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A Computational Algebraic Approach to New Exact Solutions of the Nonlinear Evolution Equations

Jianping Yu[1], Yongli Sun[2] *
[1]Department of Mathematics and Mechanics, School of Applied Science
University of Science and Technology Beijing, Beijing 100083, China
[2] Department of Mathematics and Computer Science, School of Sciences
Beijing University of Chemical Technology, Beijing 100029, China
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Abstract: This paper presents a novel method for new exact solutions of the nonlinear evolution equations. The new method is based on the property that many traveling wave solutions of nonlinear evolution equations can be written as a polynomial of two elementary functions which satisfy a projective Riccati equation, and theories of Gröbner bases. Some examples are presented to illustrate the correctness and efficiency of our method.

Keywords: nonlinear evolution equations; elliptic function solutions; mKdV equations; mKdV-Burgers equations

1 Introduction

Solving nonlinear partial differential equations is an old and important research subject, which has the theory and application value. Researchers have presented some methods for obtaining the symbolic solutions to the nonlinear evolution equations[1,2] for example, homogeneous balance method [3,4], hyperbolic function method [5], F-expansion method [6]and variable-detached method [7-10]. But, to find a general and much more efficient method becomes the important research subject. During the last years, some researchers have dealt with the exact solutions to the nonlinear evolution equations deeply, and present many methods for constructing the exact solutions [11-17]. [18] presents the definition of rank, and [19] shows that the equations with the same kind of rank have the Jacobi elliptic function solutions, and the equations with different kind of rank do not have the Jacobi elliptic sine function solutions generally.

Let the general nonlinear evolution equation be

$$H(u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, \cdots) = 0, \tag{1}$$

where H is a polynomial with respect to the variables $u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, \cdots$. Under the traveling wave transform

$$u(x,t) = u(\xi), \xi = k(x - \lambda t), \tag{2}$$

(1) can be reduced to the following ordinary differential equation

$$L(u, u^{'}, u^{''}, u^{'''}, \cdots) = 0, \tag{3}$$

where u' denotes $\frac{du}{d\xi}$.

This paper mainly deals with the approach to constructing the new exact solutions to the nonlinear evolution equations, and our method is suitable for all the nonlinear evolution equations which can be changed to the polynomial forms, whether they have the same or the different kind of rank. The method is mainly based on the theories of Gröbner bases[20], which is much efficient for solving a system of algebraic equations, projective Riccati equations [21-27], and the Jacobi elliptic equation.

*Corresponding author. *E-mail address*: sunyl@mail.buct.edu.cn
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2 Description of the Method

In order to find the solutions of equation (1), let us introduce some important equations.

The projective Riccati equations are

$$F^{'}(\xi) = pF(\xi)G(\xi), \tag{4}$$

$$G^{'}(\xi) = q + pG^2(\xi) - rF(\xi), \tag{5}$$

where p, q, r are constants.

It is well known that the Jacobi elliptic equation[**?**] can be written as

$$\phi^{'^2}(\xi) = h_0 + h_2\phi^2(\xi) + h_4\phi^4(\xi). \tag{6}$$

When

$$h_0 = h_4 = -\frac{pq}{4}, h_2 = \frac{pq}{2},$$

then we can determine the general solutions of the equations of (4) and (5) as follows: (I) If

$$G^2(\xi) = \frac{2r}{p}F(\xi) - \frac{q}{p}, \tag{7}$$

then the equations (4) and (5) have the following Jacobi elliptic function solution

$$rclF(\xi) = \frac{q}{2r} - \frac{q}{2r}\phi^2(\xi),\ \ G(\xi) = -\frac{2\phi(\xi)\phi^{'}(\xi)}{p - p\phi^2(\xi)}. \tag{8}$$

(II) If

$$G^2(\xi) = -\frac{q}{p} + \frac{2r}{p}F(\xi) - \frac{r^2}{pq}F^2(\xi), \tag{9}$$

then the equations (4) and (5) have the following Jacobi elliptic function solution

$$rclF(\xi) = \frac{q}{2r} - \frac{q}{2r}\phi(\xi),\ \ G(\xi) = -\frac{\phi^{'}(\xi)}{p - p\phi(\xi)}. \tag{10}$$

According to the above results and the theories of Gröbner bases, our algorithm can be written as follows.

