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The Asymptotic Spreading Speed of a Diffusive SIR Epidemic Model

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Abstract: In this paper, we study the asymptotic spreading speed of a diffusive SIR epidemic model with Holling-II incidence. By comparison principle and the asymptotic spreading theory of reaction-diffusion L-ogistic equation, we obtain that in this model the minimal speed of traveling waves equals to the asymptotic spreading speed.

Keywords: Asymptotic spreading speed; Epidemic model; Reaction-diffusion equations.

1 Introduction

Recently, Fu [4] and Li et al. [3] investigated the reaction-diffusion SIR epidemic model

$$\begin{cases} S_t(x,t) = d_1 S_{xx}(x,t) + \mu - \mu S(x,t) - \dfrac{\beta S(x,t)I(x,t)}{1+\alpha I(x,t)}, & (1.1a) \\ I_t(x,t) = d_2 I_{xx}(x,t) + \dfrac{\beta S(x,t)I(x,t)}{1+\alpha I(x,t)} - (\mu+\sigma)I(x,t), & (1.1b) \\ R_t(x,t) = d_3 R_{xx}(x,t) + \sigma I(x,t) - \mu R(x,t), \quad (x,t) \in \mathbb{R}\times\mathbb{R}^+, & (1.1c) \end{cases}$$

where $S(x,t)$, $I(x,t)$, $R(x,t)$ represent the population densities of the susceptible, infective, removed individuals at position x and time t, respectively. The parameters μ, β, α, σ are all positive constants in which μ denotes the death rates of susceptible, infective and removed populations. The parameter σ is the removed/recovery rate, β is the infective transmission rate and α measures the saturation level in the Holloing type II incidence function $\beta SI/(1+\alpha I)$. They proved that there exists a constant $c^* > 0$ such that this system admits nontrivial nonnegative traveling wave solutions with speed c if $c \geq c^*$ and has no nontrivial nonnegative traveling wave solutions with speed c if $c < c^*$. As we know, asymptotic spreading speed is an very important and interesting topic in the filed of mathematical epidemiology [2, 3, 5, 8–10]. For (1.1a)-(1.1c), asymptotic spreading speed is still an open problem and it is the aim of the present paper.

Noting that the function $R(x,t)$ does not appear in (1.1a) and (1.1b), we only study the subsystem of (1.1). To characterize the spreading speed of infective populations, we consider the initial value problem with the following initial condition

$$S(x,0) = 1, \qquad I(x,0) = I_0(x), \qquad x \in \mathbb{R}, \tag{1.2}$$

where I_0 is a nonnegative continuous function defined in $\mathbb{R}$ with a nonempty compact support. Throughout this paper, we assume $\beta > \mu + \sigma$. By [1, 4, 7], the minimal speed of traveling wave solutions to (1.1a) and (1.1b) can be defined as

$$c^* := \inf_{\lambda\in\mathbb{R}^+} \frac{d_2\lambda^2 + \beta - \mu - \sigma}{\lambda} = 2\sqrt{d_2(\beta-\mu-\sigma)}. \tag{1.3}$$

Theorem 1 *Let $(S,I)(x,t)$ be a solution of (1.1a) and (1.1b) with initial condition (1.2). Then c^* defined in (1.3) is the asymptotic spreading speed of $I(x,t)$ in the sense*

$$\lim_{t\to\infty} \sup_{|x|>ct} I(x,t) = 0, \quad \forall c > c^* \tag{1.4}$$

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and

$$\liminf_{t\to\infty} \inf_{|x|<ct} I(x,t) > 0, \quad \forall c \in (0, c^*). \tag{1.5}$$

2 Preliminaries

Let (S, I) be a solution of (1.1a) and (1.1b) with initial condition (1.2). The purpose of this section is to derive some a priori estimates. Hereafter we use the notation

$$X := \{\text{all uniformly continuous bounded functions defined in } \mathbb{R}\},$$
$$X^+ := \{\omega \in X \mid \omega \geq 0 \text{ in } \mathbb{R}\}.$$

