ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.34(2022) No.2/3,pp.62-69

Extremal Functions for Trudinger-Moser Inequality Involving the Anisotropic Dirichlet Norm with Logarithmic Weight

Ying Xue*, Pan Ma
School of Mathematical Sciences, Jiangsu University, Zhenjiang, 212013, P. R. China
(Received November 24 2022, accepted December 8 2022)

Abstract: In this paper, we consider the Trudinger-Moser type inequality which involves the anisotropic Dirichlet norm for weighted Sobolev space in unit Wulff ball with the weight $\omega_\beta(x) = \left(\log\left|F^0(x)\right|\right)^{\beta(N-1)}$, $\beta \in [0,1)$ in $\mathbb{R}^N$. Moreover, we obtain extremal functions for this inequality when β is sufficiently small.

Keywords: Anisotropic; extremal function; logarithmic weights; Trudinger-Moser

1 Introduction

Let Ω be a smooth and bounded domain in $\mathbb{R}^N$ with $N \geq 2$, and denote by $W_0^{1,p}(\Omega)$ the standard first order Sobolev space given by the closure of $C_0^\infty(\Omega)$ with respect to the Dirichlet norm $\|u\|_p := \left(\int_\Omega |\nabla u|^p dx\right)^{\frac{1}{p}}$. The Sobolev embedding theorem tells us $W_0^{1,N}(\Omega) \subset L^p(\Omega)$ for all $1 \leq p < \infty$, but easy examples show that $W_0^{1,N}(\Omega) \subsetneq L^\infty(\Omega)$. One knows from the works by Yudovich [1], Peetre [2], Pohozaev [3] and Trudinger [4] that $W_0^{1,N}(\Omega)$ embeds into the Orlicz space $L_\varphi(\Omega)$, with Young function $\varphi(t) = e^{|t|^{\frac{N}{N-1}}}$. This embedding was made more precise by J. Moser[5] via the following sharp inequality:

$$\sup_{u \in W_0^{1,N}(\Omega), \|\nabla u\|_{L^N(\Omega)} \leq 1} \int_\Omega e^{\alpha|u|^{\frac{N}{N-1}}} dx < C|\Omega| \Leftrightarrow \alpha \leq \alpha_N = N\omega_{N-1}^{\frac{1}{N-1}}\omega_{N-1}, \tag{1}$$

where ω_{N-1} is the surface area of the unit ball in $\mathbb{R}^N$.

This result has lead to a very rich correlation literature. Adams [6] extended Trudinger-Moser inequality to high order derivatives; Cianchi [7] studyed sharp Trudinger-Moser inequality without boundary conditions; Fontana [8] established Trudinger-Moser inequalies on manifolds; Trudinger-Moser type inequalies with weights can be found in Calanchi-Terraneo [9] and Adimurthi-Sandeep [10], for more relevant reults see [11–13].

Another natural and interesting question related to the Trudinger-Moser inequality is whether the extremal function exists or not. In this direction Carleson-Chang [14] showed that the supremum is attained when Ω is a unit ball in $\mathbb{R}^N$, that is there exists an extremal function. Then the result was extended to the domain Ω near a ball by Struwe [15]. Struwe's technique was used and extended by Flucher [16] to Ω which is the more general bounded smooth domain in $\mathbb{R}^2$. Csato and Roy [17, 18] gave an alternative proof for the result of Flucher. Later, Lin [19] generalized the existence results to general domain Ω in higher dimension. The Trudinger-Moser type inequality plays an important role in the study of nonlinear partial differential equations. So many authors are interested in this research topic, see for instance [20–23] and the references therein.

Through out this paper, for $\beta \in [0,1)$, $\omega_\beta(x) = \left|\log\left|F^0(x)\right|\right|^{\beta(N-1)}$ will denote the weight function on the Wulff ball $W_F = \left\{x \in \mathbb{R}^N, F^0(x) \leq 1\right\}$ and we denote by $W_0^{1,N}(W_F, \omega_\beta)$ the weight Sobolev space on W_F which is the completion of $C_0^\infty(W_F)$ under the norm

$$\|u\|_\beta = \left(\int_{W_F} |F(\nabla u)|^N \omega_\beta dx\right)^{\frac{1}{N}},$$

*Corresponding author. *E-mail address*: xueying6816@126.com

IJNS.2022.12.15

here the function $F(x)$ is positive, convex and homogeneous of degree 1, and its polar $F^0(x)$ represents a Finsler metric on $\mathbb{R}^N$. Denote the subspace of $W_0^{1,N}(W_F,\omega_\beta)$ all radial function by $W_{0,rad}^{1,N}(W_F,\omega_\beta)$.

