

Homotopy Perturbation Transform Method for Solving Initial Boundary Value Problems of Variable Coefficients

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Abstract: In this paper, we apply homotopy perturbation transform method for solving initial boundary value problems. This method is a combined form of the Laplace Transform method with the Homotopy Perturbation method. The nonlinear terms can be easily handled by the use of He's Polynomials. The method finds the solutions without any discretization or restrictive assumptions and free from round-off errors and therefore reduces the numerical computations to a great extent. The results reveal that the this method is very efficient, simple and can be applied to other nonlinear problems.

Keywords: Laplace transform method; Homotopy perturbation method; initial boundary value problems; He's Polynomials

1 Introduction

With the rapid development of nonlinear science, there has appeared ever-increasing interest of scientists and engineers in the analytical asymptotic techniques for nonlinear problems such as solid state physics, plasma physics, fluid mechanics and applied sciences. In many different fields of science and engineering, it is important to obtain exact or numerical solution of the linear and nonlinear partial differential equations. Searching of exact and numerical solution of nonlinear equations in science and engineering is still quite problematic that's need new methods for finding the exact and approximate solutions. There are many new approaches for nonlinear equations which were proposed, such as Bäcklund transformation [1], Hirota's bilinear method [2, 3], the homogeneous balance method [4], the Riccati expansion method [5], δ -expansion method [6], Perturbation method [7-9], Variational iteration method (VIM) [10-17], Adomain's decomposition method (ADM) [18-20], Laplace decomposition method [21] and Variational iteration decomposition method [22]. In particular, a higher dimensional initial boundary value problem with variable coefficients is of much interest. The numerical and analytical solutions of higher dimensional initial boundary value problems, linear and nonlinear, are of considerable significance for applied sciences. The following problem has been solved by various techniques such as the Adomain's decomposition method (ADM) [20], Variational iteration decomposition method (VIDM) [22] and Variational homotopy perturbation method (VHPM) [23]. Several techniques including the Adomain's decomposition method, the Variational iteration method, the weighted finite difference method, the Laplace decomposition method and the Variational iteration decomposition method have their own limitation like the calculation of Adomain's polynomials and the Lagrange's multipliers. The results obtained by these methods are divergent in most cases and which results in causing a lot of chaos. To overcome these difficulties and drawbacks such new techniques are introduced for finding the approximate results. Among all of the analytical methods in open literature the homotopy perturbation method (HPM), first proposed by He [24-29] in 1998 and was further developed and improved by him. HPM is a coupling method of homotopy and perturbation method. It is worth mentioning that HPM is applied without any discretization, restrictive assumption or transformation and is free from round-off errors. The interested authors can refer [30-37] and the references therein. The Laplace transform is totally incapable of handling nonlinear equations because of the difficulties that are caused by the nonlinear terms. Various ways have been proposed recently to deal with these nonlinearities such as Adomain's decomposition method (ADM) [38], and the Laplace decomposition method (LDM) [39-43]. Furthermore, the homotopy

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perturbation method is also combined with the well known Laplace transform method [44] to produce a highly effective technique for handling many nonlinear problems.

Motivated and inspired by the ongoing research in these areas, we consider a new method, which is called the Homotopy perturbation transform method (HPTM). The suggested HPTM provides the solution in a rapid convergent series which may leads the solution in a closed form. The advantage of this method is its capability of combining of two powerful methods for obtaining exact solution for nonlinear equations. The use of He’s polynomials in the nonlinear term was first introduced by Gorbhani [45, 46]. It is worth mentioning that the HPTM is applied without any discretization or restrictive assumptions or transformations and free from round-off errors. Unlike the method of separation of variables that require initial or boundary conditions, The HPTM provides an analytical solution by using the initial conditions only. The boundary conditions can be used only to justify the obtained results. The proposed method work efficiently and the results so far are very encouraging and reliable. We would like to emphasize that the HPTM may be considered as an important and significant refinement of the previously developed techniques and can be viewed as an alternative to the recently developed methods such as Adomian’s decomposition method (ADM), Variational iteration method (VIM) and Variational iteration decomposition method (VIDM).In this paper we have considered the effectiveness of the homotopy perturbation transform method (HPTM) for solving higher dimensional initial boundary value problems with variable coefficients. Several examples are given to verify the reliability and efficiency of the homotopy perturbation transform method.