Let $\omega_0 \in X^+$ and let ω be a solution to

$$\begin{cases} \omega_t(x,t) = d\omega_{xx}(x,t), & (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\ \omega(x,0) = \omega_0(x), & x \in \mathbb{R}. \end{cases}$$

Then $\omega(\cdot, t) \in X^+$ for all $t > 0$ due to the positivity property of the semigroup. Moreover, this positivity property also holds for

$$\omega_t(x,t) = d\omega_{xx}(x,t) - L\omega(x,t), \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \tag{2.1}$$

where L is a constant. Indeed, (2.1) can be re-written as

$$\left[e^{Lt}\omega\right]_t(x,t) = d\left[e^{Lt}\omega(\cdot,t)\right]_{xx}(x), \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+.$$

Hence $\omega(\cdot, t) \in X^+$ for all $t > 0$, if $\omega(\cdot, 0) \in X^+$.

We now derive the following a priori estimates for solutions (S, I) to system (1.1a) and (1.1b).

Lemma 2 *Let I_0 be a nonnegative nontrivial compactly supported continuous function defined in $\mathbb{R}$. Then system (1.1a) and (1.1b) with initial condition (1.2) admits a global solution (S, I) such that*

$$0 \leq S(x,t) \leq 1, \qquad 0 \leq I(x,t) \leq k, \qquad \forall (x,t) \in \mathbb{R} \times \mathbb{R}^+, \tag{2.2}$$

where $k := \frac{\beta-\mu-\sigma}{\alpha(\mu+\sigma)}$.

Proof. It is clear that (1.1a), (1.1b) and (1.2) has a solution (S, I) for $t < T$ for some $T < \infty$. Hence $S(\cdot, t), I(\cdot, t) \in X$ and $I \in (-1/\alpha, \infty)$ for all $t < T$.

First, we prove that $I(x,t) \geq 0$ for all $(x,t) \in \mathbb{R} \times (0, T)$. On the contrary, we suppose that there exist a point $P_0(x_0, t_0)$ and a constant $\rho > 0$ such that $-\frac{1}{\alpha} < I(P_0) < 0$ and $\frac{I(P_0) - \frac{1}{\alpha}}{2} < I(x,t) < 0$ for $(x,t) \in U(P_0, \rho)$. Hence, we obtain

$$\frac{\beta S(x,t)}{1 + \alpha I(x,t)} \leq \frac{\beta \hat{S}}{1 + \alpha I(x,t)} \leq \frac{2\beta \hat{S}}{\alpha I(P_0)}$$

with $\hat{S} := \max\{S(x,t) | (x,t) \in \overline{U(P_0, \rho)}\}$, which implies that there exists a constant L such that

$$\frac{\beta S(x,t)}{1 + \alpha I(x,t)} - (\mu + \sigma) \leq -L, \quad (x,t) \in U(P_0, \rho).$$

It follows from (1.1b) and $I(x,t) < 0$ for $(x,t) \in U(P_0, \rho)$ that

$$I_t(x,t) = d_2 I_{xx}(x,t) + \left[\frac{\beta S(x,t)}{1 + \alpha I(x,t)} - (\mu + \sigma)\right] I(x,t) \geq d_2 I_{xx}(x,t) - L I(x,t), \quad (x,t) \in U(P_0, \rho).$$

Now consider the problem

$$\begin{cases} \tilde{I}_t(x,t) = d_2 \tilde{I}_{xx}(x,t) - L\tilde{I}(x,t), \\ \tilde{I}(x,0) = I_0(x) \in X^+, \end{cases}$$

then we deduce the solution $\tilde{I}(x,t) \in X^+$. Combined with comparison principle, we have $I(x,t) \geq \tilde{I}(x,t) \geq 0$ for $(x,t) \in U(P_0,\rho)$, where $I(x,t)$ is the solution of the system

$$\begin{cases} I_t(x,t) = d_2 I_{xx}(x,t) + \left[\dfrac{\beta S(x,t)}{1+\alpha I(x,t)} - (\mu+\sigma)\right] I(x,t), \\ I(x,0) = I_0(x) \in X^+. \end{cases}$$