Wang-Xia [24] proved the following Trudinger-Moser type inequality

$$\sup\int_\Omega e^{\alpha|u|^{\frac{N-1}{N}}}dx \le C(N)|\Omega|,$$

for all $u\in W_0^{1,N}(\Omega)$ and $\left(\int_\Omega |F(\nabla u)|^N dx\right)^{\frac{1}{N}}\le 1$. Where $\alpha\le\alpha_N = N^{\frac{N}{N-1}}\kappa_N^{\frac{1}{N-1}}$, α_N is the optimal value in the sense that if we can find a sequence $\alpha>\alpha_N$ such that $\int_\Omega e^{\alpha|u_k|^{\frac{N-1}{N}}}dx$ is arbitrarily large. Later, Zhou [25] showed that the supremum is attained when Ω is a Wulff ball in $\mathbb{R}^2$, and in [26] they also established the Lions type concentration-compactness alternative and obtained the existence of extremal functions for the following Trudinger-Moser type inequality

$$\sup_{u\in W_0^{1,N}(\Omega),\int_\Omega F^N(\nabla u)\le 1}\int_\Omega e^{\alpha|u|^{\frac{N}{N-1}}}dx<+\infty, \tag{2}$$

for $0<\alpha\le\alpha_N=N^{\frac{N}{N-1}}\kappa_N^{\frac{1}{N-1}}$, where κ_N is the volume of a unit Wulff ball. Recently, Zhu-Chen [27] investigated Trudinger-Moser inequality which involves the anisotropic Dirichlet norm for radial Sobolev spaces with the weight ω_β in $\mathbb{R}^2$. Furthermore, we can also establish Trudinger-Moser embedding of Sobolev space with the weight ω_β in $\mathbb{R}^N$ as below.

Theorem 1 *Let $\beta\in[0,1)$, $\omega_\beta(x)=\left|\log\left|F^0(x)\right|\right|^{\beta(N-1)}$, for all $u\in W_{0,rad}^{1,N}(W_F,\omega_\beta)$. Then,*

(a)

$$\int_{W_F} e^{\alpha|u|^\gamma}dx<+\infty,$$

if and only if $\gamma\le\gamma_{\beta,N}=\frac{N}{(N-1)(1-\beta)}$.

(b)

$$\sup_{\|u\|_{\omega_\beta}\le 1,rad}\int_{W_F} e^{\alpha|u|^{\frac{N}{(N-1)(1-\beta)}}}dx<+\infty, \tag{3}$$

if and only if $\alpha\le\alpha_{\beta,N}=N\left[N^{\frac{1}{N-1}}\kappa_N^{\frac{1}{N-1}}(1-\beta)\right]^{\frac{1}{1-\beta}}$.

Remark 2 *Using the same arguments in [27], we can prove Theorem 1(a), Theorem 1(b) in the subcritical case (i.e. $\alpha<\alpha_{\beta,N}$)and the optimal of best constant of the Trudinger-Moser inequality with logarithmic weight. While for the critical case (i.e. $\alpha=\alpha_{\beta,N}$), Zhu-Chen obtained* (3) *by means of a critical radial lemma and the famous Leckband functional equality. However, we can provide a simpler proof by using the decreasing monotone(see Section 3).*

Let us denote

$$MT(N,\alpha,\beta)=\sup_{u\in W_{0,rad}^{1,N}(W_F,\omega_\beta),\|u\|_{\omega_\beta}\le 1}\int_{W_F} e^{\alpha|u|^{\frac{N}{(N-1)(1-\beta)}}}dx. \tag{4}$$

To prove the critical case of (3), it is enough to show that $MT(N,\alpha_{\beta,N},\beta)<\infty$. Moreover, in order to study the existence of maximizers for $MT(N,\alpha,\beta)$, let us define *the Trudinger-Moser type functional* on $W_{0,rad}^{1,N}(W_F,\omega_\beta)$ by

$$J_{\beta,N}(u)=\int_{W_F} e^{\alpha_{\beta,N}|u|^{\frac{N}{(1-\beta)(N-1)}}}dx.$$

If $\|u_n\|_{\omega_\beta}=1$, $u_n\rightharpoonup 0$ in $W_{0,rad}^{1,N}(W_F,\omega_\beta)$, and $\lim\limits_{n\to\infty}\int_{W_F\setminus B_r}|F(\nabla u_n)|^N\omega_\beta dx=0$, for any $r\in(0,1)$, then we call the sequence $\{u_n\}$ in $W_{0,rad}^{1,N}(W_F,\omega_\beta)$ a *normalized concentrating sequence*. The *concentrating level* $J_{\beta,N}^\delta(0)$ of the Trudinger-Moser type functional on the normalized concentrating sequences is defined by $J_{\beta,N}^\delta(0)=\sup\left\{\lim\limits_{n\to\infty}\sup J_{\beta,N}(u_n)\right\}$, and u_n is normalized concentrating sequence.

Next we want to provide an upper bound for $J_{\beta,N}^\delta(0)$, we have the following decreasing monotone result.

