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Blow up Estimates and Asymptotical Behavior of Solutions of Parabolic System with Nonhomogenous Nonlinearity

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Abstract: We are concerned with the blow up of solutions for the following parabolic equations

$$\begin{cases} u_t - \Delta u = \mu_1 |u|^{p-1} u + \alpha uv, & \text{in} \quad \mathbb{R}^N \times (0,T) \\ v_t - \Delta v = \mu_2 |v|^{p-1} v + \dfrac{\alpha}{2} u^2, & \text{in} \quad \mathbb{R}^N \times (0,T) \end{cases}$$

where $\mu_1,\ \mu_2 > 0$, $\alpha \geq 0$.In this paper, we consider the case $2 < p < \frac{N+2}{N-2}$ or $N \leq 2$, and study the growth rate for non-negative solutions. In particular, we give the limit of solutions near blow up. The main technical we used is the backward self-similar solution and some weighted energy-type estimates.

Keywords: parabolic equations; blow-up; backward self-similar solution; energy estimate.

1 Introduction

In this paper we study the blow up of solutions of nonlinear heat equations on $\mathbb{R}^N$. Throughout, the functions $u : \mathbb{R}^N \times (0,T) \to \mathbb{R}$, $v : \mathbb{R}^N \times (0,T) \to \mathbb{R}$ solve the equations

$$\left\{ \begin{array}{l} u_t - \Delta u - \mu_1 |u|^{p-1} u - \alpha uv = 0, \\ v_t - \Delta v - \mu_2 |v|^{p-1} v - \dfrac{\alpha}{2} u^2 = 0, \end{array} \right. \tag{1}$$

(u,v) it is assumed to be a classical solution, where $\mu_1,\ \mu_2 > 0$, $\alpha \geq 0$,

$$2 < p < \frac{N+2}{N-2} \quad \text{or} \quad N \leq 2. \tag{2}$$

$T < \infty$ is the maximal existence time of the L^∞ solution, i. e. for every $\tau < T$

$$\sup_{x \in \mathbb{R}^N} |u(x,t)| \to \infty, \quad \sup_{x \in \mathbb{R}^N} |v(x,t)| \to \infty, \quad \text{as } t \to T. \tag{3}$$

and the initial function $u_0 = u(x,0),\ v_0 = v(x,0) \in C^2(\mathbb{R}^N)$.

We first recall some results about single equation. Systems (1) is a natural generalization of $u_t - \Delta u - |u|^{p-1}u = 0$ or $u_t - \Delta u - u^p = 0$.

In [20], Giga and Kohn have investigated the semi-linear heat equation

$$\begin{cases} u_t - \Delta u = |u|^{p-1} u, & x \in \Omega,\ t > 0, \\ u = 0, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{4}$$

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where Ω is a bounded convex domain in $\mathbb{R}^N$ or $\Omega = \mathbb{R}^N$, they obtained

$$\sup_x |u(x,t)| \le C(T-t)^{-1/p-1}, \tag{5}$$

and

$$\lim_{t\to T}(T-t)^{1/p-1}u(a+y\sqrt{T-t},t) = 0 \quad \text{or} \quad \pm k, \tag{6}$$

where $k = (p-1)^{-1/p-1}$. To prove (5), two approaches were presented. The first method, was the use of interpolation inequalities and parabolic regularity theory which following [3], it works for $1 < p < \frac{3N+8}{3N-4}$ or $N = 1$. The second method, used a contraction argument, it can apply for subcritical cases, i. e. $2 < p < \frac{N+2}{N-2}$ or $N \le 2$. Notice that latter technical also required $u \ge 0$. We also refer to the earlier paper [16, 17].

About growth rate estimate (5), maybe first proved by Weissler in [46], where the case for radial solutions on a ball was considered. In [12], Friedman and McLeod also obtained this result with the assumption $\Delta u_0 + u_0^p \ge 0$ in a bounded convex domain.

Giga and Kohn also considered more general form:

$$u_t - \Delta u - f(u) = 0, \tag{7}$$

where $f(u) - |u|^{p-1}u$ grows as most as $|u|^q$ for some $q < p$. The equation $u_t - |u|^{p-1}u = 0$ or $u_t - \Delta u - u^p = 0$ has been studied by numerous scholars, and so many references available. For the convenience of readers, we only list a few for readers, see [12, 19, 38, 46].

Quittner and Souplet [38] studied the following parabolic system

$$\begin{cases} u_t - \Delta u = |v|^{p-1}v, & x\in\Omega,\ t>0,\\ v_t - \Delta v = |u|^{q-1}u, & x\in\Omega,\ t>0,\\ u = v = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x\in\Omega, \end{cases} \tag{8}$$

where Ω is bounded $p, q \ge 1$, $pq > 1$, $0 \le u_0, v_0 \in L^\infty(\Omega)$. There established type I blow-up rate for nondecreasing solutions in time:

$$C_1(T-t)^{-\alpha/2} \le \|u(t)\|_\infty \le C_2(T-t)^{-\alpha/2}, \quad C_3(T-t)^{-\beta/2} \le \|v(t)\|_\infty \le C_4(T-t)^{-\beta/2}, \tag{9}$$

where $\alpha = \frac{2(p+1)}{pq-1}$, $\beta = \frac{2(q+1)}{pq-1}$. This result was also proved in [6] in which the proof is based on a modification of the maximum principle arguments of [12].

Recently, systems (1) has been studied by many scholars. In [24], Aleks Jevnikar, Wang and Yang considered (1) with $2 < p < p_B(N)$, Ω is a smooth domain in $\mathbb{R}^N$ and $1 \le N \le 5$. Here $p_B(N) = \frac{N(N+2)}{(N-1)^2}$, if $N \ge 2$; $p_B(N) = \infty$, if $N = 1$. For any non-negative nontrivial solution (u, v) of (1) on $\Omega \times (0, T)$, they proved

$$u(x,t) + v(x,t) \le C\Big(C_1 + t^{-1/p-1} + (T-t)^{-1/p-1} + (dist(x,\partial\Omega))^{-2/p-1}\Big) \tag{10}$$

with $C > 0$, $C_1 \ge 0$. Particularly, $C_1 = 0$ if $p = 2$. Moreover, they obtained

$$(T-t)^{-1} = 0 \quad \text{if} \quad T = \infty \quad \text{and} \quad (dist(x,\partial\Omega))^{-2/p-1} = 0 \quad \text{if} \quad \Omega = \mathbb{R}^N. \tag{11}$$

Equations (1) can be seen as a special form of the following general parabolic system

$$\begin{cases} u_t - \Delta u = f(x,u,v,w), & x\in\Omega,\ t>T_0,\\ v_t - \Delta v = g(x,u,v,w), & x\in\Omega,\ t>T_0,\\ w_t - \Delta w = h(x,u,v,w), & x\in\Omega,\ t>T_0. \end{cases} \tag{12}$$

The system (12) has been studied in various mathematical directions. For example, local and global existence [1, 10], Hölder regularity [5], blow-up behavior [30] and Liouville type theorems [34–36].

Inspired by above papers, we analyze the blow up behaviour about solutions of (1). More specifically, we firstly investigate the non-negative solutions of (1), and derive a bound on the growth rate, our first result is

Theorem 1 *Let $(u, v)(x, t)$ be a non-negative classical solution of* (1) *with $2 < p < \frac{N+2}{N-2}$ or $N \leq 2$, $u_0,\ v_0 \in C^2(\mathbb{R}^N)$, then*

$$\sup_{\mathbb{R}^N \times [0,T)} |(u+v)(x,t)| \cdot (T-t)^{1/(p-1)} < \infty. \tag{13}$$

Remark 2 *One interesting feature of Theorem 1 is that non-negative solution (u, v) blows up at the same rate as the same rate as the solution of the ODE: $u_t - |u|^{p-1}u = 0$,*

$$u(t) = \pm k(T-t)^{-1/p-1} \quad , \quad k = (p-1)^{-1/p-1}.$$

We also analyze the asymptotic behaviour of solutions near blow up, our second result is

Theorem 3 *Let (u, v) be a classical solution of* (1) *with $2 < p < \frac{N+2}{N-2}$ or $N \leq 2$, $u_0,\ v_0 \in C^2(\mathbb{R}^N)$, then*

$$\begin{cases} \lim\limits_{t\to T} (T-t)^{1/p-1} u(a + y\sqrt{T-t}, t) = 0 \quad \text{or} \quad \pm\, k_1, \\ \lim\limits_{t\to T} (T-t)^{1/p-1} v(a + y\sqrt{T-t}, t) = 0 \quad \text{or} \quad \pm\, k_2, \end{cases} \tag{14}$$

where $k_1 = \big(\mu_1(p-1)\big)^{-1/p-1}$, $k_2 = \big(\mu_2(p-1)\big)^{-1/p-1}$. The limit (14) *is independent of $y \in \mathbb{R}^N$, and it is uniform on each compact set $|y| < C$.*

This gives the asymptotic behaviour of (u, v) in a space-time parabola prior to $(a,\ T)$ for any $a \in \mathbb{R}^N$.

Remark 4 *The method we used is the introduction of backward self-similar solutions for* (1)*. $(\tilde{u}_a, \tilde{v}_a)$ is called a backward self-similar solution for* (1) *if the function $(\tilde{u}_a, \tilde{v}_a)$ is defined by*

$$\begin{cases} \tilde{u}_a(y, s) = (T-t)^{\beta} u(a + y\sqrt{T-t}, t), \\ \tilde{v}_a(y, s) = (T-t)^{\beta} v(a + y\sqrt{T-t}, t), \\ \beta = \frac{1}{p-1},\ \ s = -\log(T-t), \end{cases} \tag{15}$$

where a is any point in $\mathbb{R}^N$. We shall often suppress the subscript a, writing $(\tilde{u}, \tilde{v})$ for $(\tilde{u}_a, \tilde{v}_a)$. Then $(\tilde{u}, \tilde{v}) = (\tilde{u}_a, \tilde{v}_a)$ satisfies

$$\begin{cases} \tilde{u}_s - \Delta\tilde{u} + \frac{1}{2} y \cdot \nabla\tilde{u} + \beta\tilde{u} - \mu_1 |\tilde{u}|^{p-1}\tilde{u} - \alpha e^{-\beta(p-2)s}\tilde{u}\tilde{v} = 0, \\ \tilde{v}_s - \Delta\tilde{v} + \frac{1}{2} y \cdot \nabla\tilde{v} + \beta\tilde{v} - \mu_2 |\tilde{v}|^{p-1}\tilde{v} - \frac{\alpha}{2} e^{-\beta(p-2)s}\tilde{u}^2 = 0, \end{cases} \tag{16}$$

and the blow up time T corresponds to $s = \infty$. Studying non-negative solutions of (1) *near blow up is therefore equivalent to analyzing the large-time asymptotics of non-negative solutions of* (16)*.* (13) *is translated into*

$$|(\tilde{u} + \tilde{v})(y, s)| \leq M \ \text{ uniformly for } \ a \in \mathbb{R}^N,\ \ |y| < C,\ \ s < \infty. \tag{17}$$

(14) *is translated into*

$$\begin{cases} \lim\limits_{s\to\infty} \tilde{u}(y, s) = 0 \quad \text{or} \quad \pm\, k_1, \\ \lim\limits_{s\to\infty} \tilde{v}(y, s) = 0 \quad \text{or} \quad \pm\, k_2, \end{cases} \tag{18}$$

uniformly for $|y| < C$.