This contradicts with the assumption that $-\frac{1}{\alpha} < I(P_0) < 0$. Therefore,

$$I(x,t) \geq 0 \quad \text{for} \quad (x,t) \in \mathbb{R} \times [0,T). \tag{2.3}$$

Secondly, we show that $S(x,t) \geq 0$ for $(x,t) \in \mathbb{R} \times [0,T)$. On the contrary, we suppose that there exist a point $P_1(x_1,t_1)$ and a constant $\rho_1 > 0$ such that $S(P_1) < 0$ and $\frac{S(P_1)}{2} < S(x,t) < 0$ for $(x,t) \in U(P_1,\rho_1)$. Hence, it follows from (1.1) and (2.3) that

$$\begin{aligned} S_t(x,t) &= d_1 S_{xx}(x,t) + \mu - \mu S(x,t) - \frac{\beta S(x,t) I(x,t)}{1+\alpha I(x,t)} \\ &\geq d_1 S_{xx}(x,t) - \mu S(x,t) - \frac{\beta S(x,t) I(x,t)}{1+\alpha I(x,t)} \\ &\geq d_1 S_{xx}(x,t) - \mu S(x,t), \quad (x,t) \in U(P_1,\rho_1), \end{aligned}$$

which together with the positivity property of the semigroup and comparison principle yields that $S(x,t) \geq 0$ for $(x,t) \in U(P_1,\rho_1)$. This leads to a contradiction. Thus $S(x,t) \geq 0$ for $(x,t) \in \mathbb{R} \times [0,T)$.

Thirdly, we verify that $S(x,t) \leq 1$ for $(x,t) \in \mathbb{R} \times [0,T)$. Set $\tilde{S}(x,t) := 1 - S(x,t)$, then we get from (1.1) that $\tilde{S}(x,t)$ satisfies

$$\tilde{S}_t(x,t) = d_1 \tilde{S}_{xx}(x,t) - \mu \tilde{S}(x,t) - \frac{\beta S(x,t) I(x,t)}{1+\alpha I(x,t)} \geq d_1 \tilde{S}_{xx}(x,t) - \mu \tilde{S}(x,t), \ \ (x,t) \in \mathbb{R} \times [0,T).$$

By the similar arguments as (2.3), we deduce that $\tilde{S}(x,t) \geq 0$ for $(x,t) \in \mathbb{R} \times [0,T)$, which is equivalent to $S(x,t) \leq 1$ for $(x,t) \in \mathbb{R} \times [0,T)$.

Finally, we prove $I(x,t) \leq k$ for $(x,t) \in \mathbb{R} \times [0,T)$. To this aim, using $S(x,t) \leq 1$ and $I(x,t) \geq 0$ for $(x,t) \in \mathbb{R} \times [0,T)$, it follows from (1.1b) that

$$\begin{aligned} I_t(x,t) &= d_2 I_{xx}(x,t) + \frac{\beta S(x,t) I(x,t)}{1+\alpha I(x,t)} - (\mu+\sigma) I(x,t) \\ &\leq d_2 I_{xx}(x,t) + \frac{\beta I(x,t)}{1+\alpha I(x,t)} - (\mu+\sigma) I(x,t). \end{aligned} \tag{2.4}$$

Now consider the problem

$$\begin{cases} \hat{I}_t(x,t) = d_2 \hat{I}_{xx}(x,t) + \dfrac{\beta \hat{I}(x,t)}{1+\alpha \hat{I}(x,t)} - (\mu+\sigma) \hat{I}(x,t), \\ \hat{I}(x,0) = I(x,0) = I_0(x). \end{cases} \tag{2.5}$$

Then we have $I(x,t) \leq \hat{I}(x,t) \leq k$ for $(x,t) \in \mathbb{R} \times [0,T)$, since $k := \frac{\beta-\mu-\sigma}{\alpha(\mu+\sigma)}$ is an upper solution of (2.5). This completes the proof. ■