Theorem 3 *The function $\beta \mapsto J^{\delta}_{\beta,N}(0)$ is decreasing on $[0,1)$.*

It is well known that $J^{\delta}_{0,N}(0) \le |W_F|\left(1+e^{1+\frac{1}{2}+\cdots+\frac{1}{N-1}}\right) < MT(N,\alpha_{0,N},0)$ (see [26]). As a consequence of Theorem 3, we obtain the following upper bound of $J^{\delta}_{\beta,N}(0)$

$$J^{\delta}_{\beta,N}(0) \le J^{\delta}_{0,N}(0) \le |W_F|\left(1+e^{1+\frac{1}{2}+\cdots+\frac{1}{N-1}}\right). \tag{5}$$

We will prove Theorem 3 by using a suitable change of variables on W_F. Combining the estimate (5) with a concentration-compactness principle of Lions type [26, Section 2], we obtain the existence of extremal functions for $MT(N,\alpha_{\beta,N},\beta)$ for $\beta \ge 0$ sufficiently small as follows.

Theorem 4 *There exists $\beta_0 \in (0,1)$ such that $MT(N,\alpha_{\beta,N},\beta)$ is attained for any $\beta \in [0,\beta_0)$.*

We organize this paper as follows. In Section 2, we introduce some useful facts which are crucial to our proofs. In Section 3, we finish the proof of Theorem 1 for the critical case, the decreasing monotonicity of $J^{\delta}_{\beta,N}(0)$ and the existence of extremal functions for $MT(N,\alpha_{\beta,N},\beta)$.

2 Preliminaries

We first give some important properties of the function $F(x)$ and its polar $F^0(x)$:

Lemma 5 *The function $F(x)$ and its polar $F^0(x)$ satisfies:*
(1) *F is homogeneous of degree 1;*
(2) *$F(tx) = |t|F(x)$ for any $t \in \mathbb{R}$, $x \in \mathbb{R}^N$;*
(3) *$C \le |\nabla F(x)| \le \frac{1}{C}$, and $C \le |\nabla F^0(x)| \le \frac{1}{C}$ for some $C > 0$ and any $x \ne 0$;*
(4) *$F(\nabla F^0(x)) = 1$, $F^0(\nabla F(x)) = 1$ for any $x \ne 0$;*
(5) *There exist two constants $0 \le m \le M < \infty$ such that $m|\xi| \le F(\xi) \le M|\xi|$;*
(6) *$F^0(x) \ge 0$.*

In this paper, we denote $W_F := \left\{x \in R^N \,\middle|\, F^0(x) \le 1\right\}$ and $\kappa_N := |W_F|$, the Lebesgue measure of W_F. We also use the notion $W_r(0) := \left\{x \in R^N \,\middle|\, F^0(x) \le 1\right\}$ and we call $W_r(0)$ a Wulff ball of radius r with center at 0. Let $W_F = W_r(0)$, then we consider inequality (3) is finite with logarithmic weight $\omega_\beta(x) = \left|\log\left|F^0(x)\right|\right|^{\beta(N-1)}$, $\beta \in [0,1)$ in dimension $N \ge 2$, and prove the existence of extremal functions for this inequality.

For any $u \in W^{1,N}_{0,rad}(W_F,\omega_\beta)$, using Hölder's inequality, we get

$$\begin{aligned}
|u(r)-u(s)| &\le \int_r^s \left|u'(t)\right| dt = \int_r^s \left|u'(t)\right| t^{\frac{N-1}{N}} |\log t|^{\frac{\beta(N-1)}{N}} t^{-\frac{N-1}{N}} |\log t|^{-\frac{\beta(N-1)}{N}} dt \\
&\le \left(\frac{1}{N\kappa_N}\right)^{\frac{1}{N}} \left(\frac{1}{1-\beta}\right)^{\frac{N-1}{N}} \left[\left(\int_r^s \left|u'(t)\right|^N t^{N-1} |\log t|^{\beta(N-1)} N\kappa_N dt\right)\right]^{\frac{1}{N}} \left(-\log\frac{r}{s}\right)^{\frac{(N-1)(1-\beta)}{N}} \\
&= \left(\frac{N}{\alpha_{\beta,N}}\right)^{\frac{(N-1)(1-\beta)}{N}} \left(\int_{W_s\setminus W_r} |F(\nabla u)|^N \omega_\beta dx\right)^{\frac{1}{N}} \left(-\log\frac{r}{s}\right)^{\frac{(N-1)(1-\beta)}{N}},
\end{aligned} \tag{6}$$

for any $0 < r < s \le 1$. In particular, for $s = 1$ we get

$$|u(r)| \le \left(\frac{N}{\alpha_{\beta,N}}\right)^{\frac{(N-1)(1-\beta)}{N}} \left(\int_{W_F\setminus W_r} |F(\nabla u)|^N \omega_\beta dx\right)^{\frac{1}{N}} (-\log r)^{\frac{(N-1)(1-\beta)}{N}}. \tag{7}$$

Next, we introduce a useful formula for change of variable on W_F. For any function $u \in W^{1,N}_{0,rad}(W_F,\omega_\beta)$ and for any $0 \le \tilde{\beta} < \beta$, denote

$$v(x) = \left(\frac{\alpha_{\beta,N}}{\alpha_{\tilde{\beta},N}}\right)^{\frac{(N-1)\left(1-\tilde{\beta}\right)}{N}} u(x)\,|u(x)|^{\frac{\beta-\tilde{\beta}}{1-\beta}}. \tag{8}$$

The following lemma is a key ingredient in our proofs below.

