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Optimal Stochastic Control for Mean-variance Portfolio with Multiple Dependent Risky Assets

Huihui Wu*
School of Mathematical Sciences, Jiangsu University, Zhenjiang, Jiangsu 212013, P. R. China.
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Abstract: The stochastic control for mean-variance portfolio is a hot topic in asset management. Recent researches mainly focus on the single asset or independent assets, but ignore the dependence among multiple risky assets. In this paper, we aim to find a portfolio weight equilibrium strategy for the asset-liability management problem under mean-variance criterion. We model the multiple risky assets jointly with the multivariate geometric Brownian motion. The dependence of assets is captured by the covariance matrix of n assets. This paper provides a sufficient condition for the existence of the equilibrium strategy, which involves a system of forward-backward stochastic differential equations. By solving these equations, we obtain an equilibrium strategy, which is shown to be unique in the system.

Keywords: Dependent assets, Equilibrium strategy, Forward-backward stochastic differential equations, Mean-variance portfolio.

1 Introduction

Asset portfolio allocation is an external topic in the financial market. All the investors try their best to maximize the earnings and minimize the risks. Markowitz (1952) [1] proposed the classic mean-variance model, which characterizes returns and risks with expectation and variance respectively. Under the mean-variance framework, early researches focused on the pre-commitment strategies, i.e., the strategies that are only optimal at the initial time [2–4]. However, the dynamic MV portfolio selection is a time-inconsistent optimal control problem, which does not admit the Bellman's optimal principle due to the non-linear term of the expectation in the objective function of MV models [5]. The equilibrium strategy for the time-inconsistent control is still a challenging problem.

From the perspective of game theory, Strotz (1955) [6] first explored the equilibrium strategies for the time-inconsistent control problems. They proposed that the original control problem can be viewed as a game where a player sought a subgame perfect Nash equilibrium at every time point. This idea is straightforward when the time setting is discrete [7, 8]. Recently, the time-inconsistent control problems in continuous time have attracted more attention, especially for the portfolio management problems. Ekeland and Pirvu (2008) [9] proposed a detailed definition of equilibrium policies in the context of portfolio management, and considered the equilibrium control under hyperbolic discounting. Yong (2012) [10] discussed a general time-inconsistent optimal control problem for stochastic differential equations with deterministic coefficients. Björk et al. (2014) [11] further derived the equilibrium strategy of mean-variance portfolio with state-dependent risk aversion.

To characterize the price movements of financial assets dynamically, Hu et al. (2012) [12] introduced a system of forward-backward stochastic differential equations (FBSDEs) into the stochastic linear-quadratic (LQ) control problem. THen, Hu et al. (2017) [13] proposed a general sufficient and necessary condition for the equilibrium strategy, which is shown to be unique. Both the open-loop and closed-loop equilibrium strategies of LQ optimal control problems for mean-field stochastic differential equations were discussed in Yong (2017) [14]. Wei and Wang (2017) [15] extended the framework to asset-liability management and derived the open-loop equilibrium strategy under a model with random coefficients. More recently, Zhang et al. (2020) [16] modeled the risky and risk-free assets jointly and discussed the

*Corresponding author. *E-mail address*: 2211902048@stmail.ujs.edu.cn (H. Wu).

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optimal debt ratio in the equilibrium strategy. Yan and Wong (2020) [17], Kang et al. (2021) [18] investigated open-loop equilibrium strategies for mean-variance problems under stochastic volatility (SV) models and so on.

In this paper, we discuss the equilibrium strategy for mean-variance portfolio with multiple dependent risky assets. A multivariate geometric Brownian motion is used to model the price movements of multiple assets jointly. We measure the dependence among different assets with the covariance matrix. This paper provides a sufficient condition to attain the equilibrium strategy in the system of FBSDEs. By solving these equations, we obtain an equilibrium strategy, which is shown to be unique in the system.

The paper is organized as follows: Section 2 introduces the main assumptions and proposes the framework of forward-backward stochastic differential equations. Section 3 provides a sufficient condition for the existence of the equilibrium strategy. We obtain the equilibrium strategy by solving the system of FBSDEs and show the solution is unique. Section 4 concludes the paper briefly and gives some thoughts on the future work.