Remark 5 *We shall adapt the method as described in [16, 17, 20] to establish* (17) *under hypothesis* (2)*. We introduce the energy*

$$J[\tilde{u}, \tilde{v}] = \int_{\mathbb{R}^N} \Big(\frac{1}{2}\big(|\nabla\tilde{u}|^2 + |\nabla\tilde{v}|^2\big) + \frac{\beta}{2}\big(|\tilde{u}|^2 + |\tilde{v}|^2\big) - \frac{1}{p+1}\big(\mu_1 |\tilde{u}|^{p+1} + \mu_2 |\tilde{v}|^{p+1}\big) - \frac{\alpha}{2} e^{-\beta(p-2)s}\tilde{u}^2\tilde{v} \Big) \rho dy.$$

To derive energy identities for $(\tilde{u}, \tilde{v})$, we rewrite the system (16) *in divergence form:*

$$\begin{cases} \rho\tilde{u}_s - \nabla\cdot(\rho\nabla\tilde{u}) + \beta\rho\tilde{u} - \mu_1\rho|\tilde{u}|^{p-1}\tilde{u} - \alpha\rho e^{-\beta(p-2)s}\tilde{u}\tilde{v} = 0, \\ \rho\tilde{v}_s - \nabla\cdot(\rho\nabla\tilde{v}) + \beta\rho\tilde{v} - \mu_2\rho|\tilde{v}|^{p-1}\tilde{v} - \frac{\alpha}{2}\rho e^{-\beta(p-2)s}\tilde{u}^2 = 0, \end{cases} \tag{19}$$

with

$$\rho(y) = \exp\left(-\frac{|y|^2}{4}\right). \tag{20}$$

The proof of (18) *is presented in Section 5, where we use energy-type estimates obtained by multiplying the first equation in* (16) *with* $\tilde{u}\rho$*,* $\tilde{u}_s\rho$ *and* $\tilde{u}_s\rho|y|^2$*, multiplying the second equation in* (16) *with* $\tilde{v}\rho$*,* $\tilde{v}_s\rho$*, and* $\tilde{v}_s\rho|y|^2$*, to show that* $|\tilde{u}_s|^2$*,* $|\tilde{v}_s|^2$*,* $|\nabla\tilde{u}|^2$ *and* $|\nabla\tilde{v}|^2$ *are both integrable in space-time. This forces each to tend to zero in* L^2*, at least for a subsequence of times* $s_j \to \infty$*. The regularizing effect of the equation and the paucity of possible limits then yields the desired conclusion.*

The rest of this paper is organized as follows: In Section 2 we give some energy identities that will be used in the remainder of the paper. In Section 3 we prove Theorem 1 where we apply the results in Section 2. In Section 4 we investigate improved weighted energy identities which are parallel to Section 2. In Section 5 we prove Theorem 3 according to Section 4. In Appendix we give proofs of some results in the previous sections.

2 Preliminaries and energy identities

In this section, we study the backward self-similar solutions for (1) and give some energy estimates for the system (16). To this purpose we define the backward self-similar solution $(\tilde{u}, \tilde{v})$ by

$$\begin{cases} \tilde{u}(y,s) = (T-t)^{\beta} u(x,t), \\ \tilde{v}(y,s) = (T-t)^{\beta} v(x,t), \end{cases} \tag{21}$$

where

$$x - a = (T-t)^{1/2} y, \quad T - t = e^{-s}, \quad \beta = \frac{1}{p-1} \quad \text{and} \quad a \in \mathbb{R}^N. \tag{22}$$

A direct computation shows that $(\tilde{u}_a, \tilde{v}_a)$ satisfies

$$\begin{cases} \tilde{u}_s - \Delta\tilde{u} + \frac{1}{2} y\cdot\nabla\tilde{u} + \beta\tilde{u} - \mu_1|\tilde{u}|^{p-1}\tilde{u} - \alpha e^{-\beta(p-2)s}\tilde{u}\tilde{v} = 0, \\ \tilde{v}_s - \Delta\tilde{v} + \frac{1}{2} y\cdot\nabla\tilde{v} + \beta\tilde{v} - \mu_2|\tilde{v}|^{p-1}\tilde{v} - \frac{\alpha}{2} e^{-\beta(p-2)s}\tilde{u}^2 = 0, \end{cases} \tag{23}$$

on the space-time domain $\{(y,s) : y \in \mathbb{R}^N, s > s_0 = -\log T\}$.

Next we shall give the estimates for the functional $J[\tilde{u},\tilde{v}]$(see Remark 5). To do this we rewrite the system (23) in divergence form

$$\begin{cases} \rho\tilde{u}_s - \nabla\cdot(\rho\nabla\tilde{u}) + \beta\rho\tilde{u} - \mu_1\rho|\tilde{u}|^{p-1}\tilde{u} - \alpha\rho e^{-\beta(p-2)s}\tilde{u}\tilde{v} = 0, \\ \rho\tilde{v}_s - \nabla\cdot(\rho\nabla\tilde{v}) + \beta\rho\tilde{v} - \mu_2\rho|\tilde{v}|^{p-1}\tilde{v} - \dfrac{\alpha}{2}\rho e^{-\beta(p-2)s}\tilde{u}^2 = 0, \end{cases} \tag{24}$$

where $\rho(y) = \exp\left(-\frac{|y|^2}{4}\right)$ is given in (20). Then we have the following conclusions for the functional $J[\tilde{u},\tilde{v}]$.

Proposition 6 *Let* $N \geq 1$ *and* $(\tilde{u},\tilde{v})$ *be the classical solution of* (23)*. For any* $a \in \mathbb{R}^N$*, there holds*

$$\begin{aligned} \frac{1}{2}\frac{d}{ds}\int_{\mathbb{R}^N}\left(|\tilde{u}|^2 + |\tilde{v}|^2\right)\rho dy = -2J[\tilde{u},\tilde{v}](s) + \frac{p-1}{p+1}\int_{\mathbb{R}^N}\left(\mu_1|\tilde{u}|^{p+1} + \mu_2|\tilde{v}|^{p+1}\right)\rho dy \\ + \frac{\alpha}{2}\int_{\mathbb{R}^N} e^{-\beta(p-2)s}\tilde{u}^2\tilde{v}\rho dy \end{aligned} \tag{25}$$

and

$$\int_{\mathbb{R}^N}\left(|\tilde{u}_s|^2 + |\tilde{v}_s|^2\right)\rho dy = -\frac{d}{ds}J[\tilde{u},\tilde{v}](s) + \frac{\alpha\beta(p-2)}{2}\int_{\mathbb{R}^N} e^{-\beta(p-2)s}\tilde{u}^2\tilde{v}\rho dy. \tag{26}$$

Proof. Multiplying the system (19) by $(\tilde{u},\tilde{v})$ and integrating by parts, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}\int_{\mathbb{R}^N}\big(|\tilde{u}|^2+|\tilde{v}|^2\big)\rho dy&=\int_{\mathbb{R}^N}(\tilde{u}\tilde{u}_s+\tilde{v}\tilde{v}_s)\rho dy\\&=\int_{\mathbb{R}^N}\tilde{u}\nabla\cdot(\rho\nabla\tilde{u})-\big(\beta\tilde{u}^2-\mu_1|\tilde{u}|^{p+1}-\alpha e^{-\beta(p-2)s}\tilde{u}^2\tilde{v}\big)\rho dy\\&+\int_{\mathbb{R}^N}\tilde{v}\nabla\cdot(\rho\nabla\tilde{v})-\big(\beta\tilde{v}^2-\mu_2|\tilde{v}|^{p+1}-\frac{\alpha}{2}e^{-\beta(p-2)s}\tilde{u}^2\tilde{v}\big)\rho dy\\&=\int_{\mathbb{R}^N}\Big(-\big(|\nabla\tilde{u}|^2+|\nabla\tilde{v}|^2\big)-\beta(\tilde{u}^2+\tilde{v}^2)+\big(\mu_1|\tilde{u}|^{p+1}+\mu_2|\tilde{v}|^{p+1}\big)+\frac{3}{2}\alpha e^{-\beta(p-2)s}\tilde{u}^2\tilde{v}\Big)\rho dy\\&=-2J[\tilde{u},\tilde{v}](s)+\frac{p-1}{p+1}\int_{\mathbb{R}^N}\big(\mu_1|\tilde{u}|^{p+1}+\mu_2|\tilde{v}|^{p+1}\big)\rho dy+\frac{\alpha}{2}\int_{\mathbb{R}^N}e^{-\beta(p-2)s}\tilde{u}^2\tilde{v}\rho dy.\end{aligned}$$

This implies (25).