Lemma 6 *Let u be a function in* $W^{1,N}_{0,rad}(W_F,\omega_\beta)$ *with* $\|u\|_{\omega_\beta}\le 1$ *and let v be defined by* (8). *Then we have* $\|v\|_{\omega_{\tilde\beta}}\le 1$ *for any* $\tilde\beta\le\beta$.

Proof.
By (7), we have

$$\begin{aligned}|F(\nabla v(x))|^N &= \left(\frac{\alpha_{\beta,N}}{\alpha_{\tilde\beta,N}}\right)^{(N-1)\left(1-\tilde\beta\right)}|F(\nabla u(x))|^N|u(x)|^{\frac{N(\beta-\tilde\beta)}{1-\beta}}\\ &\le \frac{1-\tilde\beta}{1-\beta}|F(\nabla u(x))|^N\frac{\omega_\beta(x)}{\omega_{\tilde\beta}(x)}\left(\int_{W_1\setminus W_{|x|}W_{F^0(X)}}|F(\nabla u)|^N\omega_\beta dy\right)^{\frac{\beta-\tilde\beta}{1-\beta}}.\end{aligned}$$

Thus, we get

$$\begin{aligned}\|v\|_{\omega_{\tilde\beta}} &\le \frac{1-\tilde\beta}{1-\beta}\int_{W_F}|F(\nabla u(x))|^N\omega_\beta(x)\left(\int_{W_F\setminus W_{F^0(X)}}|F(\nabla u(s))|^N\omega_\beta(s)\,ds\right)^{\frac{\beta-\tilde\beta}{1-\beta}}dr\\ &= \frac{1-\tilde\beta}{1-\beta}\int_0^1|F(\nabla u(x))|^N\omega_\beta(x)\,N\kappa_N r^{N-1}\left(\int_r^1|F(\nabla u(s))|^N\omega_\beta(s)\,N\kappa_N s^{N-1}ds\right)^{\frac{\beta-\tilde\beta}{1-\beta}}dr\\ &= -(N\kappa_N)^{\frac{1-\tilde\beta}{1-\beta}}\int_0^1\frac{d}{dr}\left[\left(\int_r^1|F(\nabla u(s))|^N\omega_\beta(s)\,N\kappa_N s^{N-1}\right)^{\frac{\beta-\tilde\beta}{1-\beta}}\right]dr\\ &= (N\kappa_N)^{\frac{1-\tilde\beta}{1-\beta}}\left(\int_0^1|F(\nabla u(s))|^N\omega_\beta(s)\,s^{N-1}\right)^{\frac{1-\tilde\beta}{1-\beta}}\\ &= \|u\|_{\omega_\beta}^{\frac{N\left(1-\tilde\beta\right)}{1-\beta}}\le 1.\end{aligned}$$

■

Besides, we have the following lemma as the concentration-compactness principle of Lions type [26] for functions in $W^{1,N}_{0,rad}(W_F,\omega_\beta)$.

Lemma 7 *Let* $\{u_n\}$ *be a sequence in* $W^{1,N}_{0,rad}(W_F,\omega_\beta)$ *such that* $\|u_n\|_{\omega_\beta}=1$ *and* $u_n\rightharpoonup u_0$ *in* $W^{1,N}_{0,rad}(W_F,\omega_\beta)$. *Then we have*

$$\lim_{n\to\infty}\sup\int_{W_F}e^{p\alpha_{\beta,N}|u|^{\frac{N}{(N-1)(1-\beta)}}}dx<\infty, \tag{9}$$

for any $p<p(u_0):=\left(1-\|u_0\|^N_{\omega_0}\right)^{-\frac{1}{(1-\beta)(N-1)}}$.

3 Extremal Functions For The Inequality In The Critical Case

From Remark 2, we only provide a simpler proof of Theorem 1(b) for the critical case $\alpha=\alpha_{\beta,N}$ in this section.
Proof of Theorem 1. Let u be any function in $W^{1,N}_{0,rad}(W_F,\omega_\beta)$ with $\|u\|_{\omega_\beta}\le 1$. Let v be the function defined by (8). By Lemma 6, we know that $\|v\|_{\omega_{\tilde\beta}}\le 1$ for any $\tilde\beta\le\beta$. By the definition of $MT(N,\alpha_{\beta,N},\beta)$ and the function v, we have

$$\int_{W_F}e^{\alpha_{\beta,N}|u|^{\frac{N}{(N-1)(1-\beta)}}}dx=\int_{W_F}e^{\alpha_{\tilde\beta,N}|v|^{\frac{N}{(N-1)\left(1-\tilde\beta\right)}}}dx\le MT\left(N,\alpha_{\tilde\beta,N},\tilde\beta\right).$$

The proceeding estimate holds for any function $u\in W^{1,N}_{0,rad}(W_F,\omega_\beta)$ with $\|u\|_{\omega_\beta}\le 1$. Therefore, we have

$$MT(N,\alpha_{\beta,N},\beta)\le MT\left(N,\alpha_{\tilde\beta,N},\tilde\beta\right),$$

for any $0 \le \tilde{\beta} \le \beta < 1$.

