ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.31(2021) No.2/3,pp.107-112

Blow up of Smooth Solutions to Compressible Boundary Layer Equations

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(Received 10 March 2021, accepted 22 May 2021)

Abstract: In this paper, we consider certain class of compactly supported C^∞ initial data for compressible boundary layer equations. We study the role of density in the blow up criteria of smooth solutions to the boundary layer problem for two dimensional compressible Navier-Stokes equations in finite time.

Keywords: Compressible Navier-Stokes equations; smooth solutions; blow up

1 Introduction

In this paper, we consider the boundary layer problems for compressible Navier-Stokes equations which are stated as follows

$$\begin{cases} u_t + uu_x + vu_y + \dfrac{\partial_x P(\rho)}{\rho} = u_{yy}, \qquad \text{in} \quad \Omega = \{(t,x,y)|t>0, x>0, y>0\}. \\ u_x + v_y = -\rho_t, \\ u(t,x,y)\big|_{t=0} = u_0(x,y), \\ u(t,x,y)\big|_{x=0} = u_1(t,y), \\ (u,v)(t,x,y)|_{y=0} = 0, \\ u(t,x,y) \longrightarrow U(t,x) \qquad \text{as} \quad y \to \infty, \end{cases} \tag{1}$$

where the density $\rho(t,x) \in [\rho_0, \rho_1]$ for some $0 < \rho_0 \le \rho_1$, the pressure $P(\rho)$ satisfies

$$P'(\rho) > 0.$$

In addition, (P, U) satisfies the Bernoulli law

$$U_t + UU_x + \frac{\partial_x P(\rho)}{\rho} = 0. \tag{2}$$

This problem was first derived by Wang and Williams [14], in which they studied the behavior and stability of boundary layers for the two-dimensional isentropic compressible flows. In particular, it degenerates to Prandtl boundary layer problems if ρ is a positive constant. Due to the loss of regularity of u in x and non-locality in y, the general well-posedness of the problem (1) is very complicated and still open even for the incompressible case, although numerous remarkable works (ref. [1, 8, 9, 11, 13, 16]) have appeared after Prandtl's pioneering proposal [12] in 1904. Under the monotonicity condition on the tangential velocity u, the local existence of classical solution and distribution solution to (1) have been recently separative proved in [6] and [15]. They overcome the above difficulties by using Crocco's transformation and Nash-Moser iteration, respectively. On the other hand, it was shown in the work of Grenier and his colleagues [2, 5] that

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Prandtl equations is actually nonlinear unstable if the monotonicity condition fails. The mechanics of the intsablity is due to the change of sign of vorticity, which leads to the appearance of critical layers for large Reynolds number. Based on this, Yan and Nguyen [7] proved that the asymptomatic boundary layer expansion is not valid. Consequently, it seems that the monotonicity condition is a necessary condition to support the valid of boundary layer theory. To investigate this assumption, E. and Engquist [3] considered the evolution of smooth solutions to Prandtl boundary layer problems with the initial data $u_0(x, y)$ given in the form

$$u_0(x, y) = -xb_0(x, y), \tag{3}$$

for which the monotonicity condition is not valid at $x = 0$. They proved that the smooth solution must blow up in the sense that either $\lim_{t\to T} \sup_y \|u_x(t, 0, y)\| = \infty$ or $\lim_{t\to T} \|u_{xy}(t, 0, 0)\| = \infty$ for some finite time $T > 0$. The goal of our paper is to study the role of the density ρ in the formation of such kind of singularity in the compressible boundary layer problems (1).

Analogue to the work in [3], we pose the initial data $u_0(x, y)$ in (3). For simplicity, we can normalize the problem under the assumption that $U = 0$, and then we can get the following equation from (2)

$$\frac{\partial_x P(\rho)}{\rho} = 0, \tag{4}$$

which implies that ρ is a function of t.

The following is our main theorem.