Algorithm GNE:

Step 1: For any given nonlinear evolution equation (1) with respect to x, t,

$$H(u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, \cdots) = 0,$$

where H is a polynomial with respect to $u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, \cdots$, by the traveling wave transform (2)

$$u(x,t) = u(\xi), \xi = x - \lambda t,$$

we can obtain (3)

$$L(u, u^{'}, u^{''}, u^{'''}, \cdots) = 0,$$

where λ is the velocity of the wave and $u' = \frac{du}{d\xi}$.

Step 2: Let

$$u(\xi) = \sum_{i=1}^{n} F^{i-1}(\xi)(a_iF(\xi) + b_iG(\xi)) + a_0 \tag{11}$$

be a solution to (3), where n can be obtained by the homogeneous balance method, $F(\xi)$ and $G(\xi)$ are the Jacobi elliptic solutions to projective Riccati equations (4) and (5), and $a_n^2 + b_n^2 \neq 0$.

Step 3: Substitute (11), (4), (5), (7) (or (9)) into (3), then collect all the coefficients of $F^i(\xi)G^j(\xi)$, and let all of them be zero, thus we can get a system of algebraic equations with respect to $\lambda, a_0, a_i, b_i (i = 1, 2, \cdots, n)$.

Step 4: With the DegRevLex ordering, we can use Gröbner bases to solve the system of algebraic equations obtained in Step 3, then $\lambda, a_0, a_i, b_i (i = 1, 2, \cdots, n)$ can be obtained.

Step 5: Substitute $a_0, a_i, b_i (i = 1, 2, \cdots, n)$ into (11), combined with (8) (or (10)) and (2), then we can obtain all the exact solutions to (1).

It is much more important to notice that we compute the Gröbner bases with the DegRevLex ordering, because the DegRevLex ordering often minimizes the amount of computation needed to produce a Gröbner basis, so if no other special properties are required, it can be the best choice of monomial order. In the following, two examples, mKdV equation and mKdV-Burgers equation, are presented to illustrate the efficiency of Algorithm GNE.

3 Examples

Example 3.1. Let us consider the mKdV euqation

$$u_t + \alpha u^2 u_x + \beta u_{xxx} = 0. \tag{12}$$

It is the equation with the same kind of rank. In view of Algorithm GNE, we first make use of the traveling wave transform (2), and integrate (12), then it becomes an ordinary differential equation as

$$k^2 \beta u^{''} + \frac{\alpha}{3} u^3 - \lambda u - c = 0, \tag{13}$$

where c is a constant. By the homogeneous balance method, we have $n = 1$, then let

$$u(\xi) = a_0 + a_1 F(\xi) + b_1 G(\xi). \tag{14}$$

Case (I): Substitute (14) into (13), and make use of (4), (5) and (7), then

$$\begin{aligned} & k^2\beta u^{''} + \frac{\alpha}{3}u^3 - \lambda u - c \\ & = \frac{1}{3}\alpha a_1^3 F^3(x) + (b_1 \alpha a_1^2 G(x) + \frac{3k^2\beta a_1 p^2 r + 2\alpha b_1^2 a_1 r + \alpha p a_0 a_1^2}{p}) F^2(x) \\ & +(\frac{(6b_1 p \alpha a_0 a_1 + 2\alpha b_1^3 r + 3b_1 p^2 k^2 \beta r)G(x)}{3p} + \frac{2\alpha b_1^2 a_0 r - k^2 \beta a_1 p^2 q + \alpha p a_0^2 a_1 - \alpha b_1^2 a_1 q - \lambda p a_1}{p}) F(x) \\ & + \frac{(-3b_1 p\lambda - \alpha b_1^3 q + 3b_1 p \alpha a_0^2) G(x)}{3p} + \frac{\alpha p a_0^3 - 3\lambda p a_0 - 3\alpha b_1^2 a_0 q - 3cp}{3p} = 0. \end{aligned} \tag{15}$$