Hence, from (2) we obtain

$$MT\left(N, \alpha_{\beta,N}, \beta\right) \le MT\left(N, \alpha_{0,N}, 0\right) < \infty.$$

This finishes the proof of Theorem 1 for the critical case $\alpha = \alpha_{\beta,N}$. ■

Next, we prove Theorem 3 on the decreasing monotonicity of the concentration level $J^{\delta}_{\beta,N}(0)$ as a function of β.

Proof of Theorem 3. We first note that for any $\beta \in [0,1)$ and for any normalized concentrating sequence $\{u_n\}$ in $W^{1,N}_{0,rad}(W_F, \omega_\beta)$, we have

$$J^{\delta}_{\beta,N}(0) \ge \lim_{n\to\infty} \sup \int_{W_F} e^{\alpha_{\beta,N}|u_n|^{\frac{N}{(N-1)(1-\beta)}}} dx \ge |W_F|. \tag{10}$$

It is sufficient to prove that for any $\tilde{\beta} \le \beta$,

$$J^{\delta}_{\beta,N}(0) \le J^{\delta}_{\tilde{\beta},N}(0). \tag{11}$$

Let $\{u_n\}$ be a normalized concentrating sequence in $W^{1,N}_{0,rad}(W_F, \omega_\beta)$. By extracting a subsequence, we may assume that the limit of $J^{\delta}_{\beta,N}(u_n)$ exists and is equal to the superior limit of the original sequence. Let v_n be the function defined by (8). Thus we have $v_n \in W^{1,N}_{0,rad}\left(W_F, \omega_{\tilde{\beta}}\right)$ and $\|v_n\|_{\omega_{\tilde{\beta}}} < 1$. Note that $u_n \to 0$ on $W_F \backslash W_r$ for any $r \in (0,1)$, this implies that $v_n \rightharpoonup 0$ in $W^{1,N}_{0,rad}\left(W_F, \omega_{\tilde{\beta}}\right)$.

Indeed, the sequence $\{v_n\}$ is bounded in $W^{1,N}_{0,rad}(W_F, \omega_\beta)$. If v is a weak limit of some subsequence of v_n, then by (6) with β being replaced by $\tilde{\beta}$, we know that this subsequence converges uniformly to v on $W_F \backslash W$ for any $r \in (0,1)$. However, $u_n \to 0$ on $W_F \backslash W_r$ for any $r \in (0,1)$, hence $v \equiv 0$. We have the following two case:

Case 1. $\lim\limits_{n\to\infty} \sup \|v_n\|_{\omega_{\tilde{\beta}}} < 1$. There exist some $a \in (0,1)$ and n_0 such that $\|v_n\|_{\omega_{\tilde{\beta}}} \le a$ for any $n \ge n_0$. Thus $e^{\alpha_{\tilde{\beta},N}|v_n|^{\frac{N}{(N-1)(1-\beta)}}}$ is bounded in $L^p(W_F)$ for some $p > 1$ by (3).

Therefore, by (10) we have

$$\lim_{n\to\infty} \int_{W_F} e^{\alpha_{\beta,N}|u_n|^{\frac{N}{(N-1)(1-\beta)}}} dx = \lim_{n\to\infty} \int_{W_F} e^{\alpha_{\tilde{\beta},N}|v_n|^{\frac{N}{(N-1)(1-\tilde{\beta})}}} dx = |W_F| \le J^{\delta}_{\tilde{\beta},N}(0).$$

Case 2. $\lim\limits_{n\to\infty} \sup \|v_n\|_{\omega_{\tilde{\beta}}} = 1$. We may extract a subsequence and assume that $\lim\limits_{n\to\infty} \|v_n\|_{\omega_{\tilde{\beta}}} = 1$. Define $\tilde{v}_n = \frac{v_n}{\|v_n\|_{\omega_{\tilde{\beta}}}}$, then $\tilde{v}_n \rightharpoonup 0$ in $W^{1,N}_{0,rad}\left(W_F, \omega_{\tilde{\beta}}\right)$ and $|v_n| \le |\tilde{v}_n|$.

If $\tilde{v}_n$ is a normalized concentrating sequence in $W^{1,N}_{0,rad}\left(W_F, \omega_{\tilde{\beta}}\right)$, then we have

$$\lim_{n\to\infty} J_{\beta,N}(u_n) = \lim_{n\to\infty} J_{\tilde{\beta},N}(v_n) \le \lim_{n\to\infty} \sup J_{\tilde{\beta},N}(\tilde{v}_n) \le J^{\delta}_{\tilde{\beta},N}(0).$$

Otherwise, there exist $a, r \in (0,1)$ and $n \ge n_0$ such that $\int_{W_r} |\bar{v}_n|^N \omega_{\tilde{\beta}} dx \le a$ for any $n \ge n_0$.