Next we multiply the system (19) by $(\tilde{u}_s,\tilde{v}_s)$ and then get

$$\begin{aligned}\int_{\mathbb{R}^N}\big(|\tilde{u}_s|^2+|\tilde{v}_s|^2\big)\rho dy=&\int_{\mathbb{R}^N}\tilde{u}_s\nabla\cdot(\rho\nabla\tilde{u})-\Big(\beta\tilde{u}\tilde{u}_s-\mu_1|\tilde{u}|^{p-1}\tilde{u}\tilde{u}_s-\alpha e^{-\beta(p-2)s}\tilde{u}\tilde{u}_s\tilde{v}\Big)\rho dy\\&+\int_{\mathbb{R}^N}\tilde{v}_s\nabla\cdot(\rho\nabla\tilde{v})-\Big(\beta\tilde{v}\tilde{v}_s-\mu_2|\tilde{v}|^{p-1}\tilde{v}\tilde{v}_s-\frac{\alpha}{2}e^{-\beta(p-2)s}\tilde{u}^2\tilde{v}_s\Big)\rho dy.\end{aligned}\tag{27}$$

From the definition of $J[\tilde{u},\tilde{v}](s)$ and integrating by parts, we have

$$\begin{aligned}\frac{d}{ds}J[\tilde{u},\tilde{v}](s)=&\frac{1}{2}\frac{d}{ds}\int_{\mathbb{R}^N}\big(|\nabla\tilde{u}|^2+|\nabla\tilde{v}|^2\big)\rho dy+\frac{\beta}{2}\frac{d}{ds}\int_{\mathbb{R}^N}\big(|\tilde{u}|^2+|\tilde{v}|^2\big)\rho dy\\&-\frac{1}{p+1}\frac{d}{ds}\int_{\mathbb{R}^N}\big(\mu_1|\tilde{u}|^{p+1}+\mu_2|\tilde{v}|^{p+1}\big)\rho dy-\frac{\alpha}{2}\frac{d}{ds}\int_{\mathbb{R}^N}e^{-\beta(p-2)s}\tilde{u}^2\tilde{v}\rho dy\\=&-\int_{\mathbb{R}^N}\big(\tilde{u}_s(\nabla\cdot\rho\nabla\tilde{u})+\tilde{v}_s(\nabla\cdot\rho\nabla\tilde{v})\big)dy+\beta\int_{\mathbb{R}^N}(\tilde{u}\tilde{u}_s+\tilde{v}\tilde{v}_s)\rho dy\\&-\int_{\mathbb{R}^N}\big(\mu_1|\tilde{u}|^{p-1}\tilde{u}\tilde{u}_s+\mu_2|\tilde{v}|^{p-1}\tilde{v}\tilde{v}_s\big)\rho dy\\&+\int_{\mathbb{R}^N}\alpha e^{-\beta(p-2)s}\big(\frac{\beta(p-2)}{2}\tilde{u}^2\tilde{v}-\tilde{u}\tilde{u}_s\tilde{v}-\frac{1}{2}\tilde{u}^2\tilde{v}_s\big)\Big)\rho dy.\end{aligned}\tag{28}$$

Combining (27) and (28), then we can easily obtain (26). ■

Next we give the estimates for $\tilde{u}_s,\tilde{v}_s,\tilde{u}$ and $\tilde{v}$ in (19).

Proposition 7 *Let $N\geq 1$ and $(\tilde{u},\tilde{v})$ be the classical solution of* (23). *For any $a\in\mathbb{R}^N$, there exists a constant $C>0$, depending only on $J[\tilde{u},\tilde{v}](s_0)$, N, and p, such that*

$$\int_{s_0}^{\infty}\int_{\mathbb{R}^N}\big(|\tilde{u}_s|^2+|\tilde{v}_s|^2\big)\rho dyds\leq C,\tag{29}$$

$$\int_{\mathbb{R}^N}\big(|\tilde{u}|^2+|\tilde{v}|^2\big)\rho dy\leq C,\tag{30}$$

$$\int_s^{s+1}\Big(\int_{\mathbb{R}^N}\big(\mu_1|\tilde{u}|^{p+1}+\mu_2|\tilde{v}|^{p+1}\big)\rho dy\Big)^2d\tau\leq C\quad\text{for all }\ s\geq s_0.\tag{31}$$

Proof. The first step is to derive bounds for $\frac{d}{ds}\int\big(|\tilde{u}|^2+|\tilde{v}|^2\big)\rho$ and $\frac{dJ}{ds}$. We begin with

$$\begin{aligned}\Big|\int_{\mathbb{R}^N}e^{-\beta(p-2)s}\tilde{u}^2\tilde{v}\rho dy\Big|&\leq Ce^{-\beta(p-2)s}\int_{\mathbb{R}^N}\big(|\tilde{u}|^3+|\tilde{v}|^3\big)\rho dy\\&\leq e^{-\beta(p-2)s}\Big(C(\varepsilon)+\varepsilon\int_{\mathbb{R}^N}\big(\mu_1|\tilde{u}|^{p+1}+\mu_2|\tilde{v}|^{p+1}\big)\rho dy\Big)\end{aligned}\tag{32}$$

for any $\varepsilon > 0$. Combining (25) and (32), we obtain

$$\int_{\mathbb{R}^N} \big(\mu_1|\tilde{u}|^{p+1} + \mu_2|\tilde{v}|^{p+1}\big)\rho dy \leq \frac{p+1}{p-1}\Big(2J[\tilde{u},\tilde{v}](s) + \int_{\mathbb{R}^N} (\tilde{u}\tilde{u}_s + \tilde{v}\tilde{v}_s)\rho dy\Big) + Ce^{-\beta(p-2)s}\int_{\mathbb{R}^N} \big(|\tilde{u}|^3 + |\tilde{v}|^3\big)\rho dy. \tag{33}$$

Then we infer from (32) and (33) that

$$\frac{1}{2}\int_{\mathbb{R}^N} \big(\mu_1|\tilde{u}|^{p+1} + \mu_2|\tilde{v}|^{p+1}\big)\rho dy \leq 2\Big(\frac{p+1}{p-1}\Big)J[\tilde{u},\tilde{v}](s) + \varepsilon\int_{\mathbb{R}^N} \big(|\tilde{u}_s|^2 + |\tilde{v}_s|^2\big)\rho dy + C(\varepsilon), \tag{34}$$

where we used the inequality $|\tilde{u}\tilde{u}_s| \leq \varepsilon\big(|\tilde{u}|^{p+1} + |\tilde{u}_s|^2\big) + C(\varepsilon)$ for $\varepsilon > 0$ small. Thus, from (26), (32) and (34)(with ε sufficiently small), we deduce that

$$\frac{d}{ds}J[\tilde{u},\tilde{v}](s) \leq -\frac{1}{2}\int_{\mathbb{R}^N} \big(|\tilde{u}_s|^2 + |\tilde{v}_s|^2\big)\rho dy + C_1e^{-\beta(p-2)s}\big(J[\tilde{u},\tilde{v}](s) + C_2\big). \tag{35}$$

Now we give the estimate of the term $\frac{d}{ds}\int \big(|\tilde{u}|^2 + |\tilde{v}|^2\big)\rho$. Combining (25) and (32), we get

$$\frac{1}{2}\frac{d}{ds}\int_{\mathbb{R}^N} \big(|\tilde{u}|^2 + |\tilde{v}|^2\big)\rho dy \geq -2J[\tilde{u},\tilde{v}](s) + \frac{1}{2}\Big(\frac{p-1}{p+1}\Big)\int_{\mathbb{R}^N} \big(\mu_1|\tilde{u}|^{p+1} + \mu_2|\tilde{v}|^{p+1}\big)\rho dy - Ce^{-\beta(p-2)s}, \tag{36}$$

where ε sufficiently small, $s \geq s_0$.

The next step is to show that J is bounded above and below. To accomplish this we claim that

$$-C_2 < J[\tilde{u},\tilde{v}](s) \leq C_3, \tag{37}$$

where

$$C_3 = -C_2 + \big(J[\tilde{u},\tilde{v}](s_0) + C_2\big)\exp\Big(\int_{s_0}^{\infty} C_1e^{-\beta(p-2)\tau}d\tau\Big) \leq C(1 + J[\tilde{u},\tilde{v}](s_0)).$$

Indeed, if the lower bound were to failed at some $s = s_1$, then $J[\tilde{u},\tilde{v}](s) \leq -C_2$ for all $s \geq s_1$, by (35). However, this and (36) imply that the term $\int(|\tilde{u}|^2 + |\tilde{v}|^2)\rho$ would blow up in finite time. This is a contradiction. We infer from (35) that

$$\frac{d}{ds}\log\big(J[\tilde{u},\tilde{v}](s) + C_2\big) \leq C_1e^{-\beta(p-2)s}.$$

Then we obtain the upper bound by integrating on s.

Finally, we prove the estimates (29)-(31). Our first bound (29) now follows easily by integrating (35) and J is bounded. Now we derive our second bound (30). To accomplish this, we let

$$f(s) = \Big(\int_{\mathbb{R}^N} \big(|\tilde{u}|^2 + |\tilde{v}|^2\big)\rho dy\Big)^{1/2}.$$

Since $\int \rho dy = (4\pi)^{N/2} < \infty$, we infer from Jensen's inequality, (36) and (37) that

$$\frac{1}{2}\frac{d}{ds}\big(f^2(s)\big) \geq -2J[\tilde{u},\tilde{v}](s) + Cf^{p+1}(s). \tag{38}$$

By (37) and (38), we have

$$\big(1 + J[\tilde{u},\tilde{v}](s_0)\big) + f(s)\frac{df}{ds} \geq Cf^{p+1}(s).$$

Moreover, we have

$$\text{either } f(s) \leq 1 \text{ or } Cf^p(s) \leq \frac{df}{ds} + \big(1 + J[\tilde{u},\tilde{v}](s_0)\big).$$

In particular,

$$\int_s^{s+1} f^{2p}(\tau)d\tau \le 1 + C\int_s^{s+1}\Big(\Big|\frac{df}{ds}\Big|^2 + J[\tilde u,\tilde v](s_0)\Big)d\tau \tag{39}$$

for any $s \ge s_0$. Also, from the definition of $f(s)$, one has

$$\frac{df}{ds} = f^{-1}\int_{\mathbb{R}^N}(\tilde u\tilde u_s + \tilde v\tilde v_s)\rho dy \le \Big(\int_{\mathbb{R}^N}\big(|\tilde u_s|^2 + |\tilde v_s|^2\big)\rho dy\Big)^{1/2}.$$

Moreover, we infer from (29) that

$$\int_s^{s+1}\Big|\frac{df}{ds}\Big|^2 d\tau \le \int_s^{s+1}\int_{\mathbb{R}^N}\big(|\tilde u_s|^2 + |\tilde v_s|^2\big)\rho dy d\tau \le C. \tag{40}$$