Because all the coefficients of (15) have to vanish, thus we get a system of algebraic equations with respect to $\lambda, a_0, a_1, b_1, c$ as follows:

$$\begin{aligned} & \alpha a_1^3 = 0,\ \ b_1 \alpha a_1^2 = 0, \\ & 3k^2\beta a_1 p^2 r + 2\alpha b_1^2 a_1 r + \alpha p a_0 a_1^2 = 0,\ \ 6b_1 p\alpha a_0 a_1 + 2\alpha b_1^3 r + 3b_1 p^2 k^2 \beta r = 0, \\ & 2\alpha b_1^2 a_0 r - k^2\beta a_1 p^2 q + \alpha p a_0^2 a_1 - \alpha b_1^2 a_1 q - \lambda p a_1 = 0, \\ & -3b_1 p \lambda - \alpha b_1^3 q + 3b_1 p \alpha a_0^2 = 0,\ \ \alpha p a_0^3 - 3\lambda p a_0 - 3\alpha b_1^2 a_0 q - 3cp = 0. \end{aligned} \tag{16}$$

With the DegRevLex ordering, the Gröbner basis for (16) is

$$GB = [3k^2\beta b_1 p^2 + 2\alpha b_1^3, 3k^2\beta a_1 p^2 + 2\alpha b_1^2 a_1, a_1^2, a_0 b_1, a_1 a_0, 2\lambda b_1 - k^2 \beta b_1 pq, 2\lambda a_1 - k^2\beta a_1 pq, 3c + 3\lambda a_0 - \alpha a_0^3].$$

By the above Gröbner basis, we can get all the solutions of (16)

$$c = 0,\ a_0 = 0,\ a_1 = 0,\ b_1 = \pm\sqrt{-\frac{3\beta}{2\alpha}},\ \lambda = \frac{k^2\beta pq}{2}. \tag{17}$$

By (8), (14) and (17), so the Jacobi elliptic function solutions to mKdV equations are given by

$$u(x,t) = \mp \frac{2}{p}\sqrt{-\frac{3\beta}{2\alpha}} \frac{\phi(k(x - \frac{k^2\beta pq}{2}t))\phi^{'}(k(x - \frac{k^2\beta pq}{2}t))}{1 - \phi^2(k(x - \frac{k^2\beta pq}{2}t))}. \tag{18}$$

Case (II): Substitute (14) into (13), and make use of (4), (5) and (9), we can obtain that:

$$
\begin{aligned}
k^2\beta u'' + \frac{\alpha}{3}u^3 - \lambda u - c = \; & (-\frac{(2k^2\beta a_1 p^2 + \alpha b_1^2 a_1)r^2}{pq} + \frac{1}{3}\alpha a_1^3)F^3(x) \\
& +(\frac{-\alpha a_0 b_1^2 r^2}{pq} + \frac{2(2k^2\beta a_1 p^2 + \alpha b_1^2 a_1)r}{p} - k^2\beta a_1 pr + \alpha a_0 a_1^2 \\
& +(\alpha b_1 a_1^2 - \frac{(2k^2\beta b_1 p^2 + \frac{1}{3}\alpha b_1^3)r^2}{pq})G(x))F^2(x) \\
& (-\frac{(2k^2\beta a_1 p^2 + \alpha b_1^2 a_1)q}{p} + \frac{2\alpha a_0 b_1^2 r}{p} + \alpha a_0^2 a_1 - \lambda a_1 + k^2\beta a_1 pq \\
& +(\frac{2(2k^2\beta b_1 p^2 + \frac{1}{3}\alpha b_1^3)r}{p} + 2\alpha a_0 b_1 a_1 - 3k^2\beta b_1 pr)G(x))F(x) \\
& (-\frac{(2k^2\beta b_1 p^2 + \frac{1}{3}\alpha b_1^3)q}{p} + 2k^2\beta b_1 pq + \alpha a_0^2 b_1 - \lambda b_1)G(x) \\
& +\frac{1}{3}\alpha a_0^3 - \lambda a_0 - \frac{\alpha a_0 b_1^2 q}{p} - c
\end{aligned}
\tag{19}
$$