2 Homotopy perturbation transform method (HPTM)

The following method has been introduced by Y.Khan and Q.Wu [47] by combining the Homotopy Perturbation Method and Laplace Transform Method for solving various types of linear and nonlinear systems of partial differential equations. To illustrate the basic idea of HPTM, we consider a general nonlinear partial differential equation with the initial conditions of the form [47].

$$\begin{aligned}
 D u(x, t) + R u(x, t) + N u(x, t) &= g(x, t), \\
 u(x, 0) &= h(x), u_t(x, 0) = f(x),
 \end{aligned}
 \tag{1}$$

where D is the second order linear differential operator $D = \partial^2 / \partial t^2$, R is the linear differential operator of less order than D , N represents the general nonlinear differential operator and $g(x, t)$ is the source term. Taking the Laplace transform (denoted in this paper by L) on both sides of Eq. (1):

$$L [D u(x, t)] + L [R u(x, t)] + L [N u(x, t)] = L [g(x, t)].
 \tag{2}$$

Using the differentiation property of the Laplace transform, we have

$$L [u(x, t)] = \frac{h(x)}{s} + \frac{f(x)}{s^2} - \frac{1}{s^2} L [R u(x, t)] + \frac{1}{s^2} L [g(x, t)] - \frac{1}{s^2} L [N u(x, t)].
 \tag{3}$$

Operating with the Laplace inverse on both sides of Eq. (9) gives

$$u(x, t) = G(x, t) - L^{-1} \left[\frac{1}{s^2} L [R u(x, t) + N u(x, t)] \right],
 \tag{4}$$

where $G(x, t)$ represents the term arising from the source term and the prescribed initial conditions. Now we apply the homotopy perturbation method

$$u(x, t) = \sum_{n=0}^{\infty} p^n u_n(x, t),
 \tag{5}$$

and the nonlinear term can be decomposed as

$$N u(x, t) = \sum_{n=0}^{\infty} p^n H_n(u),
 \tag{6}$$

for some He’s polynomials $H_n(u)$ (see [45, 46]) that are given by

$$H_n(u_0, u_1, \dots, u_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left[N \left(\sum_{i=0}^{\infty} p^i u_i \right) \right]_{p=0}, \quad n = 0, 1, 2, 3, \dots,
 \tag{7}$$

Substituting Eq. (6) and Eq. (5) in Eq. (4) we get

$$\sum_{n=0}^{\infty} p^n u_n(x, t) = G(x, t) - p \left(L^{-1} \left[\frac{1}{s^2} L \left[R \sum_{n=0}^{\infty} p^n u_n(x, t) + \sum_{n=0}^{\infty} p^n H_n(u) \right] \right] \right), \quad (8)$$

which is the coupling of the Laplace transform and the homotopy perturbation method using He's polynomials. Comparing the coefficient of like powers of p , the following approximations are obtained.

$$\begin{aligned} p^0 : u_0(x, t) &= G(x, t), \\ p^1 : u_1(x, t) &= -\frac{1}{s^2} L [R u_0(x, t) + H_0(u)], \\ p^2 : u_2(x, t) &= -\frac{1}{s^2} L [R u_1(x, t) + H_1(u)], \\ p^3 : u_3(x, t) &= -\frac{1}{s^2} L [R u_2(x, t) + H_2(u)], \\ &\vdots \end{aligned} \quad (9)$$

3 Application

In this section, we apply the homotopy perturbation transform method (HPTM) for solving various types of initial boundary value problems.