Thus, we know that the RHD of (39) and (40) are bounded by quantities depending only on p, n and $J[\tilde u,\tilde v](s_0)$. These estimates lead to (30) by using the Sobolev inequality

$$\|f\|_{L^\infty} \le C\big(\|f'\|_{L^2} + \|f\|_{L^2}\big)^\alpha\|f\|_{L^{2p}}^{1-\alpha}, \quad \alpha = \frac{1}{p+1},$$

where the function $f(s)$ is defined on $(0,1)$. Now the equality (25) can be rewritten as

$$\begin{aligned}\frac{p-1}{p+1}\int_{\mathbb{R}^N}\big(\mu_1|\tilde u|^{p+1} + \mu_2|\tilde v|^{p+1}\big)\rho dy = \int_{\mathbb{R}^N}(\tilde u\tilde u_s + \tilde v\tilde v_s)\rho dy + 2J[\tilde u,\tilde v](s)\\ - \frac{\alpha}{2}\int_{\mathbb{R}^N}e^{-\beta(p-2)s}\tilde u^2\tilde v\rho dy.\end{aligned}$$

By using Schwarz's inequality, (32) and (37), we conclude that

$$\begin{aligned}\Big(\frac{p-1}{p+1}\Big)^2\Big(\int_{\mathbb{R}^N}\big(\mu_1|\tilde u|^{p+1} + \mu_2|\tilde v|^{p+1}\big)\rho dy\Big)^2 \le C\Big(f^2(s)\int_{\mathbb{R}^N}\big(|\tilde u_s|^2 + |\tilde v_s|^2\big)\rho dy\Big)\\ + 1 + J[\tilde u,\tilde v](s_0).\end{aligned} \tag{41}$$

Since $f(s)$ is bounded, it follows that the integration of (41) with respect to s yields (31). ■

Notice that the constant C in Proposition 7 depends on the initial energy $J[\tilde u,\tilde v](s_0)$ which is associated with the choice of a, we expect that the estimate could be uniform over $a \in \mathbb{R}^N$. Then the next lemma proves this conclusion.

Proposition 8 *Let $N \ge 1$, then there holds $J[\tilde u,\tilde v](s_0) \le C$, where the constant $C > 0$ is independent of $a \in \mathbb{R}^N$.*

Proof. Recall that

$$\begin{aligned}J[\tilde u,\tilde v](s_0) = \int_{\mathbb{R}^N}\Big(\frac{1}{2}\big(|\nabla\tilde u|^2 + |\nabla\tilde v|^2\big)(y,s_0) + \frac{\beta}{2}\big(|\tilde u|^2 + |\tilde v|^2\big)(y,s_0)\\ - \frac{1}{p+1}\big(\mu_1|\tilde u|^{p+1} + \mu_2|\tilde v|^{p+1}\big)(y,s_0) - \frac{\alpha}{2}e^{-\beta(p-2)s_0}\tilde u^2\tilde v(y,s_0)\Big)\rho dy.\end{aligned}$$

We only consider the term $\int\big(|\tilde u|^2+|\tilde v|^2\big)\rho dy$, and the other terms in $J[\tilde u,\tilde v](s_0)$ can be handled similarly. By the definition of $(\tilde u,\tilde v)$, we have

$$\int_{\mathbb{R}^N}\big(|\tilde u|^2 + |\tilde v|^2\big)\rho dy = T^{2/(p-1)}\int_{\mathbb{R}^N}\big(|u_0|^2 + |v_0|^2\big)\rho\Big(\frac{x-a}{\sqrt{T}}\Big)\frac{dx}{T^{n/2}}.$$

Since u_0, v_0 is bounded on $\mathbb{R}^N$, it follows that

$$\int_{\mathbb{R}^N}\big(|\tilde u|^2 + |\tilde v|^2\big)\rho dy \le T^{2/(p-1)}(4\pi)^{n/2}\sup\big(|u_0|^2 + |v_0|^2\big) \le C,$$

where $C > 0$ is independent of $a \in \mathbb{R}^N$. ■

3 Proof of Theorem 1

In this section we shall borrow an idea of [16, 17] to give the proof of Theorem 1 by using the contradiction argument. Precisely, the purpose here is to prove that any nonnegative blow up solution should satisfy

$$\overline{\lim_{t\to T}} \sup_{x\in\mathbb{R}^N} (T-t)^{\beta}|(u+v)(x,t)| < \infty. \tag{42}$$

As described in Remark 4, it is equivalent to controlling $(\tilde{u}_a, \tilde{v}_a)(0,s)$ as s tends to infinity

$$\overline{\lim_{s\to\infty}} \sup_{a\in\mathbb{R}^N} |(\tilde{u}+\tilde{u})(0,s)| < \infty. \tag{43}$$

In order to prove (43), we shall achieve this by using the regularizing effect of the equations to bootstrap from the integral estimates of Proposition 7.

Proof of Theorem 1. If the conclusion (42) fails, then there is an increasing sequence of times $t_k \to T$ such that

$$\sup_{\mathbb{R}^N\times[0,t_k)} (u+v)(x,t)\cdot(T-t)^{\beta} = \sup_{x\in\mathbb{R}^N} (u+v)(x,t_k)\cdot(T-t_k)^{\beta} = M_k, \tag{44}$$

and $M_k \to \infty$ as $k \to \infty$. We may choose $x_k \in \mathbb{R}^N$ such that

$$\frac{1}{2}M_k \le (u+v)(x_k,t_k)\cdot(T-t_k)^{\beta} \le M_k. \tag{45}$$

Let $(\tilde{u}_k, \tilde{v}_k)(y,s)$ be the rescaled solution around x_k, defined by (21)-(22) with $a = x_k$. Let $s_k = -\log(T-t_k)$. It follows that (44)-(45) become

$$0 \le (\tilde{u}_k+\tilde{v}_k)(y,s) \le M_k \ \text{ when } \ s \le s_k, \quad \frac{1}{2}M_k \le (\tilde{u}_k+\tilde{v}_k)(0,s_k) \le M_k. \tag{46}$$

If $\delta > 0$ is chosen small enough, then the domain of $(\tilde{u}_k, \tilde{v}_k)$ includes the cylinder

$$\Omega(\delta,s) = \{(y,s) : |y| < \delta,\ -\delta^2 < s - s_k \le 0\}.$$

We rescale $(\tilde{u}_k, \tilde{v}_k)$ by

$$\begin{cases} \hat{u}_k(z,\tau) = {\lambda_k}^{2/(p-1)}\tilde{u}_k(\lambda_k z, {\lambda_k}^2\tau + s_k), \\ \hat{v}_k(z,\tau) = {\lambda_k}^{2/(p-1)}\tilde{v}_k(\lambda_k z, {\lambda_k}^2\tau + s_k), \end{cases}$$

with $\lambda_k \to 0$ determined by ${\lambda_k}^{2/(p-1)}M_k = 1$. It is clear that each $(\hat{u}_k, \hat{v}_k)$ is defined on the rescaled cylinder $\hat{\Omega}(\delta/\lambda_k)$, where

$$\hat{\Omega}(r) = \{(z,\tau) : |z| < r, -r^2 < \tau \le 0\},$$

Moreover, the (46) becomes

$$0 \le \hat{u}_k + \hat{v}_k \le 1 \ \text{ on } \ \hat{\Omega}\left(\frac{\delta}{\lambda_k}\right) \quad \text{and} \quad \frac{1}{2} \le (\hat{u}_k+\hat{v}_k)(0,0) \le 1. \tag{47}$$

We can choose a subsequence, still denoted by $\{(\hat{u}_k, \hat{v}_k)\}$ satisfies

$$0 \le \hat{u}_k,\ \hat{v}_k \le 1 \ \text{ on } \ \hat{\Omega}\left(\frac{\delta}{\lambda_k}\right) \quad \text{and} \quad \frac{1}{4} \le \hat{u}_k(0,0),\ \hat{v}_k(0,0) \le 1. \tag{48}$$

On the other hand, it is obviously that the system of $(\hat{u}_k, \hat{v}_k)$ is given by

$$\begin{cases} \hat{u}_{k\tau} - \Delta\hat{u}_k - \mu_1\hat{u}_k^p &= -{\lambda_k}^2\big(\frac{1}{2}z\cdot\nabla\hat{u}_k + \beta\hat{u}_k\big) + \alpha{\lambda_k}^{\frac{2\beta}{p-2}}e^{-\beta(p-2)s}\hat{u}_k\hat{v}_k, \\ \hat{v}_{k\tau} - \Delta\hat{v}_k - \mu_2\hat{v}_k^p &= -{\lambda_k}^2\big(\frac{1}{2}z\cdot\nabla\hat{v}_k + \beta\hat{v}_k\big) + \frac{\alpha}{2}{\lambda_k}^{\frac{2\beta}{p-2}}e^{-\beta(p-2)s}\hat{u}_k^2. \end{cases} \tag{49}$$

Now we want to pass the limit $k \to \infty$ in the first equation of (49)(taking a subsequence if necessary). This requires some a priori bounds. Using the L^q regularity theory for parabolic equations ([25, Chapter 4, p. 355, formula (10.12)]), we conclude from (48) and (49) that $\nabla\hat{u}_k$, $\nabla^2\hat{u}_k$ and $\hat{u}_{k\tau}$ are bounded in $L^q\big(\hat{\Omega}(r)\big)$ for each $q<\infty$ and $r>0$, the bound being independent of k. Therefore, by Sobolev's inequality, $\hat{u}_k$ is Hölder continuous, so that Schauder's estimates apply [11, Chapter 3, p. 64, Theorem 5]. We conclude finally that $\nabla^2\hat{u}_k$ and $\hat{u}_{k\tau}$ are Hölder continuous on each $\hat{\Omega}(r)$, uniformly with respect to k. By the Arzela-Ascoli theorem and a diagonal argument, there is a subsequence converging uniformly to a limit u^* on each $\hat{\Omega}(r)$. This u^* is defined on $\mathbb{R}^N\times(-\infty,0)$ and it satisfies

$$0\le u^*\le 1,\ \ u^*(0,0)\ge\frac{1}{4},\ \ u^*_\tau-\Delta u^*-\mu_1(u^*)^p=0, \tag{50}$$

by passing the limit in (48) and (49).