Since all the coefficients of (19) have to be zero, thus we get a system of algebraic equations with respect to $\lambda, a_0, a_1, b_1, c$ as follows:

$$
\begin{aligned}
& -\frac{(2k^2\beta a_1 p^2 + \alpha b_1^2 a_1)r^2}{pq} + \frac{1}{3}\alpha a_1^3 = 0, \\
& \frac{-\alpha a_0 b_1^2 r^2}{pq} + \frac{2(2k^2\beta a_1 p^2 + \alpha b_1^2 a_1)r}{p} - k^2\beta a_1 pr + \alpha a_0 a_1^2 = 0, \\
& \alpha b_1 a_1^2 - \frac{(2k^2\beta b_1 p^2 + \frac{1}{3}\alpha b_1^3)r^2}{pq} = 0, \\
& -\frac{(2k^2\beta a_1 p^2 + \alpha b_1^2 a_1)q}{p} + \frac{2\alpha a_0 b_1^2 r}{p} + \alpha a_0^2 a_1 - \lambda a_1 + k^2\beta a_1 pq = 0, \\
& \frac{2(2k^2\beta b_1 p^2 + \frac{1}{3}\alpha b_1^3)r}{p} + 2\alpha a_0 b_1 a_1 - 3k^2\beta b_1 pr = 0, \\
& -\frac{(2k^2\beta b_1 p^2 + \frac{1}{3}\alpha b_1^3)q}{p} + 2k^2\beta b_1 pq + \alpha a_0^2 b_1 - \lambda b_1 = 0, \\
& \frac{1}{3}\alpha a_0^3 - \lambda a_0 - \frac{\alpha a_0 b_1^2 q}{p} - c = 0.
\end{aligned}
\tag{20}
$$

With the DegRevLex ordering, the Gröbner basis for the system of algebraic equations (20) is $GB = [3k^2\beta b_1 p^2 + 2\alpha b_1^3, -3b_1\beta k^2 r^2 p + 2\alpha b_1 a_1^2 q, -6r^2p^2k^2\beta a_1 - 3r^2\alpha b_1^2 a_1 + \alpha a_1^3 pq, a_0 b_1, qpa_1^2 + 2ra_1 pa_0 + r^2 b_1^2, -k^2\beta b_1 pq + 2\lambda b_1, 2\lambda a_1 - k^2\beta a_1 pq, 3c + 3\lambda a_0 - \alpha a_0^3]$.

Then all the solutions to (20) are given by

$$c = 0,\ a_0 = \mp\frac{1}{2}kq\sqrt{\frac{6\beta p}{\alpha q}},\ a_1 = \pm kr\sqrt{\frac{6\beta p}{\alpha q}},\ b_1 = 0,\ \lambda = \frac{k^2\beta pq}{2}. \tag{21}$$

$$c = 0,\ a_0 = 0,\ a_1 = \pm kr\sqrt{\frac{3\beta p}{2\alpha q}},\ b_1 = \pm kp\sqrt{\frac{-3\beta}{2\alpha}},\ \lambda = \frac{k^2\beta pq}{2}. \tag{22}$$

By (10), (14), (21) and (22), the Jacobi elliptic function solutions to mKdV are

$$
\begin{aligned}
u_1(x,t) &= \mp\frac{1}{2}kq\sqrt{\frac{6\beta p}{\alpha q}} \pm kr\sqrt{\frac{6\beta p}{\alpha q}}(\frac{q}{2r} - \frac{q}{2r}\phi(k(x - \frac{k^2\beta pq}{2}t))) \\
&= \mp\frac{1}{2}kq\sqrt{\frac{6\beta p}{\alpha q}}\phi(k(x - \frac{k^2\beta pq}{2}t)).
\end{aligned}
$$

