$, we have the so-called kth-order deformation equation

$$L_j[u_{i,k}(x,t) - \chi_k u_{i,k-1}(x,t)] = \hbar R_{j,k}(\vec{u}_{i,k-1}(x,t)), \quad i = 1 \cdots m;\ j = 1 \cdots n, \tag{9}$$

subject to the initial conditions

$$L_j(0) = 0,$$

where

$$R_{j,k}(\vec{u}_{i,k-1}(x,t)) = \frac{1}{(k-1)!}\frac{\partial^{k-1} N_j[\phi_1(x,t,q), \phi_2(x,t,q), \cdots, \phi_m(x,t,q)]}{\partial q^{k-1}}\Big|_{q=0}, \tag{10}$$

and

$$\chi_{\mathrm{m}} = \begin{cases} 0 & m \leq 1, \\ 1 & m > 1. \end{cases} \tag{11}$$

It should be emphasized that $u_{i,k}(x,t)$ is governed by the linear equations (9) and (10) with the linear boundary conditions that come from the original problem. These equations can be easily solved by symbolic computation softwares such as Maple and Mathematica.

3 Homotopy Padé Method

Traditionally the $[m, n]$ Padé for $u(x, t)$ is in the form

$$\frac{\sum_{k=0}^{m} F_k(x)t^k}{1+\sum_{k=1}^{n} F_{m+1+k}(x)t^k},$$

or

$$\frac{\sum_{k=0}^{m} G_k(t)x^k}{1+\sum_{k=1}^{n} G_{k+m+1}(t)x^k},$$

where $F_k(r)$ and $G_k(t)$ are functions.

In Homotopy Padé approximation, we employ the traditional Padé technique to the series (5) for the embedding parameter q to gain the $[m, n]$ Padé approximation in the form of

$$\frac{\sum_{k=0}^{m} w_k(x,t)q^k}{1+\sum_{k=1}^{n} w_{m+k+1}(x,t)q^k}, \tag{12}$$

where $w_k(t, x)$ is a function and for $i = 0, 1, \cdots, m, m+2, \cdots, m+n+1$, $w_i(x, t)$ is determined by product of the denominator of the above expression in the $\sum_{i=0}^{m+n} u_i(x,t)q^i$ and equating the powers of $q^i,\ i = 0, 1, \cdots, m+n$. Thus we have $m+n+1$ equations and $m+n+1$ unknowns $w_i(x,t),\ \ i = 0, 1, \cdots, m, m+2, \cdots, m+n+1$. By setting $q = 1$ in (12) the so-called $[m, n]$ Homotopy Padé approximation in the following form is yield.

$$\frac{\sum_{k=0}^{m} w_k(x,t)}{1+\sum_{k=1}^{n} w_{m+k+1}(x,t)}. \tag{13}$$

It is found that the $[m, n]$ homotopy Padé approximation often converges faster than the corresponding traditional $[m, n]$ Padé approximation and in many cases the $[m, m]$ Homotopy Padé approximation is independent of the auxiliary parameter $\hbar$.

4 Application

In this section we apply HAM and HPadéM to solve the BLP system (1). In all cases, we assume that the initial guesses $u_0(x, y, t) = u(x, y, 0)$ and $v_0(x, y, t) = v(x, y, 0)$ i.e. are the initial conditions, and we use the auxiliary linear operator $L_j = \frac{\partial}{\partial t}$ and the auxiliary function $H(x, y, t) = 1$. We give approximations of compute error terms to show the efficiency of HAM and HPadéM.

4.1 (2+1)-dimensional Boiti-Leon-Pempinelli system

Let us consider the BLP system (1) with the exact solutions

$$\begin{aligned} u(x,y,t) &= b_1 + b_1 \tanh\xi, \\ v(x,y,t) &= c_0 + p\tanh\xi, \end{aligned} \tag{14}$$

where $\xi = b_1x + py + 2b_1^2t + l$; b_1, c_0, p and l are arbitary constants. For simplicity, $b_1 = p = 1, c_0 = 0.1$ and $l = 1.5$ are used for constants. Then, the (14) takes the following form

$$\begin{aligned} u(x,y,t) &= 1 + \tanh(x+y+2t+1.5), \\ v(x,y,t) &= 0.1 + \tanh(x+y+2t+1.5). \end{aligned} \tag{15}$$