Example 3.1. Consider the two dimensional initial boundary value problem [20, 22, 23]

$$u_{tt} = \frac{1}{2} y^2 u_{xx} + \frac{1}{2} x^2 u_{yy}, \quad x > 0, \quad 0 < y < 1, \quad t > 0, \quad (10)$$

subject to the initial conditions

$$u(x, y, 0) = x^2 + y^2, \quad u_t(x, y, 0) = -(x^2 + y^2), \quad (11)$$

and boundary conditions

$$\begin{aligned} u(0, y, t) &= y^2 e^{-t}, \quad u(1, y, t) = (1 + y^2) e^{-t}, \\ u(x, 0, t) &= x^2 e^{-t}, \quad u(x, 1, t) = (1 + x^2) e^{-t}. \end{aligned} \quad (12)$$

Taking Laplace transform both of sides subject to the initial condition, we get

$$L[u(x, y, t)] = \frac{1}{s} (x^2 + y^2) + \frac{1}{2s^2} [L(y^2 u_{xx}) + L(x^2 u_{yy})]. \quad (13)$$

Taking Inverse Laplace transform we get

$$u(x, y, t) = (x^2 + y^2) + L^{-1} \left[\frac{1}{2s^2} L(y^2 u_{xx}) + \frac{1}{2s^2} L(x^2 u_{yy}) \right]. \quad (14)$$

Now we apply the homotopy perturbation method in the form

$$u(x, t) = \sum_{n=0}^{\infty} p^n u_n(x, y, t), \quad (15)$$

Eq. (14), now reduces to

$$\begin{aligned} \sum_{n=0}^{\infty} p^n u_n(x, y, t) &= (x^2 + y^2) + p L^{-1} \left[\frac{1}{2s^2} L \left(y^2 \left(\sum_{n=0}^{\infty} p^n u_n(x, y, t) \right)_{xx} \right) \right] \\ &+ p L^{-1} \left[\frac{1}{2s^2} L \left(x^2 \left(\sum_{n=0}^{\infty} p^n u_n(x, y, t) \right)_{yy} \right) \right]. \end{aligned} \quad (16)$$

Comparing the coefficients of various powers of p , we get

$$\begin{aligned} p^0 : u_0(x, y, t) &= (x^2 + y^2) (1 - t), \\ p^1 : u_1(x, y, t) &= (x^2 + y^2) \left(\frac{t^2}{2!} - \frac{t^3}{3!} \right), \\ p^2 : u_2(x, y, t) &= (x^2 + y^2) \left(\frac{t^4}{4!} - \frac{t^5}{5!} \right), \end{aligned} \quad (17)$$

proceeding in a similar manner, we get

$$\begin{aligned}
 p^3 : u_3(x, y, t) &= (x^2 + y^2) \left(\frac{t^6}{6!} - \frac{t^7}{7!} \right), \\
 p^4 : u_4(x, y, t) &= (x^2 + y^2) \left(\frac{t^8}{8!} - \frac{t^9}{9!} \right), \\
 &\vdots
 \end{aligned}
 \tag{18}$$

Then the series solution is given by

$$u(x, y, t) = (x^2 + y^2) \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} + \dots \right) = (x^2 + y^2) e^{-t},
 \tag{19}$$

which is an exact solution and is same as obtained by ADM [20], VIDM [22] and VHPM [23].

Example 3.2 Consider the three dimensional initial boundary value problem [20, 22, 23]

$$u_{tt} = \frac{1}{45}x^2u_{xx} + \frac{1}{45}y^2u_{yy} + \frac{1}{45}z^2u_{zz} - u, \quad x > 0, \quad 0 < y < 1, \quad t > 0,
 \tag{20}$$

subject to the initial conditions

$$u(x, y, z, 0) = 0, \quad u_t(x, y, z, 0) = x^6y^6z^6,$$

and the Neumann boundary conditions

$$\begin{aligned}
 u_x(0, y, z, t) &= 0, \quad u_x(1, y, z, t) = 6y^6z^6 \sinh t \\
 u_y(x, 0, z, t) &= 0, \quad u_y(x, 1, z, t) = 6x^6z^6 \sinh t \\
 u_z(x, y, 0, t) &= 0, \quad u_z(x, y, 1, t) = 6x^6y^6 \sinh t
 \end{aligned}
 \tag{21}$$

By applying the aforesaid method subject to the initial conditions, we have

$$u(x, y, z, t) = x^6y^6z^6t + pL^{-1} \left[\frac{1}{s^2}L \left(\frac{1}{45}x^2u_{xx} \right) + \frac{1}{s^2}L \left(\frac{1}{45}y^2u_{yy} \right) \right] + pL^{-1} \left[\frac{1}{s^2}L \left(\frac{1}{45}z^2u_{zz} \right) - \frac{1}{s^2}L(u) \right].
 \tag{22}$$