$$\begin{aligned} u_2(x,t) &= \pm kr\sqrt{\frac{3\beta p}{2\alpha q}}(\frac{q}{2r} - \frac{q}{2r}\phi(k(x-\frac{k^2\beta pq}{2}t)) \pm kp\sqrt{\frac{-3\beta}{2\alpha}}\frac{-\phi^{'}(k(x-\frac{k^2\beta pq}{2}t))}{p - p\phi(k(x-\frac{k^2\beta pq}{2}t))} \\ &= \pm(\frac{kq}{2}\sqrt{\frac{3\beta p}{2\alpha q}}(1-\phi(k(x-\frac{k^2\beta pq}{2}t))) - k\sqrt{\frac{-3\beta}{2\alpha}}\frac{\phi^{'}(k(x-\frac{k^2\beta pq}{2}t))}{1-\phi(k(x-\frac{k^2\beta pq}{2}t))}). \end{aligned}$$

Example 3.2. Consider the mKdV-Burgers equation with different kind of rank

$$u_t + u^2u_x + \alpha u_{xx} + \beta u_{xxx} = 0. \tag{23}$$

Similar to Example 3.1, we first make use of the traveling wave transform (2), and integrate (23), then it becomes the following ordinary differential equation

$$k^2\beta u^{''} + \alpha k u^{'} + \frac{1}{3}u^3 - \lambda u - c = 0, \tag{24}$$

where c is an integrating constant. By the homogeneous balance method, we have $n = 1$, then

$$u(\xi) = a_0 + a_1F(\xi) + b_1G(\xi). \tag{25}$$

Case (I): Substitute (25) into (24), and make use of (4), (5) and (7), then

$$\begin{aligned} &k^2\beta u^{''} + \alpha k u^{'} + \frac{1}{3}u^3 - \lambda u - c \\ &= \frac{1}{3}a_1^3F^3(x) \\ &+(-k^2\beta a_1pr + a_0a_1^2 + \frac{2(b_1^2a_1 + 2k^2\beta a_1p^2)r}{p} + b_1a_1^2G(x))F^2(x) \\ &+(-\alpha kb_1r + a_0^2a_1 - \lambda a_1 + k^2\beta a_1pq + \frac{2(\alpha kb_1p + a_0b_1^2)r}{p} - \frac{(b_1^2a_1 + 2k^2\beta a_1p^2)q}{p} \\ &+(\frac{2(2k^2\beta b_1p^2 + \frac{1}{3}b_1^3)r}{p} + \alpha ka_1p + 2a_0b_1a_1 - 3k^2\beta b_1pr)G(x))F(x) \\ &+(-\lambda b_1 + a_0^2b_1 + 2k^2\beta b_1pq - \frac{(2k^2\beta b_1p^2 + \frac{1}{3}b_1^3)q}{p})G(x) \\ &+\alpha kb_1q - c + \frac{1}{3}a_0^3 - \frac{(\alpha kb_1p + a_0b_1^2)q}{p} - \lambda a_0. \end{aligned} \tag{26}$$

Since all the coefficients of (26) have to be zero, thus we get a system of algebraic equations with respect to $\lambda, a_0, a_1, b_1, c$ as follows:

$$\begin{aligned} &\frac{1}{3}a_1^3 = 0, \\ &-k^2\beta a_1pr + a_0a_1^2 + \frac{2(b_1^2a_1 + 2k^2\beta a_1p^2)r}{p} = 0, \\ &b_1a_1^2 = 0, \\ &-\alpha kb_1r + a_0^2a_1 - \lambda a_1 + k^2\beta a_1pq + \frac{2(\alpha kb_1p + a_0b_1^2)r}{p} - \frac{(b_1^2a_1 + 2k^2\beta a_1p^2)q}{p} = 0, \\ &\frac{2(2k^2\beta b_1p^2 + \frac{1}{3}b_1^3)r}{p} + \alpha ka_1p + 2a_0b_1a_1 - 3k^2\beta b_1pr = 0, \\ &-\lambda b_1 + a_0^2b_1 + 2k^2\beta b_1pq - \frac{(2k^2\beta b_1p^2 + \frac{1}{3}b_1^3)q}{p} = 0, \\ &\alpha kb_1q - c + \frac{1}{3}a_0^3 - \frac{(\alpha kb_1p + a_0b_1^2)q}{p} - \lambda a_0 = 0. \end{aligned} \tag{27}$$