Define the function

$$\bar{v}_n(x) = \begin{cases} \tilde{v}_n(x) - \tilde{v}_n(r) & , \quad |x| \le r, \\ 0 & , \quad 0 < |x| < 1. \end{cases}$$

It is clear that $\bar{v}_n(x) \in W^{1,N}_{0,rad}\left(W_F, \omega_{\tilde{\beta}}\right)$ and $\|\bar{v}_n\|^N_{\omega_{\tilde{\beta}}} \le a < 1$ for any $n \ge n_0$. For $\varepsilon > 0$ sufficiently small such that $(1+\varepsilon)\, a^{\frac{1}{(N-1)(1-\tilde{\beta})}} < 1$ on W_r , we get

$$\tilde{v}_n(x) = \bar{v}_n(x) + \tilde{v}_n(r),$$

and

$$|\tilde{v}_n(x)|^{\frac{N}{(1-\tilde{\beta})(N-1)}} \le (1+\varepsilon)\, |\bar{v}_n(x)|^{\frac{N}{(1-\tilde{\beta})(N-1)}} + C\left(N, \tilde{\beta}, \varepsilon\right) |\tilde{v}_n(r)|^{\frac{N}{(1-\tilde{\beta})(N-1)}},$$

where $C\left(N, \tilde{\beta}, \varepsilon\right) = \left(1 - (1+\varepsilon)^{-\frac{(1-\tilde{\beta})(N-1)}{N\tilde{\beta}+1-\tilde{\beta}}}\right)^{-\frac{(1-\tilde{\beta})(N-1)}{N\tilde{\beta}+1-\tilde{\beta}}}$. Choosing the appropriate a and ε, and applying Trudinger-Moser inequality (3) for the weight $\omega_{\tilde{\beta}}$, we obtain that $e^{\alpha_{\tilde{\beta},N}(1+\varepsilon)|\bar{v}_n(x)|^{\frac{N}{(N-1)(1-\tilde{\beta})}}}$ is bounded in $L^p(W_F)$ for some $p > 1$. Note that $\tilde{v}_n(r) \to 0$ as $n \to \infty$. Hence, $e^{\alpha_{\tilde{\beta},N}|\tilde{v}_n(x)|^{\frac{N}{(N-1)(1-\tilde{\beta})}}}$ is bounded in $L^p(W_r)$ for some $p > 1$.

Therefore, we have

$$\lim_{n\to\infty}\int_{W_r} e^{\alpha_{\tilde\beta,N}|\tilde v_n(x)|^{\frac{N}{(N-1)(1-\tilde\beta)}}}dx=|W_r|\,,$$

on $W_F\backslash W$. By (7) and Lebesgue's dominated convergence theorem, we get

$$\lim_{n\to\infty}\int_{W_F\backslash W_r} e^{\alpha_{\tilde\beta,N}|\tilde v_n(x)|^{\frac{N}{(N-1)(1-\tilde\beta)}}}dx=|W_F|-|W_r|\,.$$

Combining the above two estimates, we can obtain

$$\lim_{n\to\infty}\int_{W_F} e^{\alpha_{\tilde\beta,N}|\tilde v_n(x)|^{\frac{N}{(N-1)(1-\tilde\beta)}}}dx=|W_F|\,. \tag{12}$$

From (12) and (10), we have

$$\lim_{n\to\infty}J_{\beta,N}(u_n)=\lim_{n\to\infty}J_{\tilde\beta,N}(v_n)\le|W_F|\le J^{\delta}_{\tilde\beta,N}(0)\,.$$

Summarizing, we get

$$\lim_{n\to\infty}\sup J_{\beta,N}(u_n)\le J^{\delta}_{\tilde\beta,N}(0)\,,$$

for any normalized concentrating sequence $u_n\in W^{1,N}_{0,rad}(W_F,\omega_\beta)$. The proof of Theorem 3 is completed. ∎

In order to prove Theorem 4, we need the following lemma.

Lemma 8 *The function $\beta\mapsto MT(N,\alpha_{\beta,N},\beta)$ is continuous at $\beta=0$, i.e.*

$$\lim_{\beta\to0}MT(N,\alpha_{\beta,N},\beta)=MT(N,\alpha_{0,N},0)\,.$$

Proof.

It follows from the proof of Theorem 1 that

$$\lim_{\beta\to0}\sup MT(N,\alpha_{\beta,N},\beta)\le MT(N,\alpha_{0,N},0)\,.$$

Thus, it is enough to show that

$$\lim_{\beta\to0}\inf MT(N,\alpha_{\beta,N},\beta)\ge MT(N,\alpha_{0,N},0)\,.$$

It is well know that $MT(N,\alpha_{0,N},0)$ is attained by a function $u_0\in C^1(W_F)\cap W^{1,N}_{0,rad}(W_F)$ with $\|F(\nabla u_0)\|_N=1$. Hence, for any $\varepsilon>0$, $\lim_{\beta\to0}\int_{W_F}|F(\nabla u)|^N\omega_\beta dx=\int_{W_F}|F(\nabla u)|^N dx=1$ and there exists some $\beta_\varepsilon>0$ such that $\|u_0\|_{\omega_\beta}\le1+\varepsilon$ for any $\beta\le\beta_\varepsilon$.