3 Proof of Theorem 1

First, we prove (1.4). One can infer from Lemma 2 and equation (1.1b) that

$$\begin{aligned} I_t(x,t) &= d_2 I_{xx}(x,t) + \frac{\beta S(x,t) I(x,t)}{1+\alpha I(x,t)} - (\mu+\sigma) I(x,t) \\ &\leq d_2 I_{xx}(x,t) + \frac{I(x,t)}{1+\alpha I(x,t)} \{\beta - (\mu+\sigma)[1+\alpha I(x,t)]\} \\ &\leq d_2 I_{xx}(x,t) + I(x,t)[\beta - \mu - \sigma - \alpha(\mu+\sigma) I(x,t)], \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \end{aligned}$$

which combined with comparison principle and the asymptotic spreading theory for reaction-diffusion Logistic equation [6] yields (1.4) holds.

Now we show (1.5). Denote $\tilde{S}(x,t) := 1 - S(x,t)$, then (1.1) can be written as

$$\begin{aligned}\tilde{S}_t(x,t) &= d_1\tilde{S}_{xx}(x,t) - \mu\tilde{S}(x,t) + \frac{\beta S(x,t)I(x,t)}{1+\alpha I(x,t)}\\ &\le d_1\tilde{S}_{xx}(x,t) - \mu\tilde{S}(x,t) + \beta I(x,t), \quad (x,t)\in\mathbb{R}\times\mathbb{R}^+,\end{aligned} \tag{3.1}$$

where we have used $S(x,t) \le 1$ and $I(x,t) \ge 0$ in $\mathbb{R}\times\mathbb{R}^+$. This together with comparison principle gives that $\tilde{S}(x,t) \le \acute{S}(x,t)$ for $(x,t)\in\mathbb{R}\times\mathbb{R}^+$, where $\acute{S}(x,t)$ satisfies

$$\begin{cases}\acute{S}_t(x,t) = d_1\acute{S}_{xx}(x,t) - \mu\acute{S}(x,t) + \beta I(x,t), \quad (x,t)\in\mathbb{R}\times\mathbb{R}^+,\\ \acute{S}(x,0) = 0.\end{cases}$$

Using the theory of linear operator semigroup leads to

$$\acute{S}(x,t) = T(t)\acute{S}(x,0) + \beta\int_0^t T(t-r)I(y,r)dr, \tag{3.2}$$

where

$$\big(T(t)\acute{S}\big)(x) = \frac{e^{-\mu t}}{\sqrt{4\pi d_1 t}}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4d_1 t}}\acute{S}(y)dy.$$

Next, for given $\varepsilon\in(0,c^*)$, we choose a constant $\delta\in(0,\frac{d_2(\beta-\mu-\sigma)}{\beta})$ small enough such that

$$\hat{c} := 2\sqrt{d_2(\beta-\mu-\sigma)-\beta\delta}\in(c^*-\varepsilon,c^*). \tag{3.3}$$

For this δ, we claim that there are a sufficiently large $\tau > 0$ and a constant $q > 0$ such that

$$\acute{S}(x,t) \le \delta + qI(x,t), \quad (x,t)\in\mathbb{R}\times[\tau,\infty). \tag{3.4}$$

To derive (3.4), we choose

$$\tau = \tau(\delta) = \frac{1}{\mu}\ln\left(\frac{2\beta k}{\mu\delta}\right).$$

Then by a simple calculation, we obtain

$$\beta\int_0^{t_0-\tau}\frac{e^{-\mu(t_0-r)}}{\sqrt{4\pi d_1(t_0-r)}}\int_{\mathbb{R}} e^{-\frac{(x_0-y)^2}{4d_1(t_0-r)}}I(y,r)dydr \le \frac{\delta}{2}, \quad \forall t_0\ge\tau. \tag{3.5}$$