Theorem 1 *Assume that $\rho_t \le 0$, and u_0 takes the form $u_0(x, y) = -xb_0(x, y)$ where $a_0(y) = -u_{0x}(0, y)$ satisfies the conditions of Lemma 2, then smooth solutions of* (1) *do not exist globally in time.*

Setting $\omega = u(t, 0, y)$ with u as a smooth solution of (1), it satisfies

$$\begin{cases} \omega_t + u_x(t, 0, y)\omega + v(t, 0, y)\omega_y = \omega_{yy}, \\ w(0, y) = 0, \qquad \omega(t, 0) = 0. \end{cases} \tag{5}$$

Applying the maximal principle of the heat equation to problem (5), we have

$$\omega \equiv 0.$$

Therefore, we can reasonably assume that

$$u(t, x, y) = -xb(t, x, y), \tag{6}$$

with $b(0, x, y) = b_0(x, y)$ and denote

$$a(t, y) := b(t, 0, y) = -u_x(t, 0, y).$$

Henceforth, based on (6), the equations in problem (1) can be transformed into

$$\begin{cases} b_t = b(b + xb_x) - vb_y + b_{yy}, \\ v = -\rho_t y - \int_0^y u_x(t, x, z)\mathrm{d}z. \end{cases} \tag{7}$$

In particular, at $x = 0$, problem (1) can be reformulated by

$$\begin{cases} a_t = a^2 + a_{yy} - (\int_0^y a(t, z)\,\mathrm{d}z - \rho_t y)a_y, \\ a|_{t=0} = a_0(y), \\ a(t, 0) = 0, \\ a(t, y) \longrightarrow 0 \qquad \text{as} \qquad y \to 0. \end{cases} \tag{8}$$

2 Proof of the theorem

In this section, we first consider the blow up of the solution a to problem (8) in finite time in order to prove Theorem 1. By developing the idea provided in [3], we define

$$F(t) = \int_0^\infty a^2 \mathrm{d}y, \qquad E(t) = \int_0^\infty (\frac{1}{2}a_y^2 - \frac{1}{4}a^3)\mathrm{d}y. \tag{9}$$

Now,we have the following Lemma for a

Lemma 2 *Suppose that the initial data $a_0(y) \geq 0$ of (8). If $E(0) < 0$ and $\rho_t \leq 0$, then there exists a finite $T \in (0, \infty)$ such that*

$$\textit{either} \ \lim_{t\to T} \sup_y |a| = \infty, \quad \textit{or} \quad \lim_{t\to T} |a_y(t, 0)| = \infty.$$

Proof. First, as a result of the maximal principle, we have

$$a \geq 0, \text{ and } \quad a_y(t, 0) \geq 0.$$

From (8), it follows by direct calculations

$$\begin{aligned}
\dot{F}(t) =& 2\int_0^\infty a\Big(a^2 + a_{yy} - a_y(\int_0^y a(t,z)\,\mathrm{d}z - \rho_t y)\Big)\mathrm{d}y \\
=& -2\int_0^\infty a_y^2(t,y)\,\mathrm{d}y + \int_0^\infty (3a - \rho_t)a^2\,\mathrm{d}y \\
\leq& \big(3\sup_y |a| + \max|\rho_t|\big)\int_0^\infty a^2\,\mathrm{d}y.
\end{aligned} \tag{10}$$

Hence, once $\sup_y |a| < \infty$, then we have $F(t)$ is bounded, denoted by $C(t)$. Meanwhile, there holds

$$\begin{aligned}
\frac{d}{dt}\int_0^\infty a(t,y)\,\mathrm{d}y =& \int_0^\infty \Big(a^2 + a_{yy} - (\int_0^y a(t,z)\,\mathrm{d}z - y\rho_t)a_y\Big)\mathrm{d}y \\
=& -a_y(t,0) + \int_0^\infty (2a^2 - \rho_t a)(t,y)\mathrm{d}y \\
\leq& 2F(t) - \rho_t\int_0^\infty a\mathrm{d}y,
\end{aligned} \tag{11}$$

Therefore, by Gronwall's inequality, we have

$$\int_0^\infty a(t,y)\,\mathrm{d}y \leq C(t).$$

This implies that $\int_0^y a\,\mathrm{d}y$ is uniformly bounded with respect to y.

Combining it with the assumption $\max_{y,t\leq T} |a| < \infty$ and (8), we know that a is exponentially decay with respect to y, which in turn indicates $E(t)$ and $F(t)$ are well-defined.