Finally, in order to reach a contradiction we shall show that u^* is independent of τ. By the change of variables formula

$$\iint_{\hat{\Omega}(\delta/\lambda_k)}|\hat{u}_{k\tau}|^2dzd\tau={\lambda_k}^\sigma\iint_{\Omega(\delta,s_k)}|\tilde{u}_{ks}|^2dyds$$

with $\sigma=-N+2+\dfrac{4}{p-1}$. If $N\le 2$ or $p<\dfrac{N+2}{N-2}$, then $\sigma>0$. On the other hand,

$$\iint_{\Omega(\delta,s_k)}|\tilde{u}_{ks}|^2dyds\le C(\delta)\int_{s_0}^{\infty}\int_{\mathbb{R}^N}|\tilde{u}_{ks}|^2\rho(y)dyds,$$

where $C(\delta)=\exp(\dfrac{\delta^2}{4})$. By Propositions 7 and 8, we have

$$\int_{s_0}^{\infty}\int_{\mathbb{R}^N}|\tilde{u}_{ks}|^2\rho dyds\le C, \tag{51}$$

with C independent of k. We conclude that

$$\iint_{\hat{\Omega}(\delta/\lambda_k)}|\hat{u}_{k\tau}|^2dzd\tau\le C{\lambda_k}^\sigma,\quad \sigma>0.$$

As $k\to\infty$ this gives

$$\iint_{\hat{\Omega}(r)}|\hat{u}_{k\tau}|^2dzd\tau\to 0\quad\forall\, r>0,$$

and so $u^*_\tau\equiv 0$. We have obtained a nonnegative solution of

$$\Delta u^*+\mu_1(u^*)^p=0\quad\text{in }\mathbb{R}^N,$$

where $u^*(0)\ge\dfrac{1}{2}$. This contradicts the nonexistence theorems of [15, 16]. Then the proof is complete. ■

4 Improved the energy estimates

In this section, we define a weighted energy functional by

$$\begin{aligned}J_2[\tilde{u},\tilde{v}](s)=\int_{\mathbb{R}^N}\Big(\frac{1}{2}\big(|\nabla\tilde{u}|^2+|\nabla\tilde{v}|^2\big)+\frac{\beta}{2}\big(|\tilde{u}|^2+|\tilde{v}|^2\big)-\frac{1}{p+1}\big(\mu_1|\tilde{u}|^{p+1}+\mu_2|\tilde{v}|^{p+1}\big)\\ -\frac{\alpha}{2}e^{-\beta(p-2)s}\tilde{u}^2\tilde{v}\Big)\rho|y|^2dy.\end{aligned} \tag{52}$$

This is the analogue of $J[\tilde{u},\tilde{v}]$ with weight ρ replaced by an improved weight $\rho|y|^2$. We shall derive some similar energy identities as in Proposition 6.

Proposition 9 *Let $N\ge 1$ and $(\tilde{u},\tilde{v})$ be the classical solution of* (23). *For any $a\in\mathbb{R}^N$, there holds*

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}\int_{\mathbb{R}^N}\big(|\tilde{u}|^2+|\tilde{v}|^2\big)\rho|y|^2dy=&-2J_2[\tilde{u},\tilde{v}]+\frac{p-1}{p+1}\int_{\mathbb{R}^N}\big(\mu_1|\tilde{u}|^{p+1}+\mu_2|\tilde{v}|^{p+1}\big)\rho|y|^2dy\\ &+\frac{\alpha}{2}\int_{\mathbb{R}^N}e^{-\beta(p-2)s}\tilde{u}^2\tilde{v}\rho|y|^2dy+\int_{\mathbb{R}^N}\Big(N-\frac{1}{2}|y|^2\Big)\big(|\tilde{u}|^2+|\tilde{v}|^2\big)\rho dy\end{aligned} \tag{53}$$

and

$$\int_{\mathbb{R}^N} \big(|\tilde{u}_s|^2 + |\tilde{v}_s|^2\big)\rho|y|^2 dy = -\frac{d}{ds}J_2[\tilde{u}, \tilde{v}] - 2\int_{\mathbb{R}^N} \big((y\cdot\nabla\tilde{u})\tilde{u}_s + (y\cdot\nabla\tilde{v})\tilde{v}_s\big)\rho dy + \frac{\alpha\beta(p-2)}{2}\int_{\mathbb{R}^N} e^{-\beta(p-2)s}\tilde{u}^2\tilde{v}\rho|y|^2 dy. \tag{54}$$

Proof. Multiplying the first equation in (23) with $|y|^2\tilde{u}$ and multiplying the second equation in (23) with $|y|^2\tilde{v}$, then integrating by parts gives

$$\begin{aligned}
&\frac{1}{2}\frac{d}{ds}\int_{\mathbb{R}^N} \big(|\tilde{u}|^2 + |\tilde{v}|^2\big)\rho|y|^2 dy \\
&= \int_{\mathbb{R}^N} |y|^2\tilde{u}\Big(\nabla\cdot(\rho\nabla\tilde{u}) - \beta\rho\tilde{u} + \mu_1\rho|\tilde{u}|^{p-1}\tilde{u} + \alpha\rho e^{-\beta(p-2)s}\tilde{u}\tilde{v}\Big)dy \\
&\quad + \int_{\mathbb{R}^N} |y|^2\tilde{v}\Big(\nabla\cdot(\rho\nabla\tilde{v}) - \beta\rho\tilde{v} + \mu_2\rho|\tilde{v}|^{p-1}\tilde{v} + \frac{\alpha}{2}\rho e^{-\beta(p-2)s}\tilde{u}^2\Big)dy \\
&= -\int_{\mathbb{R}^N} \rho|y|^2\Big(\big(|\nabla\tilde{u}|^2 + |\nabla\tilde{v}|^2\big) + \frac{1}{p-1}\big(|\tilde{u}|^2 + |\tilde{v}|^2\big) - \big(\mu_1|\tilde{u}|^{p+1} + \mu_2|\tilde{v}|^{p+1}\big) \\
&\qquad - \frac{3\alpha}{2}e^{-\beta(p-2)s}\tilde{u}^2\tilde{v}\Big)dy - \int_{\mathbb{R}^N} \rho\nabla(|y|^2)\cdot\big(\tilde{u}\nabla\tilde{u} + \tilde{v}\nabla\tilde{v}\big)dy \\
&= -2J_2[\tilde{u}, \tilde{v}] + \frac{p-1}{p+1}\int_{\mathbb{R}^N} \big(\mu_1|\tilde{u}|^{p+1} + \mu_2|\tilde{v}|^{p+1}\big)\rho|y|^2 dy \\
&\quad - \int_{\mathbb{R}^N} \rho y\cdot\nabla\big(|\tilde{u}|^2 + |\tilde{v}|^2\big)dy + \frac{\alpha}{2}\int_{\mathbb{R}^N} e^{-\beta(p-2)s}\tilde{u}^2\tilde{v}\rho|y|^2 dy.
\end{aligned}$$

Since $\mathrm{div}(\rho y) = \Big(N - \frac{1}{2}|y|^2\Big)\rho$, it follows that (53) holds.

A direct computation shows that

$$\begin{aligned}
\frac{dJ_2}{ds} = \int_{\mathbb{R}^N} \Big(&\big(\nabla\tilde{u}\cdot\nabla\tilde{u}_s + \nabla\tilde{v}\cdot\nabla\tilde{v}_s\big) + \beta(\tilde{u}\tilde{u}_s + \tilde{v}\tilde{v}_s) - \big(\mu_1|\tilde{u}|^{p-1}\tilde{u}\tilde{u}_s + \mu_2|\tilde{v}|^{p-1}\tilde{v}\tilde{v}_s\big) \\
&+ \alpha e^{-\beta(p-2)s}\big(\frac{\beta(p-2)}{2}\tilde{u}^2\tilde{v} - \tilde{u}\tilde{u}_s\tilde{v} - \frac{1}{2}\tilde{u}^2\tilde{v}_s\big)\Big)\rho|y|^2 dy.
\end{aligned}$$

Multiplying the first equation in (23) with $|y|^2\tilde{u}_s$ and the second equation in (23) with $|y|^2\tilde{v}_s$, then integrating by parts gives

$$\begin{aligned}
&\int_{\mathbb{R}^N} \big(|\tilde{u}_s|^2 + |\tilde{v}_s|^2\big)\rho|y|^2 dy \\
&= \int_{\mathbb{R}^N} |y|^2\tilde{u}_s\Big(\nabla\cdot(\rho\nabla\tilde{u}) - \beta\rho\tilde{u} + \mu_1\rho|\tilde{u}|^{p-1}\tilde{u} + \alpha\rho e^{-\beta(p-2)s}\tilde{u}\tilde{v}\Big)dy \\
&\quad + \int_{\mathbb{R}^N} |y|^2\tilde{v}_s\Big(\nabla\cdot(\rho\nabla\tilde{v}) - \beta\rho\tilde{v} + \mu_2\rho|\tilde{v}|^{p-1}\tilde{v} + \frac{\alpha}{2}\rho e^{-\beta(p-2)s}\tilde{u}^2\Big)dy \\
&= -\int_{\mathbb{R}^N} \Big(\big(\nabla\tilde{u}\cdot\nabla\tilde{u}_s + \nabla\tilde{v}\cdot\nabla\tilde{v}_s\big) + \beta(\tilde{u}\tilde{u}_s + \tilde{v}\tilde{v}_s) - \big(\mu_1|\tilde{u}|^{p-1}\tilde{u}\tilde{u}_s + \mu_2|\tilde{v}|^{p-1}\tilde{v}\tilde{v}_s\big) \\
&\quad - \alpha e^{-\beta(p-2)s}\big(\tilde{u}\tilde{u}_s\tilde{v} + \frac{1}{2}\tilde{u}^2\tilde{v}_s\big)\Big)\rho|y|^2 dy - 2\int_{\mathbb{R}^N} \big((y\cdot\nabla\tilde{u})\tilde{u}_s + (y\cdot\nabla\tilde{v})\tilde{v}_s\big)\rho dy.
\end{aligned}$$

This is (54). ■

The following Proposition focus on the estimates of the integral $\big((y\cdot\nabla\tilde{u})\tilde{u}_s + (y\cdot\nabla\tilde{v})\tilde{v}_s\big)$ by using the similar strategy as in [19].