If $\acute{S}(x_0,t_0)\le\delta$, then (3.4) holds obviously for any given point $(x_0,t_0)\in\mathbb{R}\times[\tau,\infty)$. If $\acute{S}(x_0,t_0)>\delta$ for any given point $(x_0,t_0)\in\mathbb{R}\times[\tau,\infty)$, then it follows from (3.2) that

$$\begin{aligned}\acute{S}(x_0,t_0) &= \beta\int_0^{t_0}\frac{e^{-\mu(t_0-r)}}{\sqrt{4\pi d_1(t_0-r)}}\int_{\mathbb{R}} e^{-\frac{(x_0-y)^2}{4d_1(t_0-r)}}I(y,r)dydr\\ &= \beta\int_0^{t_0-\tau}\frac{e^{-\mu(t_0-r)}}{\sqrt{4\pi d_1(t_0-r)}}\int_{\mathbb{R}} e^{-\frac{(x_0-y)^2}{4d_1(t_0-r)}}I(y,r)dydr\\ &\quad+\beta\int_{t_0-\tau}^{t_0}\frac{e^{-\mu(t_0-r)}}{\sqrt{4\pi d_1(t_0-r)}}\int_{\mathbb{R}} e^{-\frac{(x_0-y)^2}{4d_1(t_0-r)}}I(y,r)dydr,\end{aligned} \tag{3.6}$$

which together with (3.5) implies that

$$\beta\int_{t_0-\tau}^{t_0}\frac{e^{-\mu(t_0-r)}}{\sqrt{4\pi d_1(t_0-r)}}\int_{\mathbb{R}} e^{-\frac{(x_0-y)^2}{4d_1(t_0-r)}}I(y,r)dydr > \frac{\delta}{2}.$$

Hence, there exist a constant $\nu = \nu(\delta) > 0$ and a point $(y_0, r_0) \in \mathbb{R} \times [t_0 - \tau, t_0]$ such that $I(y_0, r_0) \geq \nu$. Since $I(x,t)$ is uniformly continuous in $\mathbb{R} \times \mathbb{R}^+$, there exists a constant $\rho_2 > 0$ such that

$$I(y, r_0) \geq \frac{\nu}{2}, \quad \forall y \in [y_0 - \rho_2, y_0 + \rho_2].$$

Considering the problem

$$\begin{cases} z_r(y,r) = d_2 z_{yy}(y,r) - (\mu+\sigma) z(y,r), \quad (y,r) \in \mathbb{R} \times (r_0, \infty), \\ z(y, r_0) = \underline{I}(y), \quad y \in \mathbb{R}, \end{cases}$$

where $\underline{I}(y)$ is a uniformly continuous nonnegative function defined in $\mathbb{R}$ such that $\underline{I}(y) \leq \frac{\nu}{2}$ in $\mathbb{R}$ and

$$\underline{I}(y) = \frac{\nu}{2}, \quad \forall y \in [y_0 - \frac{\rho_2}{2}, y_0 + \frac{\rho_2}{2}], \qquad \underline{I}(y) = 0, \quad \forall |y - y_0| \geq \rho_2,$$

we have that $z(y,r) > 0$ for $(y,r) \in \mathbb{R} \times (r_0, \infty)$. Choose $R > 0$ and $\eta \in (0, \tau)$ such that

$$x_0 \in [y_0 - R, y_0 + R] \quad \text{and} \quad t_0 \in [r_0 + \eta, r_0 + \tau].$$

Hence

$$N := \min\{z(y,r) \mid y \in [y_0 - R, y_0 + R], r \in [r_0 + \eta, r_0 + \tau]\} > 0.$$

Moreover, by comparison principle, we obtain that $I(y,r) \geq z(y,r)$ for $(y,r) \in \mathbb{R} \times [r_0, \infty)$, since I satisfies $I(y, r_0) \geq \underline{I}(y)$ for all $y \in \mathbb{R}$ and

$$I_r(y,r) \geq d_2 I_{yy}(y,r) - (\mu+\sigma) I(y,r), \quad (y,r) \in \mathbb{R} \times (r_0, \infty).$$

In particular, we have $I(x_0, t_0) \geq z(x_0, t_0) \geq N$ for $(x_0, t_0) \in [y_0 - R, y_0 + R] \times [r_0 + \eta, r_0 + \tau]$. Note from (3.5) and (3.6) that

$$\begin{aligned} \acute{S}(x_0, t_0) &\leq \frac{\delta}{2} + \beta k \int_{t_0 - \tau}^{t_0} e^{-\mu(t_0 - r)} dr \\ &= \frac{\delta}{2} + \beta k \int_0^{\tau} e^{-\mu r} dr \\ &\leq \delta + \frac{\beta k}{\mu} \\ &= \delta + \frac{\beta k}{\mu N} \cdot N \\ &\leq \delta + q I(x_0, t_0), \end{aligned}$$

where $q := \frac{\beta k}{\mu N}$. Consequently, we conclude that (3.4) holds.