Now we apply the homotopy perturbation method in the form

$$u(x, t) = \sum_{n=0}^{\infty} p^n u_n(x, y, z, t),
 \tag{23}$$

Then Eq. (22) reduces to

$$\begin{aligned}
 \sum_{n=0}^{\infty} p^n u_n(x, y, z, t) &= x^6y^6z^6t + pL^{-1} \left[\frac{1}{45s^2}L \left(x^2 \left(\sum_{n=0}^{\infty} p^n u_n(x, y, z, t) \right)_{xx} \right) \right] \\
 &+ pL^{-1} \left[\frac{1}{45s^2}L \left(y^2 \left(\sum_{n=0}^{\infty} p^n u_n(x, y, z, t) \right)_{yy} \right) \right] + pL^{-1} \left[\frac{1}{45s^2}L \left(z^2 \left(\sum_{n=0}^{\infty} p^n u_n(x, y, z, t) \right)_{zz} \right) \right] \\
 &- pL^{-1} \left[\frac{1}{s^2}L \left(\sum_{n=0}^{\infty} p^n u_n(x, y, z, t) \right) \right].
 \end{aligned}
 \tag{24}$$

Comparing the coefficient of various powers of p , we get

$$\begin{aligned}
 p^0 : u_0(x, y, z, t) &= x^6y^6z^6t, \\
 p^1 : u_1(x, y, z, t) &= x^6y^6z^6 \frac{t^3}{3!}, \\
 p^2 : u_2(x, y, z, t) &= x^6y^6z^6 \frac{t^5}{5!},
 \end{aligned}
 \tag{25}$$

Proceeding in a similar manner, we get

$$\begin{aligned}
 p^3 : u_3(x, y, z, t) &= x^6y^6z^6 \frac{t^7}{7!}, \\
 p^4 : u_4(x, y, z, t) &= x^6y^6z^6 \frac{t^9}{9!}, \\
 &\vdots
 \end{aligned}
 \tag{26}$$

Then the series solution in closed form is given by

$$u(x, y, z, t) = x^6 y^6 z^6 \left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \frac{t^7}{7!} + \dots \right) = x^6 y^6 z^6 \sinh t, \quad (27)$$

which is an exact solution and is same as obtained by ADM [20], VIDM [22] and VHPM [23].

Example 3.3 Consider the two dimensional nonlinear inhomogeneous initial boundary value problem [20, 22, 23]

$$u_{tt} = 2x^2 + 2y^2 + \frac{15}{2} (xu_{xx}^2 + yu_{yy}^2), \quad x > 0, \quad 0 < y < 1, \quad t > 0, \quad (28)$$

with the initial conditions

$$u(x, y, 0) = 0, \quad u_t(x, y, 0) = 0, \quad (29)$$

and the boundary conditions

$$\begin{aligned} u(0, y, t) &= y^2 t^2 + y t^6, \quad u(1, y, t) = (1 + y^2) t^2 + (1 + y) t^6, \\ u(x, 0, t) &= x^2 t^2 + x t^6, \quad u(x, 1, t) = (1 + x^2) t^2 + (1 + x) t^6. \end{aligned} \quad (30)$$

By applying the aforesaid method subject to the initial conditions, we get

$$\begin{aligned} \sum_{n=0}^{\infty} p^n u_n(x, y, t) &= (x^2 + y^2) t^2 + pL^{-1} \left[\frac{1}{s^2} L \left(\frac{15}{2} x \left(\sum_{n=0}^{\infty} p^n u_n(x, y, t) \right)_{xx}^2 \right) \right] \\ &+ pL^{-1} \left[\frac{1}{s^2} L \left(\frac{15}{2} y \left(\sum_{n=0}^{\infty} p^n u_n(x, y, t) \right)_{yy}^2 \right) \right]. \end{aligned} \quad (31)$$

Comparing the coefficients of various powers of p , we get

$$\begin{aligned} p^0 : \quad u_0(x, y, t) &= (x^2 + y^2) t^2, \\ p^1 : \quad u_1(x, y, t) &= (x + y) t^6, \\ p^2 : \quad u_2(x, y, t) &= 0. \end{aligned} \quad (32)$$

Proceeding in a similar manner, we get

$$p^3 : \quad u_3(x, y, t) = 0, \quad (33)$$

in general $u_n(x, y, t) = 0, \quad n \geq 2$ Therefore the complete solution is given by

$$u(x, y, t) = (x^2 + y^2) t^2 + (x + y) t^6, \quad (34)$$

which is same as obtained by ADM [20], VIDM [22] and VHPM [23].