Proposition 10 *Let $N \geq 1$ and $(\tilde{u}, \tilde{v})$ be the classical solution of (23). For any $a \in \mathbb{R}^N$, there holds*

$$\begin{aligned}
&\frac{1}{2}\frac{d}{ds}\int_{\mathbb{R}^N}\Big(\frac{1}{2}|y|^2 - N\Big)\big(|\tilde{u}|^2 + |\tilde{v}|^2\big)\rho dy - (p+1)\int_{\mathbb{R}^N}\big((y\cdot\nabla\tilde{u})\tilde{u}_s + (y\cdot\nabla\tilde{v})\tilde{v}_s\big)\rho dy \\
&\quad = \int_{\mathbb{R}^N}\big(|\nabla\tilde{u}|^2 + |\nabla\tilde{v}|^2\big)\rho\Big(c(N,p) + \frac{p-1}{4}|y|^2\Big)dy + \int_{\mathbb{R}^N}\Big[\alpha\Big(\frac{3}{4}\tilde{u}^2\tilde{v}|y|^2 - N\tilde{u}\tilde{v} - \frac{1}{2}\tilde{u}^2\Big) \\
&\qquad - \alpha(p+1)\Big((y\cdot\nabla\tilde{u})\tilde{u}\tilde{v} + \frac{1}{2}(y\cdot\nabla\tilde{v})\tilde{u}^2\Big)\Big]e^{-\beta(p-2)s}\rho dy,
\end{aligned} \tag{55}$$

with $c(N,p) = \frac{1}{2}\big((2-N)p + (N+2)\big)$.

Proof. We multiply the first equation in (23) with $\bar{u} = y\cdot\nabla\tilde{u}$ and the second equation in (23) with $\bar{v} = y\cdot\nabla\tilde{v}$. Then by integrating and we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N}\big((y\cdot\nabla\tilde{u})\tilde{u}_s + (y\cdot\nabla\tilde{v})\tilde{v}_s\big)\rho dy \\
&\quad = \int_{\mathbb{R}^N}\bar{u}\Big(\nabla\cdot(\rho\nabla\tilde{u}) - \beta\rho\tilde{u} + \mu_1\rho|\tilde{u}|^{p-1}\tilde{u} + \alpha\rho e^{-\beta(p-2)s}\tilde{u}\tilde{v}\Big)dy \\
&\qquad + \int_{\mathbb{R}^N}\bar{v}\Big(\nabla\cdot(\rho\nabla\tilde{v}) - \beta\rho\tilde{v} + \mu_2\rho|\tilde{v}|^{p-1}\tilde{v} + \frac{\alpha}{2}\rho e^{-\beta(p-2)s}\tilde{u}^2\Big)dy \\
&\quad = -\int_{\mathbb{R}^N}\rho\Big(\nabla\bar{u}\cdot\nabla\tilde{u} + \beta\tilde{u}\bar{u} - \mu_1|\tilde{u}|^{p-1}\tilde{u}\bar{u}\Big)dy \\
&\qquad - \int_{\mathbb{R}^N}\rho\Big(\nabla\bar{v}\cdot\nabla\tilde{v} + \beta\tilde{v}\bar{v} - \mu_2|\tilde{v}|^{p-1}\tilde{v}\bar{v}\Big)dy \\
&\qquad + \alpha\int_{\mathbb{R}^N}\bar{u}\rho e^{-\beta(p-2)s}\tilde{u}\tilde{v} + \frac{\alpha}{2}\int_{\mathbb{R}^N}\bar{v}\rho e^{-\beta(p-2)s}\tilde{u}^2 dy.
\end{aligned}$$

As in [19], a direct computation shows that

$$\begin{aligned}
\int_{\mathbb{R}^N}\rho\big(\nabla\bar{u}\cdot\nabla\tilde{u}\big)dy &= \int_{\mathbb{R}^N}\rho|\nabla\tilde{u}|^2 dy - \frac{1}{2}\int_{\mathbb{R}^N}\Big(N - \frac{1}{2}|y|^2\Big)\rho|\nabla\tilde{u}|^2 dy, \\
\int_{\mathbb{R}^N}\rho\bar{u}\tilde{u}dy &= -\frac{1}{2}\int_{\mathbb{R}^N}|\tilde{u}|^2\rho\Big(N - \frac{1}{2}|y|^2\Big)dy, \\
-\int_{\mathbb{R}^N}\rho|\tilde{u}|^{p-1}\tilde{u}\bar{u}dy &= \frac{1}{p+1}\int_{\mathbb{R}^N}|\tilde{u}|^{p+1}\rho\Big(N - \frac{1}{2}|y|^2\Big)dy.
\end{aligned}$$

Similarly, one has

$$\begin{aligned}
\int_{\mathbb{R}^N}\rho\big(\nabla\bar{v}\cdot\nabla\tilde{v}\big)dy &= \int_{\mathbb{R}^N}\rho|\nabla\tilde{v}|^2 dy - \frac{1}{2}\int_{\mathbb{R}^N}\Big(N - \frac{1}{2}|y|^2\Big)\rho|\nabla\tilde{v}|^2 dy, \\
\int_{\mathbb{R}^N}\rho\bar{v}\tilde{v}dy &= -\frac{1}{2}\int_{\mathbb{R}^N}|\tilde{v}|^2\rho\Big(N - \frac{1}{2}|y|^2\Big)dy, \\
-\int_{\mathbb{R}^N}\rho|\tilde{v}|^{p-1}\tilde{v}\bar{v}dy &= \frac{1}{p+1}\int_{\mathbb{R}^N}|\tilde{v}|^{p+1}\rho\Big(N - \frac{1}{2}|y|^2\Big)dy.
\end{aligned}$$

Combining these identities, one gets

$$\begin{aligned}
-\int_{\mathbb{R}^N}\big((y\cdot\nabla\tilde{u})\tilde{u}_s + (y\cdot\nabla\tilde{v})\tilde{v}_s\big)\rho dy &= \int_{\mathbb{R}^N}\rho\big(|\nabla\tilde{u}|^2 + |\nabla\tilde{v}|^2\big)dy + \int_{\mathbb{R}^N}\rho\Big(\frac{1}{2}|y|^2 - N\Big) \\
&\quad \Big(\frac{1}{2}\big(|\nabla\tilde{u}|^2 + |\nabla\tilde{v}|^2\big) + \frac{\beta}{2}\big(|\tilde{u}|^2 + |\tilde{v}|^2\big) - \frac{1}{p+1}\big(\mu_1|\tilde{u}|^{p+1} + \mu_2|\tilde{v}|^{p+1}\big)\Big)dy \\
&\quad - \alpha\int_{\mathbb{R}^N}\bar{u}\rho e^{-\beta(p-2)s}\tilde{u}\tilde{v} - \frac{\alpha}{2}\int_{\mathbb{R}^N}\bar{v}\rho e^{-\beta(p-2)s}\tilde{u}^2 dy.
\end{aligned} \tag{56}$$

A linear combination of (53) and (25) shows that

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}\int_{\mathbb{R}^N}\Big(\frac{1}{2}|y|^2-N\Big)\big(|\tilde{u}|^2+|\tilde{v}|^2\big)\rho dy=&-\int_{\mathbb{R}^N}\rho\Big(\frac{1}{2}|y|^2-N\Big)\Big(\big(|\nabla\tilde{u}|^2+|\nabla\tilde{v}|^2\big)\\&+\frac{\beta(p+1)}{2}\big(|\tilde{u}|^2+|\tilde{v}|^2\big)-\big(\mu_1|\tilde{u}|^{p+1}+\mu_2|\tilde{v}|^{p+1}\big)\Big)dy+\frac{3\alpha}{4}\int_{\mathbb{R}^N}e^{-\beta(p-2)s}\tilde{u}^2\tilde{v}\rho|y|^2dy.\end{aligned}\tag{57}$$

By adding (57) and $(p+1)\cdot$(56), and then one can obtain (55). ■

From the above estimates, we are ready to obtain the crucial integrability of $\big(|\tilde{u}_s|^2+|\tilde{v}_s|^2\big)+\big(|\nabla\tilde{u}|^2+|\nabla\tilde{v}|^2\big)$.

Proposition 11 *Let* $2<p<\dfrac{N+2}{N-2}$ *or* $N\le 2$*, and* $(\tilde{u},\tilde{v})$ *be the classical solution of* (23)*. For any* $a\in\mathbb{R}^N$*, there holds*

$$\int_{s_0}^{\infty}\int_{\mathbb{R}^N}\Big(\big(|\tilde{u}_s|^2+|\tilde{v}_s|^2\big)+\big(|\nabla\tilde{u}|^2+|\nabla\tilde{v}|^2\big)\Big)\big(1+|y|^2\big)\rho dyds<\infty.\tag{58}$$

Proof. Let

$$\tilde{J}_2[\tilde{u},\tilde{v}]=J_2[\tilde{u},\tilde{v}]-\frac{1}{2}\int_{\mathbb{R}^N}\Big(\frac{1}{2}|y|^2-N\Big)\big(|\tilde{u}|^2+|\tilde{v}|^2\big)\rho dy.$$

We claim that

$$\begin{aligned}\frac{d}{ds}(\tilde{J}_2+CJ+C)\le&-C\int_{\mathbb{R}^N}\Big(\big(|\tilde{u}_s|^2+|\tilde{v}_s|^2\big)+\big(|\nabla\tilde{u}|^2+|\nabla\tilde{v}|^2\big)\Big)\big(1+|y|^2\big)\rho dy\\&+Ce^{-\beta(p-2)s}(\tilde{J}_2+CJ+C)\end{aligned}$$

for a suitable choice of $C>0$. If this claim holds, then we know that $\tilde{J}_2+CJ$ is bounded. In order to prove this claim, we first need to give the estimates of $\dfrac{d}{ds}\int(|\tilde{u}|^2+|\tilde{v}|^2)\rho|y|^2$ and $\dfrac{dJ_2}{ds}$. Similar to (36), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}\int_{\mathbb{R}^N}\big(|\tilde{u}|^2+|\tilde{v}|^2\big)\rho|y|^2dy\ge-2J_2[\tilde{u},\tilde{v}]&+\frac{1}{2}\Big(\frac{p-1}{p+1}\Big)\int_{\mathbb{R}^N}\big(\mu_1|\tilde{u}|^{p+1}+\mu_2|\tilde{v}|^{p+1}\big)\rho|y|^2dy\\&+\int_{\mathbb{R}^N}\Big(N-\frac{1}{2}|y|^2\Big)\big(|\tilde{u}|^2+|\tilde{v}|^2\big)\rho dy-Ce^{-\beta(p-2)s}.\end{aligned}\tag{59}$$