For application of the homotopy analysis method, we choose the initial approximations

$$\begin{aligned} u_0(x,y,t) &= 1 + \tanh(x+y+1.5), \\ v_0(x,y,t) &= 0.1 + \tanh(x+y+1.5). \end{aligned} \tag{16}$$

Employing HAM with mentioned parameters in section 2, we have the following zero-order deformation equations

$$\begin{aligned} (1-q)L_i[\phi_{1t} - u(x,y,0)] &= q\hbar[\phi_{1ty} - (\phi_1^2 - \phi_{1x})_{xy} + 2\phi_{2xxx}], \\ (1-q)L_i[\phi_{2t} - v(x,y,0)] &= q\hbar[\phi_{2t} - \phi_{2x} - 2\phi_1\phi_{2x}]. \end{aligned} \tag{17}$$

Subsequently solving the Nth order deformation equations one has

$$
\begin{aligned}
u_0(x,y,t) &= 1+\tanh(x+y+1.5),\\
v_0(x,y,t) &= 0.1+\tanh(x+y+1.5),\\
\\
u_1(x,y,t) &= 2\hbar t(-1+\tanh^2(x+y+1.5)),\\
v_1(x,y,t) &= -4\hbar t\tanh(x+y+1.5)(-1+\tanh^2(x+y+1.5)),\\
\\
u_2(x,y,t) &= -2\hbar t\text{sech}^2(x+y+1.5)\\
&\quad +6\hbar t\tanh(x+y+1.5)\text{sech}^4(x+y+1.5)\\
&\quad +4\hbar t\text{sech}^2(x+y+1.5)\\
&\quad -5\hbar t\text{sech}^4(x+y+1.5)\\
&\quad +2\hbar\tanh(x+y+1.5)\text{sech}^2(x+y+1.5)),\\
v_2(x,y,t) &= 8\hbar t(\tanh(x+y+1.5)\text{sech}^2(x+y+1.5)\\
&\quad -12\hbar t\tanh(x+y+1.5)\text{sech}^4(x+y+1.5)\\
&\quad +10\hbar t\text{sech}^2(x+y+1.5)\\
&\quad -63\hbar t\text{sech}^4(x+y+1.5)\\
&\quad +60\hbar t\text{sech}^6(x+y+1.5)\\
&\quad +\hbar\tanh(x+y+1.5)\text{sech}^2(x+y+1.5)),
\end{aligned}
$$

and so on.
We use an 9-term approximation and set

$$U_{HAM}=u_0+u_1+u_2+...+u_8, \quad \text{and} \quad V_{HAM}=v_0+v_1+v_2+...+v_8.$$

We declare the results for 8th order HAM approximations in Tables 1 and 2. The results obtained with $\hbar=-0.7$ are better than $\hbar=-1$. Hence, the outputs of HAM are better than the Homotopy perturbation method. The influence of $\hbar$ on the convergence of the solution series are given in Figure 1. Moreover the absolute error for u is drawn in Figure 2. It is easy to see that in order to have a good approximation, $\hbar$ has to be chosen in $-1.1<\hbar<0$. This means that for these values of $\hbar$ the series (7) converges to the exact solution (1). In Table 3, the absolute errors of approximation results are given with $[4,4]$ HPad$\acute{e}$M. It is shown the HPad$\acute{e}$M accelerate the convergence of the related series.

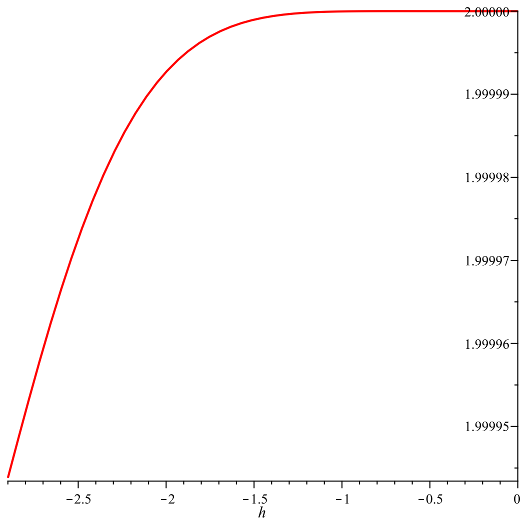

Figure 1: The $\hbar$ curve of BLP system for u(7,4,0.3) obtained from the 8th order HAM.