Next, we will show that

$$\lim_{t\to T} |a_y(t,0)| = \infty, \quad \text{for some } 0 < T < \infty. \tag{12}$$

Otherwise, it is easy to obtain that

$$\begin{aligned}
\frac{d}{dt}F(t) =& -2\int_0^\infty a_y^2(t,y)\,\mathrm{d}y + \int_0^\infty (3a^3 - \rho_t a^2)\,\mathrm{d}y, \\
\frac{d}{dt}\int_0^\infty \frac{1}{2}a_y^2(t,y)\,\mathrm{d}y =& \int_0^\infty a_y\Big(a_{yyy} + (a+\rho_t)a_y - (\int_0^y a\mathrm{d}y - y\rho_t)a_{yy}\Big)\mathrm{d}y \\
=& \int_0^\infty (-a_{yy}^2 + \frac{3}{2}aa_y^2 + \frac{1}{2}\rho_t a_y^2)(t,y)\,\mathrm{d}y, \\
\frac{d}{dt}\int_0^\infty \frac{1}{3}a^3(t,y)\,\mathrm{d}y =& \int_0^\infty a^2\Big(a_{yy} + a^2 - (\int_0^y a(t,z)\,\mathrm{d}z - y\rho_t)a_y\Big)\mathrm{d}y \\
=& \int_0^\infty \Big(-2aa_y^2 + \frac{4}{3}a^4 - \frac{1}{3}\rho_t a^3\Big)\,\mathrm{d}y,
\end{aligned} \tag{13}$$

in which the boundary condition $a(t,0)=a_{yy}(t,0)=0$ is used.

Since $\rho_t\le 0$, we have

$$\begin{aligned}\dot{E}(t)=&\int_0^\infty \Big(-a_{yy}^2+3aa_y^2+\frac{1}{2}\rho_t a_y^2-a^4+\frac{1}{4}\rho_t a^3\Big)\,\mathrm{d}y\\ =&\int_0^\infty \Big(-a_{yy}^2-\frac{3}{2}a^2a_{yy}-a^4+\rho_t(\frac{1}{2}a_y^2+\frac{1}{4}a^3)\Big)\,\mathrm{d}y\\ =&\int_0^\infty \Big(-(a_{yy}-\frac{3}{4}a^2)^2-\frac{7}{16}a^4-\rho_t(\frac{1}{2}a_y^2+\frac{1}{4}a^3)\Big)\,\mathrm{d}y\\ \le&0,\end{aligned}$$

Thus, there holds

$$E(t)\le 0, \tag{14}$$

provided $E(0)\le 0$. That means,

$$\int a_y^2\mathrm{d}y\le\frac{1}{2}\int a^3\mathrm{d}y. \tag{15}$$

Next, let's consider the auxiliary functional

$$G_\beta(t)=-\frac{E}{F^\beta}\ge 0$$

with $\beta\in(1,\frac{5}{4})$. It can be calculated that

$$\frac{d}{dt}G_\beta(t)=F^{-\beta-1}(\beta\dot{F}E-\dot{E}F), \tag{16}$$

where

$$\begin{aligned}-F\dot{E} \;=\;& \int_0^\infty a^2\mathrm{d}y\int_0^\infty (a_{yy}+\frac{3}{4}a^2)^2\,\mathrm{d}y+\frac{7}{16}\int_0^\infty a^2\mathrm{d}y\int_0^\infty a^4\mathrm{d}y\\ &-\rho_t\int_0^\infty a^2\mathrm{d}y\int_0^\infty(\frac{1}{2}a_y^2+\frac{1}{4}a^3)\mathrm{d}y\\ \ge\;& \Big(\int_0^\infty a(a_{yy}+\frac{3}{4}a^2)\mathrm{d}y\Big)^2+\frac{7}{16}\Big(\int_0^\infty a^3\,\mathrm{d}y\Big)^2-\rho_t\int_0^\infty a^2\,\mathrm{d}y\int_0^\infty\Big(\frac{1}{2}a_y^2+\frac{1}{4}a^3\Big)\mathrm{d}y\end{aligned}$$

By using the equation in (8), we have

$$\begin{aligned}-F\dot{E} \;\ge\;& \Big(\int_0^\infty a\big(a_t-\frac{1}{4}a^2+a_y(\int_0^y a\mathrm{d}z-y\rho_t)\big)\,\mathrm{d}y\Big)^2+\frac{7}{16}(\int_0^\infty a^3\,\mathrm{d}y)^2\\ &-\rho_t\int_0^\infty a^2\,\mathrm{d}y\int_0^\infty(\frac{1}{2}a_y^2+\frac{1}{4}a^3)\,\mathrm{d}y\\ =\;& \frac{1}{4}(F_t-\frac{3}{2}\int_0^\infty a^3\,\mathrm{d}z+\rho_t\int_0^\infty a^2\,\mathrm{d}y)^2+\frac{7}{16}(\int_0^\infty a^3\,\mathrm{d}y)^2\\ &-\rho_t\int_0^\infty a^2\,\mathrm{d}y\int_0^\infty(\frac{1}{2}a_y^2+\frac{1}{4}a^3)\,\mathrm{d}y.\end{aligned} \tag{17}$$