Hence, for any $\beta\le\beta_\varepsilon$, we get

$$MT(N,\alpha_{\beta,N},\beta)\ge\int_{W_F}e^{\alpha_{\beta,N}\left(\frac{|u_0|}{1+\varepsilon}\right)^{\frac{N}{(N-1)(1-\beta)}}}dx.$$

Let $\beta\to0$, $\varepsilon\to0$. Using again Fatou's lemma, we have

$$\begin{aligned}\lim_{\beta\to0}\inf MT(N,\alpha_{\beta,N},\beta)&\ge\int_{W_F}e^{\alpha_{0,N}|u_0|^{\frac{N}{(N-1)}}}dx=\int_{W_F}e^{\alpha|u_0|^{\frac{N}{(N-1)}}}dx\\&=MT(N,\alpha_{0,N},0)\,.\end{aligned}$$

Therefore,

$$\lim_{\beta\to0}MT(N,\alpha_{\beta,N},\beta)=MT(N,\alpha_{0,N},0)\,.$$

∎

Now, we are ready to prove Theorem 4.

Proof of Theorem 4. It is well know [26] that

$$MT\left(N,\alpha_{0,N},0\right) > |W_F|\left(1+e^{1+\frac{1}{2}+\cdots+\frac{1}{N-1}}\right) \geq J^{\delta}_{0,N}\left(0\right).$$

By Lemma 8 and Theorem 3, there exists some $\beta_0 \in (0,1)$ such that

$$MT\left(N,\alpha_{0,N},0\right) > J^{\delta}_{0,N}\left(0\right) \geq J^{\delta}_{\beta,N}\left(0\right). \tag{13}$$

for any $\beta \in [0,\beta_0)$.

Let $\{u_n\}$ be a maximizing sequence for $MT\left(N,\alpha_{\beta,N},\beta\right)$. We can assume that $u_n \rightharpoonup u_0$ in $W^{1,N}_{0,rad}\left(W_F,\omega_\beta\right)$ and $u_n \to u_0$ on $W_F \backslash W_r$ for any $r \in (0,1)$ by (6) for some $u_0 \in W^{1,N}_{0,rad}\left(W_F,\omega_\beta\right)$.

Suppose that $u_0 \equiv 0$. First we claim that $\{u_n\}$ is not a normalized concentrating sequence. Suppose that $\{u_n\}$ is a normalized concentrating sequence, then

$$MT\left(N,\alpha_{\beta,N},\beta\right) = \lim_{n\to\infty} J_{\beta,N}\left(u_n\right) \leq J^{\delta}_{\beta,N}\left(0\right),$$

which contradicts (13).

There exist $a, r \in (0,1)$ and n_0 such that

$$\int_{W_r} \left|F\left(\nabla u\right)\right|^N \omega_\beta dx \leq a,$$

for any $n \geq n_0$. Repeating the proof of (12) in this case, we have

$$MT\left(N,\alpha_{\beta,N},\beta\right) = \lim_{n\to\infty} J_{\beta,N}\left(u_n\right) = |W_F| \leq J^{\delta}_{\beta,N}\left(0\right),$$

which again contradicts (13). Thus we must have $u_0 \not\equiv 0$.

Now, by Lemma 7, $e^{\alpha_{\beta,N}|u_n|^{\frac{N}{(N-1)(1-\beta)}}}$ is bounded in $L^p\left(W_F\right)$ for some $p > 1$. Therefore, it holds that

$$MT\left(N,\alpha_{\beta,N},\beta\right) = \lim_{n\to\infty} \int_{W_F} e^{\alpha_{\beta,N}|u_n|^{\frac{N}{(N-1)(1-\beta)}}} dx = \int_{W_F} e^{\alpha_{\beta,N}|u_0|^{\frac{N}{(N-1)(1-\beta)}}} dx.$$

Since $u_n \rightharpoonup u_0$ in $W^{1,N}_{0,rad}\left(W_F,\omega_\beta\right)$, we have $\|u_0\|_{\omega_\beta} \leq 1$.

If $\|u_0\|_{\omega_\beta} < 1$, then we get

$$\begin{aligned} MT\left(N,\alpha_{\beta,N},\beta\right) &= \int_{W_F} e^{\alpha_{\beta,N}|u_0|^{\frac{N}{(N-1)(1-\beta)}}} dx < \int_{W_F} e^{\alpha_{\beta,N}\left(\frac{|u_0|}{\|u_0\|_{\omega_\beta}}\right)^{\frac{N}{(N-1)(1-\beta)}}} dx \\ &\leq MT\left(N,\alpha_{\beta,N},\beta\right), \end{aligned}$$

which is impossible. Therefore, $\|u_0\|_{\omega_\beta} = 1$ and u_0 is an extremal function for $MT\left(N,\alpha_{\beta,N},\beta\right)$. ■

References

[1] V.I. Yudovich. Some estimates connected with integral operators and with solutions of elliptic equations. *Translated in Soviet Math. Dokl*, 2(1961):746–749.