With the DegRevLex ordering, the Gröbner basis for the system of equations (27) is obtained as:

$GB = [3k^2\beta b_1p^2 + 2b_1^3, 3k^2\beta a_1p^2 + 2b_1^2a_1, a_1^2, -\alpha b_1^2 + 3b_1ka_0\beta p, -a_1\alpha b_1 + 3a_1ka_0\beta p, (-3\beta^2k^2qp + \alpha^2)b_1 + 6\lambda b_1\beta, (-3\beta^2k^2qp + \alpha^2)a_1 + 6\beta\lambda a_1, 6c - 2a_0^3 - 3\alpha kb_1q + 6\lambda a_0]$.

By the above Gröbner basis, we obtain the solutions to (27) as follows:

$$\begin{aligned} &c = \pm\frac{(9\beta^2k^2qp + \alpha^2)\alpha}{27\beta^2}\sqrt{-\frac{3\beta}{2}},\ a_0 = \pm\frac{\alpha}{3\beta}\sqrt{-\frac{3\beta}{2}},\ a_1 = 0, \\ &b_1 = \pm pk\sqrt{-\frac{3\beta}{2}},\ \lambda = \frac{3\beta^2k^2qp - \alpha^2}{6\beta}. \end{aligned} \tag{28}$$

According to (8), (25) and (28), the solutions to the equation (23) are given by

$$u(x,t) = \pm\frac{\alpha}{\beta}\sqrt{-\frac{\beta}{6}} \mp k\sqrt{-6\beta}\frac{\phi(k(x - \frac{3\beta^2k^2qp - \alpha^2}{6\beta}t))\phi'(k(x - \frac{3\beta^2k^2qp - \alpha^2}{6\beta}t))}{1 - \phi^2(k(x - \frac{3\beta^2k^2qp - \alpha^2}{6\beta}t))} \tag{29}$$

Case (II): Substitute (25) into (24), and make use of (4), (5) and (9), then

$$\begin{aligned} &k^2\beta u'' + \alpha ku' + \frac{1}{3}u^3 - \lambda u - c \\ &= (-\frac{(b_1^2a_1 + 2k^2\beta a_1p^2)r^2}{pq} + \frac{1}{3}a_1^3)F(x)^3 \\ &+(-\frac{(\alpha kb_1p + a_0b_1^2)r^2}{pq} - k^2\beta a_1pr + a_0a_1^2 + \frac{2(b_1^2a_1 + 2k^2\beta a_1p^2)r}{p} \\ &+(b_1a_1^2 - \frac{(2k^2\beta b_1p^2 + \frac{1}{3}b_1^3)r^2}{pq})G(x))F^2(x) \\ &+(-\alpha kb_1r + a_0^2a_1 - \lambda a_1 + k^2\beta a_1pq + \frac{2(\alpha kb_1p + a_0b_1^2)r}{p} - \frac{(b_1^2a_1 + 2k^2\beta a_1p^2)q}{p} \\ &+(\frac{2(2k^2\beta b_1p^2 + \frac{1}{3}b_1^3)r}{p} + \alpha ka_1p + 2a_0b_1a_1 - 3k^2\beta b_1pr)G(x))F(x) \\ &(-\lambda b_1 + a_0^2b_1 + 2k^2\beta b_1pq - \frac{(2k^2\beta b_1p^2 + \frac{1}{3}b_1^3)q}{p})G(x) \\ &-\lambda a_0 + \frac{1}{3}a_0^3 + \alpha kb_1q - c - \frac{(\alpha kb_1p + a_0b_1^2)q}{p}. \end{aligned} \tag{30}$$