Due to (15), it can be obtained that

$$\begin{aligned}&F_t-\frac{3}{2}\int_0^\infty a^3(t,z)\,\mathrm{d}z+\rho_t\int_0^\infty a^2(t,z)\,\mathrm{d}z\\ =&-2\int_0^\infty a_y^2+\frac{3}{2}\int_0^\infty a^3\mathrm{d}y\ge\frac{1}{2}\int_0^\infty a^3\mathrm{d}y\ge 0.\end{aligned} \tag{18}$$

Meanwhile, we also have

$$F_t - \frac{3}{2}\int_0^\infty a^3 \, \mathrm{d}y + \rho_t \int_0^\infty a^2 \, \mathrm{d}y = -6E + \int_0^\infty a_y^2 \geq -6E \geq 0, \tag{19}$$

$$F_t \geq 2\int_0^\infty a^3 dy. \tag{20}$$

Then, it follows from (17) that

$$\begin{aligned} -F\dot{E}(t) \quad \geq \quad & -\frac{3}{2}E(F_t - \frac{3}{2}\int_0^\infty a^3 \, \mathrm{d}y + \rho_t \int_0^\infty a^2 \, \mathrm{d}y) - \frac{7}{4}E\int_0^\infty a^3 \, \mathrm{d}y \\ & -\rho_t \int_0^\infty a^2 \mathrm{d}y \int_0^\infty (\frac{1}{2}a_y^2 + \frac{1}{4}a^3). \end{aligned} \tag{21}$$

By using (19), we obtain

$$\begin{aligned} -F\dot{E}(t) + \beta\dot{F}E \quad \geq \quad & (\frac{3}{2} - \beta)(-E)F_t + \frac{1}{2}E\int_0^\infty a^3 dy - \frac{3}{2}E\int_0^\infty a^2 dy \\ & -\rho_t \int_0^\infty a^2 \mathrm{d}y \int_0^\infty (\frac{1}{2}a_y^2 + \frac{1}{4}a^3) \\ \geq \quad & (\frac{5}{2} - 2\beta)(-E)\int_0^\infty a^3 dy - \rho_t \int_0^\infty a^2 dy(-4E + \frac{3}{2}E) \\ \geq \quad & 0 \end{aligned} \tag{22}$$

as $\beta \in (1, \frac{5}{4})$.

Consequently, $G_\beta(t)$ is a function increasing in t, thereby

$$\frac{d}{dt}F \geq -6E \geq 6G_\beta(0)F^\beta, \tag{23}$$

which implies F must be infinite at some $0 < T < \infty$. Therefore, it holds

$$\sup_y |a| = \infty, \quad \text{at} \quad t = T.$$

It is contradict to the assumption, therefore, there exits some finite $T > 0$ such that

$$\lim_{t \to T} |a_y(t, 0)| = \infty, \tag{24}$$

which ends the proof of Lemma 2. ■

Remark 3 *since* $\frac{dF}{dt} \geq CF^\beta$*, then* $\frac{d}{dt}(\frac{1}{1-\beta}F^{1-\beta}) \geq C$*. Due to* $\beta \in (1, \frac{5}{4})$*, therefore,*

$$F^{1-\beta} \leq (Ct + \frac{F^{1-\beta}}{1-\beta}|_{t=0})(1-\beta) = F_0^{1-\beta} - C(\beta - 1)t,$$

which means

$$F^{\beta-1} \geq \frac{1}{F_0^{1-\beta} - C(\beta-1)t} \longrightarrow \infty \qquad \text{as} \qquad t \to T := \frac{F_0^{1-\beta}}{C(\beta-1)}. \tag{25}$$

We note that smooth solutions of (8) are unique. Hence, at $x = 0$,we have

$$sup_{y>0}|\frac{u(t,x,y)}{x}| \to +\infty$$

as $t \to T$, which implies that

$$sup_{x,y}|u_x| \to +\infty$$

as $t \to T$. This comlpetes the proof of Theorem 1.

3 Conclusion

In this paper, under the condition that the inital tangential velocity does not satisfy the monotonicity and the density decreases with time, we prove that smooth sloutions of the compressible Navier-Stokes equations blow up in finite time.

Acknowledgments

This research was supported by the National Natural Science Foundation of China (No. 11901249).

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