Example 3.4 Consider the three dimensional nonlinear initial boundary value problem [20, 22, 23]

$$u_{tt} = (2 - t^2) + u - (e^{-x} u_{xx}^2 + e^{-y} u_{yy}^2 + e^{-z} u_{zz}^2), \quad x > 0, \quad 0 < y < 1, \quad t > 0, \quad (35)$$

subject to the initials conditions

$$u(x, y, z, 0) = e^x + e^y + e^z, \quad u_t(x, y, z, 0) = 0, \quad (36)$$

and the Neumann boundary conditions

$$\begin{aligned} u_x(0, y, z, t) &= 1, \quad u_x(1, y, z, t) = e, \\ u_y(x, 0, z, t) &= 0, \quad u_y(x, 1, z, t) = e, \\ u_z(x, y, 0, t) &= 1, \quad u_z(x, y, 1, t) = e. \end{aligned} \quad (37)$$

By applying the aforesaid method, subject to the initial conditions we have

$$\begin{aligned} \sum_{n=0}^{\infty} p^n u_n(x, y, z, t) &= (e^x + e^y + e^z) + \left(t^2 - \frac{t^4}{12} \right) + pL^{-1} \left[\frac{1}{s^2} L \left(\sum_{n=0}^{\infty} p^n u_n(x, y, z, t) \right) \right] \\ &- pL^{-1} \left[\frac{1}{s^2} L \left(e^{-x} \left(\sum_{n=0}^{\infty} p^n u_n(x, y, z, t) \right)_{xx}^2 \right) \right] - pL^{-1} \left[\frac{1}{s^2} L \left(e^{-y} \left(\sum_{n=0}^{\infty} p^n u_n(x, y, z, t) \right)_{yy}^2 \right) \right] \\ &- pL^{-1} \left[\frac{1}{s^2} L \left(e^{-z} \left(\sum_{n=0}^{\infty} p^n u_n(x, y, z, t) \right)_{zz}^2 \right) \right]. \end{aligned} \quad (38)$$

Comparing the coefficients of various powers of p , we get

$$\begin{aligned} p^0 : u_0(x, y, z, t) &= (e^x + e^y + e^z) + t^2 - \frac{t^4}{12}, \\ p^1 : u_1(x, y, z, t) &= \frac{t^4}{12} - \frac{t^6}{360}, \\ p^2 : u_2(x, y, z, t) &= \frac{t^6}{360} - \frac{t^8}{20160}, \\ &\vdots \end{aligned} \quad (39)$$

for obtaining the approximate solution, some terms that appear in u_{2i-1} are canceled out with the terms appearing in u_{2i} , then the complete solution is given by

$$u(x, y, z, t) = (e^x + e^y + e^z) + t^2. \quad (40)$$

which is same as obtained by ADM [20], VIDM [22] and VHPM [23].

4 Conclusions

In this paper we applied the homotopy perturbation transform method (HPTM) for solving initial boundary value problems. In previous papers [20, 22, 40, 41, 42, 43] many authors have already used Adomian's polynomials to decompose the non linear terms in equations. The solution procedure by using He's polynomials [45, 46] is simple, but the calculation of Adomian's polynomials is complex. It is worth mentioning that the method is capable of reducing the volume of the computational work as compared to the classical method while still maintaining the high level of accuracy of the numerical results. The fact that the HPTM solves nonlinear problems without using the Adomian's polynomials can be considered as a clear advantage of this algorithm over the decomposition method. Also the proposed scheme exploits full advantage of Variational iteration method (VIM), Adomian's decomposition method (ADM) and Variational iteration decomposition method (VIDM). Finally, we conclude that HPTM can be considered as a nice refinement in existing numerical technique and might find wide applications.

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