[2] J. Peetre. Espaces d interpolation et torme de Soboleff. *Ann.* Inst. Fourier (Grenoble), 16(1966):279–317.

[3] S.I. Pohozaev. The Sobolev embedding in the case pl = n. *Proc. of the Technical Scientific Conference on Advances of Scientific Research*, 1964–1965.

[4] N.S. Trudinger. On imbeddings into Orlicz spaces and some applications. *Math. Mech*, 17(1967):473–483.

[5] J. Moser. A sharp form of an inequality by N. Trudinger. Indiana Univ. *Math*, 20(1970-71):1077–1092.

[6] D.R. Adams. A sharp inequality of J. Moser for higher order derivatives. *Ann. of Math*,128(2)(1988):385–398.

[7] A. Cianchi. Moser-Trudinger inequalities without boundary conditions and isoperimetric problems. Indiana Univ. *Math*, 54(3)(2005):669–705.

[8] L. Fontana. Sharp borderline Sobolev inequalities on compact Riemannian manifolds. *Comment. Math. Helv*, 68(3)(1993):415–454.

[9] M. Calanchi, E. Terraneo. Non-radial maximizers for functionals with exponential nonlinearity in R2. *Adv. Nonlinear Stud*, 5(2005):337–350.

[10] Adimurthi, K. Sandeep. A singular Moser-Trudinger embedding and its applications. *NoDEA Nonlinear Differential Equations Appl*, 13(2007), no. 5-6, 585–603.

[11] M. Calanchi. Some weighted inequalities Trudinger of Trudinger-Moser type. *Birkhauser/Springer.* Cham, 2014.

[12] M. Calanchi, B. Ruf. Trudinger-Moser type inequalities with logarithmic weights in dimension N. *Nonlinear Anal*, 121(2015):403–411.

[13] V.H. Nguyen. Remarks on the Moser-Trudinger type inequality with logarithmic weights in dimension N. *Proc. Am. Math. Soc*, 147(2019):5183–5193.

[14] L. Carleson, S-Y A. Chang. On the existence of an extremal function for an inequality of J. Moser (English, with French summary). *Bull. Sci. Math*, (2)110(1986), no. 2, 113–127.

[15] M. Struwe. Critical points of embeddings of $H_0^{1,n}$ into Orlicz spaces (English, with French summary). *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 5 (1988), no. 5, 425–464.

[16] M. Flucher. Extremal functions for the Trudinger-Moser inequality in 2 dimensions. *Math. Helv*, 67(1992), no. 3, 471–497.

[17] G. Csato, P. Roy. Singular Moser-Trudinger inequality on simply connected domains. *Comm. Partial Differential Equations*, 41(2016), no. 5, 838–847.

[18] G. Csato, P. Roy, and V. H. Nguyen. Extremals for the singular Moser-Trudinger inequality via n-harmonic transplantation. *arXiv:1801.03932v3*, 2018.

[19] K-C Lin. Extremal functions for Moser's inequality. *Trans. Amer. Math. Soc*, 348(1996), no. 7, 2663–2671.

[20] D.G. De Figueiredo, O.H. Miyagaki, B. Ruf. Elliptic equations in R2 with nonlinearities in the critical growth range *Djairo G. de Figueiredo-Selected Papers. Springer. Cham*, 1995:423–437.

[21] D G. de Figueiredo, J M. do Ó, and B. Ruf. On an inequality by N. Trudinger and J. Moser and related elliptic equations. *Comm. Pure Appl. Math*, 55(2002), no. 2, 135–152.

[22] F. Robert, J. Wei. Asymptotic behavior of a fourth order mean field equation with Dirichlet boundary condition. *Indiana University mathematics journal*, 2008:2039–2060.

[23] M. Calanchi, E. Massa, B. Ruf. Weighted Trudinger-Moser inequalities and associated Liouville type equations. *Proceedings of the American Mathematical Society*, 146(12)(2018):5243–5256.

[24] G. F. Wang, C. Xia. Blow-up analysis of a Finsler-Liouville equation in two dimensions. *J. Differential Equations*, 252(2012):1668–1700.

[25] C. Zhou, C. Zhou. Extremal functions of Moser-Trudinger inequality involving Finsler-Laplacian. *Communications on Pure & Applied Analysis*, 17(6)(2018):2309.

[26] C. Zhou C. Zhou. Moser-Trudinger inequality involving the anisotropic Dirichlet norm $\left(\int_\Omega F^N(\nabla u)\,dx\right)^{\frac{1}{N}}$ on $W_0^{1,N}(\Omega)$. *Journal of Functional Analysis*, 276(9)(2019):2901–2935.

[27] M. Zhu, W. Chen. Trudinger-Moser Inequality Involving the Anisotropic Norm with Logarithmic Weights in Dimension Two. *Mathematica Applicata*, 35(04)(2022):766–775.