Furthermore, similar to (34), we have

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}^N}\big(\mu_1|\tilde{u}|^{p+1}+\mu_2|\tilde{v}|^{p+1}\big)\rho|y|^2dy\\&\qquad\le 2\Big(\frac{p+1}{p-1}\Big)\tilde{J}_2[\tilde{u},\tilde{v}]+\varepsilon\int_{\mathbb{R}^N}\big(|\tilde{u}_s|^2+|\tilde{v}_s|^2\big)\rho|y|^2dy+C(\varepsilon).\end{aligned}\tag{60}$$

Also, (54) leads to

$$\begin{aligned}\frac{dJ_2}{ds}\le-\frac{1}{2}\int_{\mathbb{R}^N}\big(|\tilde{u}_s|^2+|\tilde{v}_s|^2\big)\rho|y|^2dy&-2\int_{\mathbb{R}^N}\rho\big((y\cdot\nabla\tilde{u})\tilde{u}_s+(y\cdot\nabla\tilde{v})\tilde{v}_s\big)dy\\&+Ce^{-\beta(p-2)s}(\tilde{J}_2+C).\end{aligned}\tag{61}$$

Next we estimate the last two terms in (55) and define

$$\begin{aligned}f(s)=\alpha\int_{\mathbb{R}^N}\Big(\frac{3}{4}\tilde{u}^2\tilde{v}|y|^2-N\tilde{u}\tilde{v}-\frac{1}{2}\tilde{u}^2\Big)e^{-\beta(p-2)s}\rho dy&\\-\alpha(p+1)\int_{\mathbb{R}^N}\Big((y\cdot\nabla\tilde{u})\tilde{u}\tilde{v}+\frac{1}{2}(y\cdot\nabla\tilde{v})\tilde{u}^2\Big)e^{-\beta(p-2)s}\rho dy.&\end{aligned}$$

Integrating by parts in the last integral term of $f(s)$ gives

$$\begin{aligned}|f(s)|&\le Ce^{-\beta(p-2)s}\Big(1+\int_{\mathbb{R}^N}\big(\mu_1|\tilde{u}|^{p+1}+\mu_2|\tilde{v}|^{p+1}\big)\big(|y|^2+1\big)\rho dy\Big)\\&\le\varepsilon\int_{\mathbb{R}^N}\big(|\tilde{u}_s|^2+|\tilde{v}_s|^2\big)\rho\big(|y|^2+1\big)dy+Ce^{-\beta(p-2)s}(\tilde{J}_2+C).\end{aligned}\tag{62}$$

Here we make use of (34), (37) and (60) in the last step.

Now consider $\tilde{J}_2 + CJ$. We infer from (54) and (55) that

$$\begin{aligned}\frac{d\tilde{J}_2}{ds} =& -\int_{\mathbb{R}^N}\big(|\tilde{u}_s|^2+|\tilde{v}_s|^2\big)\rho|y|^2dy-(p+3)\int_{\mathbb{R}^N}\rho\big((y\cdot\nabla\tilde{u})\tilde{u}_s+(y\cdot\nabla\tilde{v})\tilde{v}_s\big)dy\\ &-\int_{\mathbb{R}^N}\big(|\nabla\tilde{u}|^2+|\nabla\tilde{v}|^2\big)\rho\Big(c_2+\frac{p-1}{4}|y|^2\Big)dy+\frac{\alpha\beta(p-2)}{2}\int_{\mathbb{R}^N}e^{-\beta(p-2)s}\tilde{u}^2\tilde{v}\rho|y|^2dy\\ &-\alpha\int_{\mathbb{R}^N}\Big(\frac{3}{4}\tilde{u}^2\tilde{v}|y|^2-N\tilde{u}\tilde{v}-\frac{1}{2}\tilde{u}^2\Big)e^{-\beta(p-2)s}\rho dy+\alpha(p+1)\int_{\mathbb{R}^N}\Big((y\cdot\nabla\tilde{u})\tilde{u}\tilde{v}\\ &+\frac{1}{2}(y\cdot\nabla\tilde{v})\tilde{u}^2\Big)e^{-\beta(p-2)s}\rho dy.\end{aligned}$$

For the term which contains $e^{-\beta(p-2)s}$, we can be handled it similarly as (62). We mainly control the second term by applying the Cauchy-Schwarz inequality: for any $\varepsilon > 0$,

$$\left|\int\rho\big((y\cdot\nabla\tilde{u})\tilde{u}_s+(y\cdot\nabla\tilde{v})\tilde{v}_s\big)dy\right|\le\varepsilon\int\big(|\nabla\tilde{u}|^2+|\nabla\tilde{v}|^2\big)\rho|y|^2dy+\frac{1}{4\varepsilon}\int\big(|\tilde{u}_s|^2+|\tilde{v}_s|^2\big)\rho dy.$$

Choosing ε small enough such that $\frac{p-1}{4}-(p+3)\varepsilon=\delta>0$, Then we conclude that

$$\begin{aligned}\frac{d\tilde{J}_2}{ds}\le&-\int_{\mathbb{R}^N}\Big(\big(|\tilde{u}_s|^2+|\tilde{v}_s|^2\big)|y|^2+\delta\big(|\nabla\tilde{u}|^2+|\nabla\tilde{v}|^2\big)|y|^2+c_2\big(|\nabla\tilde{u}|^2+|\nabla\tilde{v}|^2\big)\Big)\rho dy\\ &+\frac{p+3}{4\varepsilon}\int_{\mathbb{R}^N}\big(|\tilde{u}_s|^2+|\tilde{v}_s|^2\big)\rho dy+\varepsilon\int_{\mathbb{R}^N}\big(|\tilde{u}_s|^2+|\tilde{v}_s|^2\big)\rho(|y|^2+1)dy+Ce^{-\beta(p-2)s}(\tilde{J}_2+C).\end{aligned}$$

Thus, we deduce from (26) that

$$\begin{aligned}\frac{d\tilde{J}_2}{ds}+C\frac{dJ}{ds}\le&-C\int_{\mathbb{R}^N}\Big(\big(|\tilde{u}_s|^2+|\tilde{v}_s|^2\big)+\big(|\nabla\tilde{u}|^2+|\nabla\tilde{v}|^2\big)\Big)\rho(|y|^2+1)dy\\ &+C\int_{\mathbb{R}^N}e^{-\beta(p-2)s}\tilde{u}^2\tilde{v}\rho dy+Ce^{-\beta(p-2)s}(\tilde{J}_2+C)\\ \le&-C\int_{\mathbb{R}^N}\Big(\big(|\tilde{u}_s|^2+|\tilde{v}_s|^2\big)+\big(|\nabla\tilde{u}|^2+|\nabla\tilde{v}|^2\big)\Big)(1+|y|^2)\rho dy\\ &+Ce^{-\beta(p-2)s}(\tilde{J}_2+CJ+C),\end{aligned}$$

where the last inequality is obtained by using (32)-(34). Since J is bounded, this yields the desired estimate

$$\begin{aligned}\frac{d}{ds}(\tilde{J}_2+CJ+C)\le&-C\int_{\mathbb{R}^N}\Big(\big(|\tilde{u}_s|^2+|\tilde{v}_s|^2\big)+\big(|\nabla\tilde{u}|^2+|\nabla\tilde{v}|^2\big)\Big)\big(1+|y|^2\big)\rho dy\\ &+Ce^{-\beta(p-2)s}(\tilde{J}_2+CJ+C)\end{aligned}\tag{63}$$

for a suitable choice of $C > 0$. It follows as in the proof of Proposition 7 that $\tilde{J}_2 + CJ + C$ is bounded above and below by using (59)-(60) and (63). The assertion (58) now results from (63) by integration in s, and the proof is completed. ■

5 Proof of Theorem 3

In this section we focus on the proof of Theorem 3 by using the estimates in Section 4. To this purpose we denote $U := (u, v)^{\perp}$, then the system (1) can be rewritten as

$$U_t-\Delta U=F(U),\tag{64}$$

where $U_t = (u_t, v_t)^{\perp}$, $\Delta U = (\Delta u, \Delta v)^{\perp}$, $F(U) = (\mu_1|u|^{p-1}u + \alpha uv, \mu_2|v|^{p-1}v + \frac{\alpha}{2}u^2)^{\perp}$. For any $a \in \mathbb{R}^N$, we let

$$\tilde{U}_a(y,s)=(T-t)^{\beta}U(a+y\sqrt{T-t},t),\tag{65}$$

where $\beta = \dfrac{1}{p-1}$, $s = -\log(T-t)$. Then $\tilde{U} = \tilde{U}_a$ solves the following equation

$$\tilde{U}_s - \Delta\tilde{U} + \frac{1}{2}y\cdot\nabla\tilde{U} + \beta\tilde{U} + G(\tilde{U}) = 0, \tag{66}$$

where

$$\tilde{U}_s = \begin{bmatrix}\tilde{u}_s\\ \tilde{v}_s\end{bmatrix}, \Delta\tilde{U} = \begin{bmatrix}\Delta\tilde{u}\\ \Delta\tilde{v}\end{bmatrix}, y\cdot\nabla\tilde{U} = \begin{bmatrix}y\cdot\nabla\tilde{u}\\ y\cdot\nabla\tilde{v}\end{bmatrix}, G(\tilde{U}) = \begin{bmatrix}-\mu_1|\tilde{u}|^{p-1}\tilde{u} - \alpha e^{-\beta(p-2)s}\tilde{u}\tilde{v}\\ -\mu_2|\tilde{v}|^{p-1}\tilde{v} - \dfrac{\alpha}{2}e^{-\beta(p-2)s}\tilde{u}^2\end{bmatrix}.$$

Then Theorem 3 is equivalent to the following results.

Theorem 12 *Assume that* $2 < p < \dfrac{N+2}{N-2}$ *or* $N \le 2$*, then*

$$\lim_{t\to T}(T-t)^{\beta}U(a + y\sqrt{T-t}, t) = 0 \quad \text{or} \quad \kappa, \tag{67}$$

where $\kappa = \begin{bmatrix}\pm k_1\\ \pm k_2\end{bmatrix} = \begin{bmatrix}\pm\big(\mu_1(p-1)\big)^{-1/p-1}\\ \pm\big(\mu_2(p-1)\big)^{-1/p-1}\end{bmatrix}$*. The limit* (67) *is independent of* $y \in \mathbb{R}^N$*, and it is uniform on each compact set* $|y| < C$*.*

As explained in the introduction, this is really an assertion about the large time asymptotics of $\tilde{U} = \tilde{U}_a$:

$$\lim_{t\to T}\tilde{U} = 0 \text{ or } \kappa \quad \text{uniformly for } |y| < C, \tag{68}$$

and the limits are the constant solutions of

$$\beta\tilde{U} + H(\tilde{U}) = 0,$$

where $H(\tilde{U}) = (-\mu_1|\tilde{u}|^{p-1}\tilde{u}, -\mu_2|\tilde{v}|^{p-1}\tilde{v})$. Before starting the proof, we first prepare an auxiliary lemmas(see [20]).