Table 1: Absolute error of BLP system for 8th order HAM with $\hbar = -0.7$ and $t = 0.3$

$x\|y$	y=1	y=4	y=7	y=10
$\|u_a - u_e\|$				
1	3.83E-01	1.06E-03	2.62E-06	6.51E-09
4	1.06E-03	2.62E-06	6.51E-09	1.61E-11
7	2.62E-06	6.51E-09	1.61E-11	4.00E-14
10	6.51E-09	1.61E-11	4.00E-14	9.91E-17
$\|v_a - v_e\|$				
1	5.30E-05	6.27E-09	1.48E-11	3.66E-14
4	6.27E-09	1.48E-11	3.66E-14	9.07E-17
7	1.48E-11	3.66E-14	9.07E-17	2.25E-19
10	3.66E-14	9.07E-17	2.25E-19	5.57E-22

Table 2: Absolute error of BLP system for 8th order HPM with $t = 0.3$

$x\|y$	y=1	y=4	y=7	y=10
$\|u_a - u_e\|$				
1	1.16E-00	3.29E-03	8.15E-06	2.02E-08
4	3.29E-03	8.15E-06	2.02E-08	5.01E-11
7	8.15E-06	2.02E-08	5.01E-11	1.24E-13
10	2.02E-08	5.01E-11	1.24E-13	3.08E-16
$\|v_a - v_e\|$				
1	1.82E-04	7.88E-08	1.93E-10	4.78E-13
4	7.88E-08	1.93E-10	4.78E-13	1.18E-15
7	1.93E-10	4.78E-13	1.18E-15	2.94E-18
10	4.78E-13	1.18E-15	2.94E-18	7.28E-21

Table 3: Absolute error of BLP system for $[4, 4]$ HPad$\acute{e}$M with $t = 0.3$

$x\|y$	y=1	y=4	y=7	y=10
$\|u_a - u_e\|$				
1	1.34E-03	3.83E-07	9.95E-10	2.44E-12
4	3.83E-07	9.95E-10	2.44E-12	6.05E-15
7	9.95E-10	2.44E-12	6.05E-15	1.50E-17
10	2.44E-12	6.05E-15	1.50E-17	3.72E-20
$\|v_a - v_e\|$				
1	5.69E-06	4.54E-10	2.57E-13	9.27E-18
4	4.54E-10	2.57E-13	9.27E-18	4.44E-21
7	2.57E-13	9.27E-18	4.44E-21	1.09E-23
10	9.27E-18	4.44E-21	1.09E-23	2.70E-26

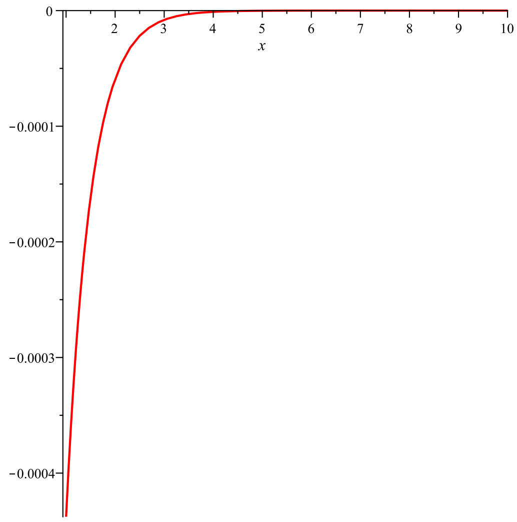

Figure 2: Absolute error curve of BLP system with 8th order HAM.

5 Conclusion

In this paper, we approximate the solutions of the (2+1)-dimensional Boiti-Leon-Pempinelli (BLP) system by the HAM and HPadéM. The convergence region for our approximation, are determined by the parameter $\hbar$, which provides us a great freedom to choose convenient value for it. It is illustrated efficiency and accuracy of proposed methods by implementing on the mentioned equations. It is shown the HPadéM accelerate the convergence of the related series.

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