Because all the coefficients of (30) have to vanish, thus we get another system of algebraic equations with respect to $\lambda, a_0, a_1, b_1, c$ as follows:

$$\begin{aligned} &-\frac{(b_1^2a_1 + 2k^2\beta a_1p^2)r^2}{pq} + \frac{1}{3}a_1^3 = 0, \\ &-\frac{(\alpha kb_1p + a_0b_1^2)r^2}{pq} - k^2\beta a_1pr + a_0a_1^2 + \frac{2(b_1^2a_1 + 2k^2\beta a_1p^2)r}{p} = 0, \\ &b_1a_1^2 - \frac{(2k^2\beta b_1p^2 + \frac{1}{3}b_1^3)r^2}{pq} = 0, \\ &-\alpha kb_1r + a_0^2a_1 - \lambda a_1 + k^2\beta a_1pq + \frac{2(\alpha kb_1p + a_0b_1^2)r}{p} - \frac{(b_1^2a_1 + 2k^2\beta a_1p^2)q}{p} = 0, \\ &\frac{2(2k^2\beta b_1p^2 + \frac{1}{3}b_1^3)r}{p} + \alpha ka_1p + 2a_0b_1a_1 - 3k^2\beta b_1pr = 0, \\ &-\lambda b_1 + a_0^2b_1 + 2k^2\beta b_1pq - \frac{(2k^2\beta b_1p^2 + \frac{1}{3}b_1^3)q}{p} = 0, \\ &-\lambda a_0 + \frac{1}{3}a_0^3 + \alpha kb_1q - c - \frac{(\alpha kb_1p + a_0b_1^2)q}{p} = 0. \end{aligned} \tag{31}$$

Then, with the DegRevLex ordering, the Gröbner basis for the system of equations (31) is:

$GB = [2b_1^3+3k^2\beta b_1p^2, 2b_1^2a_1+3k^2\beta a_1p^2, a_1^2pq+r^2b_1^2, 3a_0\beta b_1kp-b_1^2\alpha, 3a_1\beta kpa_0-a_1b_1\alpha, (-3\beta^2k^2pq+\alpha^2)b_1+6\lambda b_1\beta, (-3\beta^2k^2pq+\alpha^2)a_1+6\lambda a_1\beta, 6c+6\lambda a_0-2a_0^3-3\alpha kb_1q]$.

From the above Gröbner basis, we get all the solutions to (31):

$$\begin{array}{l} c=\pm\dfrac{(9\beta^2k^2qp+\alpha^2)\alpha}{27\beta^2}\sqrt{-\dfrac{3\beta}{2}},\ a_0=\pm\dfrac{\alpha}{3\beta}\sqrt{-\dfrac{3\beta}{2}},\ a_1=\pm kr\sqrt{\dfrac{3\beta p}{2q}} \\ b_1=\pm pk\sqrt{-\dfrac{3\beta}{2}},\ \lambda=\dfrac{3\beta^2k^2qp-\alpha^2}{6\beta}. \end{array} \tag{32}$$

Then by (8), (25) and (32), the general solutions to the equation (23) are

$$\begin{aligned} u(x,t) &= \pm\frac{\alpha}{3\beta}\sqrt{-\frac{3\beta}{2}}\pm\frac{kq}{2}\sqrt{\frac{3\beta p}{2q}}(1-\phi(k(x-\frac{3\beta^2qp-\alpha^2}{6\beta t})) \\ &\mp k\sqrt{-\frac{3\beta}{2}}\frac{\phi^{'}(k(x-\dfrac{3\beta^2k^2qp-\alpha^2}{6\beta}t))}{1-\phi(k(x-\dfrac{3\beta^2k^2qp-\alpha^2}{6\beta}t))} \end{aligned} \tag{33}$$

4 Conclusions

This paper presents a computational method for constructing the new exact solutions to the nonlinear evolution equations. Our method is mainly based on the theories of Gröbner bases with DegRevLex ordering and the relations between the Jacobi elliptic function equation

$$\phi^{'2}(\xi)=h_0+h_2\phi^2(\xi)+h_4\phi^4(\xi)$$

and the projective Riccati equations

$$\begin{array}{l} F^{'}(\xi)=pF(\xi)G(\xi), \\ G^{'}(\xi)=q+pG^2(\xi)-rF(\xi). \end{array}$$

We can not only obtain the new exact solutions of the nonlinear evolution equations with the same kind of rank, but also the Jacobi elliptic solutions to the nonlinear evolution equations with the different kind of rank. The correctness and efficiency of the method are illustrated by solving mKdV equation and mKdV-Burgers equation.

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