Lemma 13 *If* v *solves*

$$\begin{aligned} v_t - \Delta v &= 0 \quad \text{in } B\times(0,\tau)\\ v &= g \quad \text{on } \partial B\times(0,\tau)\\ v &= 0 \quad \text{at } t = 0, \end{aligned} \tag{69}$$

where B *is a ball in* $\mathbb{R}^n$ *and* $\tau > 0$*, then*

$$\sup_{0<t<\tau}\|v\|^2_{H^1(B)} \le C\int_0^{\tau}\Big(\|g\|^2_{H^1(\partial B)} + \|g_t\|^2_{L^2(\partial B)}\Big)dt. \tag{70}$$

The constant C *remains uniform as long as* τ *and the radius of* B *stay bounded away from* 0 *and* ∞*.*

The next lemma shows the L^∞-norm estimate for $\tilde{U}$.

Proposition 14 *Suppose that* $\tilde{U}$ *solves* (66)*, with* $2 < p < \dfrac{N+2}{N-2}$ *or* $N \le 2$*. For any* $K > 0$ *there exist* $\delta > 0$ *and* $M > 0$ *such that*

$$\int_0^1\int_{B(R)}|\tilde{U}_s|^2dyds + \sup_{0<s<1}\Big(\int_{B(R)}|\tilde{U}|^2dy\Big) < K \tag{71}$$

and

$$\int_0^1\int_{B(R)}|\nabla\tilde{U}|^2dyds < \delta. \tag{72}$$

Then we have

$$|\tilde{U}(y,s)| < M \quad \text{for } (y,s)\in B\left(\frac{R}{2}\right)\times(\frac{1}{2},1), \tag{73}$$

where the constants δ *and* M *depend only on* K*,* R*,* p*, and* N*.*

Proof. Given R and $K > 0$, suppose that $\tilde{U}$ solves (66) on $B(R) \times (0,1)$ and satisfies (71)-(72). The main smallness condition on δ will appear later, but we require $\delta \leq K$ now. The hypotheses (71)-(72) assure that for some r, $\frac{3R}{4} < r < R$,

$$\int_0^1 \Big(\|\tilde{U}\|^2_{H^1(\partial B(r))} + \|\tilde{U}_s\|^2_{L^2(\partial B(r))} \Big) dt \leq \frac{8}{R} K. \tag{74}$$

We choose such an r, and keep it fixed for the rest of the proof. The first step is to control the growth of

$$\|\tilde{U}\|^2_{1,r}(s) = \int_{B(r)} \big(|\tilde{U}|^2(y,s) + |\nabla \tilde{U}|^2(y,s) \big) dy.$$

This would be standard if $\tilde{U} = 0$ on $\partial B(r)$. The effect of nonzero boundary data will be handled by using Lemma 13. We assert that if $\|\tilde{U}\|^2_{1,r}(s_1) = c_1 < \infty$, then there exist $\varepsilon_1 < \frac{1}{4}$, $c_1^{'} > 0$ (depending on N, p, R, and K) such that

$$\|\tilde{U}\|_{1,r}(s) \leq c_1^{'} \quad \text{on } [s_1, \min(s_1 + \varepsilon_1, 1)]. \tag{75}$$

Indeed, let $v(y,s)$ solves

$$\begin{aligned} v_t - \Delta v &= 0 \quad \text{in } B(r) \times (s_1, 1) \\ v &= \tilde{U} \quad \text{on } \partial B(r) \times (s_1, 1) \\ v &= 0 \quad \text{at } s = s_1, \end{aligned}$$

and notice that

$$\sup_{s_1 < s < 1} \|v\|_{1,r} \leq c_2$$

by Lemma 13 and (74). We write (66) in the integral form

$$\tilde{U}(s) = e^{(s-s_1)\Delta} \tilde{U}(s_1) + \int_{s_1}^{s} e^{(s-\tau)\Delta} \Big(-G(\tilde{U}) - \frac{1}{2} y \cdot \nabla \tilde{U} - \beta \tilde{U} \Big) d\tau \tag{76}$$

where $\tilde{U}(s) = \tilde{U}(\cdot, s)$, and $e^{\tau\Delta}$ is the semigroup generated by $-\Delta$ with zero Dirichlet data on $\partial B(r)$. Since $2 < p < \frac{N+2}{N-2}$ or $N \leq 2$, there exists $q > p$ such that

$$\frac{1}{q} \geq \frac{1}{2} - \frac{1}{N} \quad \text{and } 0 < \theta = \frac{N}{2} \Big(\frac{p}{q} - \frac{1}{2} + \frac{1}{N} \Big) < 1.$$

For such q, we have

$$\|e^{\tau\Delta} f\|_{1,r} \leq C(\tau^{-\theta} + 1) \|f\|_{L^{q/p}(B(r))},$$

(see e.g. [43]). Moreover, we have

$$\begin{aligned} &\|e^{\tau\Delta} f\|_{1,r} \leq C(\tau^{-1/2} + 1) \|f\|_{L^2(B(r))}, \; \|e^{\tau\Delta} f\|_{1,r} \leq C \|f\|_{1,r}, \\ &\| |f|^{p-1} f \|_{L^{q/p}(B(r))} \leq C \|f\|^p_{1,r}. \end{aligned}$$

Applying these estimates to the RHS of (76), we see that

$$L(\tau) = \sup_{s_1 \leq s \leq \tau} \|\tilde{U}\|_{1,r}$$

satisfies

$$L(\tau) \leq c_1 + c_2 + c_3 \Big((\tau - s_1)^{1-\theta} \big(L(\tau)^p + L(\tau)^2 \big) + (\tau - s_1)^{1/2} L(\tau) \Big) \tag{77}$$

for each $\tau \leq 1$. Let $c_1^{'} = 2(c_1 + c_2)$, we choose $\varepsilon_1 < \frac{1}{4}$ so that

$$c_3 \big({\varepsilon_1}^{1-\theta} ({c_1^{'}}^p + {c_1^{'}}^2) + {\varepsilon_1}^{1/2} c_1^{'} \big) < c_1 + c_2.$$

Then it follows from (77) that $L(\tau) \le c_1^{'}$ whenever $\tau \le \min(1, s_1 + \varepsilon_1)$, and this establishes (75).

Next we continue the second step. Let $c_1 = (2K)^{1/2}$, $c_1^{'}$ and ε_1 as in (75), and $\delta \le K\varepsilon_1$, we claim that

$$\|\tilde{U}\|_{1,r}(s) \le c_1^{'} \text{ if } \frac{1}{4} \le s \le 1. \tag{78}$$

Indeed, for any s in this range, (72) assures the existence of some $s_1 \in (s - \varepsilon_1, s)$ such that

$$\int_{B(r)} |\nabla\tilde{U}(y, s_1)|^2 dy \le \frac{\delta}{\varepsilon_1} \le K.$$

Therefore $\|\tilde{U}\|_{1,r}^2(s_1) \le 2K = c_1^2$, making use of (71), and so (78) follows from (75).

The final step consists of passing from the H^1 bound (78) to a uniform estimate (73). This is an immediate consequence of parabolic regularity theory, and we omit the details here. ■

Now we are ready to prove Theorem 12.

Proof of Theorem 12. It suffices to establish (68). Let $\{s_j\}$ be any sequence tending to infinity, and consider $\tilde{U}_j(y, s) = \tilde{U}(y, s + s_j)$. The analogues of (71) and (72) hold for $\tilde{U}_j$ on any cylinder, at least when s_j and K are large enough, by Propositions 7 and 11. We conclude from Proposition 14 that

$$|\tilde{U}_j| \le M(R) \quad \text{on } B(R) \times (-R, \infty)$$

for sufficiently large s_j, and hence by parabolic regularity that

$$|\nabla\tilde{U}_j|,\ |\nabla^2\tilde{U}_j|,\ |\tilde{U}_{js}| \le M'(R) \quad \text{on } B\left(\frac{R}{2}\right) \times (-\frac{R}{2}, \infty).$$

By a standard diagonal argument, there is a subsequence (still denoted $\tilde{U}_j$) converging uniformly on every compact subset of $\mathbb{R}^{N+1}$ to a locally Lipschitz continuous function $\tilde{U}_\infty$, which solves (66) in all $\mathbb{R}^{N+1}$ (see Lemma 1 of [19]). Applying Propositions 7 and 11 again, we can obtain

$$\int_{-R}^{\infty}\int_{B(R)} |\nabla\tilde{U}_j|^2 dyds = \int_{-R+s_j}^{\infty}\int_{B(R)} |\nabla\tilde{U}|^2 dyds \to 0$$

$$\int_{-R}^{\infty}\int_{B(R)} |\tilde{U}_{js}|^2 dyds = \int_{-R+s_j}^{\infty}\int_{B(R)} |\tilde{U}_s|^2 dyds \to 0$$

as $j \to \infty$. It follows that $\tilde{U}_\infty$ is independent of both y and s, and hence that $\tilde{U}_\infty$ equals 0 or κ.

It remains to show that $\tilde{U}_\infty$ is independent of the choice of $\{s_j\}$. Consider another sequence $\{\bar{s}_j\}$, with $\bar{U}_j(y, s) = \tilde{U}(y, s + \bar{s}_j)$ converging to $\bar{U}_\infty$, also equals 0 or κ. If $\tilde{U}_\infty \ne \bar{U}_\infty$, then by the continuity of $\tilde{U}(0, s)$ there would be a sequence $\{\theta_j\}$ for which $\tilde{U}(0, \theta_j) \to D$, with $0 < |D| < |\kappa|$. But by the last paragraph, a subsequence of $\{\tilde{U}(y, s + \theta_j)\}$ will converge uniformly to 0 or κ, contradicting the fact that $\tilde{U}(0, 0 + \theta_j) \to D$. Therefore $\tilde{U}_\infty = \bar{U}_\infty$, and the proof is complete. ■

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