Using $S(x,t) := 1 - \tilde{S}(x,t)$, we obtain from (1.1b) and (3.4) that

$$\begin{aligned} I_t(x,t) &= d_2 I_{xx}(x,t) + \frac{I(x,t)}{1 + \alpha I(x,t)} \{\beta[1 - \tilde{S}(x,t)] - \mu - \sigma - \alpha(\mu+\sigma) I(x,t)\} \\ &\geq d_2 I_{xx}(x,t) + \frac{I(x,t)}{1 + \alpha I(x,t)} \{\beta - \mu - \sigma - \beta[\delta + q I(x,t)] - \alpha(\mu+\sigma) I(x,t)\} \\ &= d_2 I_{xx}(x,t) + \frac{I(x,t)}{1 + \alpha I(x,t)} (\beta - \mu - \sigma - \beta\delta) - \frac{I(x,t)}{1 + \alpha I(x,t)} [\beta q + \alpha(\mu+\sigma)] I(x,t) \\ &\geq d_2 I_{xx}(x,t) + I(x,t)[1 - \alpha I(x,t)](\beta - \mu - \sigma - \beta\delta) - I(x,t)[\beta q + \alpha(\mu+\sigma)] I(x,t) \\ &= d_2 I_{xx}(x,t) + I(x,t)\{\beta - \mu - \sigma - \beta\delta - [\alpha(\beta - \mu - \sigma - \beta\delta) + [\beta q + \alpha(\mu+\sigma)]] I(x,t)\} \end{aligned}$$

due to $(1 - \alpha I) \leq 1/(1 + \alpha I) \leq 1$ and $\beta - \mu - \sigma - \beta\delta > 0$. Hence, the above inequality is equivalent to

$$I_t(x,t) \geq d_2 I_{xx}(x,t) + \varrho I(x,t) \left[\frac{\beta - \mu - \sigma - \beta\delta}{\varrho} - I(x,t) \right], \quad (x,t) \in \mathbb{R} \times [\tau, \infty), \tag{3.7}$$

where $\varrho := \alpha(\beta - \mu - \sigma - \beta\delta) + [\beta q + \alpha(\mu + \sigma)] > 0$.

Finally, consider the problem

$$\begin{cases} \check{I}_t(x,t) = d_2 I_{xx}(x,t) + \varrho I(x,t)\Big[\dfrac{\beta - \mu - \sigma - \beta\delta}{\varrho} - I(x,t)\Big], & (x,t) \in \mathbb{R} \times (\tau, \infty), \\ \check{I}(x,\tau) = I(x,\tau), \quad x \in \mathbb{R}. \end{cases}$$

Applying the upper and lower solution technique for reaction-diffusion equations, we conclude that

$$I(x,t) \geq \check{I}(x,t), \quad (x,t) \in \mathbb{R} \times [\tau, \infty). \tag{3.8}$$

Furthermore, combining with (3.8) and the asymptotic spreading theory for reaction-diffusion Logistic equation yields that

$$\liminf_{t\to\infty} \inf_{|x|<ct} I(x,t) \geq \liminf_{t\to\infty} \inf_{|x|<ct} \check{I}(x,t) = \lim_{t\to\infty} \inf_{|x|<ct} \check{I}(x,t) = \frac{\beta - \mu - \sigma - \beta\delta}{\varrho} > 0, \quad \forall c \in (0, \hat{c}).$$

By the arbitrariness of $\varepsilon \in (0, c^*)$, we obtain that

$$\liminf_{t\to\infty} \inf_{|x|<ct} I(x,t) > 0, \quad \forall c \in (0, c^*).$$

This finishes the proof of Theorem 1.

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