2 Problem Formulation

Assume that the financial market consists of n dependent risky assets $\boldsymbol{P}(t) = (P_1(t), \cdots, P_n(t))'$, where $P_i(t)$ denotes the price of the i-th asset at time t. The dynamic system of $\boldsymbol{P}(t)$ can be characterized by using a multivariate geometric Brownian motion:

$$d\boldsymbol{P}(t) = \boldsymbol{P}(t) \circ [\boldsymbol{\mu} dt + \boldsymbol{\Sigma} d\boldsymbol{B}(t)], \tag{1}$$

where $\boldsymbol{\mu} = (\mu_1, \mu_2, \cdots, \mu_n)'$ denotes the expected return, $\boldsymbol{B}(t) = (B_1(t), B_2(t), \cdots, B_n(t))'$ denotes the Brownian motion for each asset, and $\circ$ denotes the Hadamard product. Here $\boldsymbol{\Sigma}$ is the covariance matrix, which characterizes the dependence among difference assets:

$$\boldsymbol{\Sigma} = (\sigma_{ij})_{1 \le i,j \le n} = \begin{pmatrix} \sigma_{11} & \cdots & \sigma_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \cdots & \sigma_{nn} \end{pmatrix}.$$

Let $X(t) = \boldsymbol{w}(t)'\boldsymbol{P}(t)$ be the total assets of the portfolio with weights $\boldsymbol{w}(t) = (w_1(t), \cdots, w_n(t))'$. Then, we have the following lemma:

Lemma 1 The dynamic of total assets $X(t)$ can be characterized as

$$dX(t) = \{X(t) + \boldsymbol{w}(t)'[\boldsymbol{P}(t) \circ (\boldsymbol{\mu} - \boldsymbol{1})]\}dt + \boldsymbol{w}(t)'[\boldsymbol{P}(t) \circ \boldsymbol{\Sigma} d\boldsymbol{B}(t)], \tag{2}$$

where $\boldsymbol{1} = (1, 1, \cdots, 1)'$ is a n-dimensional vector.

For the proofs of lemmas and theorems, see Appendix for details.

For any initial state $X(t)$, $t \in [0, T)$, the investors are assumed to follow the mean-variance criteria over terminal wealth $X(T)$. It means that they should take the optimal equilibrium strategy $\bar{\boldsymbol{w}}(\cdot)$ to balance the variance and expectation dynamically, i.e.,

$$\begin{aligned} \bar{\boldsymbol{w}}(\cdot) &= \arg\min_{\boldsymbol{w}(\cdot)} J(\boldsymbol{w}(\cdot); X(t)) \\ &= \arg\min_{\boldsymbol{w}(\cdot)} \frac{1}{2}\text{Var}_t[X(T)] - \lambda E_t[X(T)], \end{aligned} \tag{3}$$

where $\boldsymbol{w}(\cdot)$ is a deterministic process on the interval $[t, T]$. Here $\lambda > 0$ is a constant and $E_t[\cdot] = E[\cdot|t]$.

3 Properties of Equilibrium Strategy

3.1 A sufficient condition

In this section, we derive a sufficient condition to ensure the existence of the equilibrium strategy. Assume that $\bar{\boldsymbol{w}}(\cdot)$ is the given strategy and $\bar{X}(\cdot)$ is the corresponding process of total asset. For any $t \in [0, T)$ and a tiny value $\varepsilon > 0$, we consider a perturbation strategy as follows,

$$\boldsymbol{w}^{t,\varepsilon,\nu}(s) = \bar{\boldsymbol{w}}(s) + \boldsymbol{\nu}(s) I_{[t,t+\varepsilon]}(s), \tag{4}$$

where $I_{[t,t+\varepsilon]}(\cdot)$ is the indicator function over $[t, t+\varepsilon]$, $\boldsymbol{\nu}(s) = (\nu_1(s), \cdots, \nu_n(s))'$ is a deterministic and bounded process on the interval $[0, T]$. It means that the perturbation strategy differs from the given strategy only on the interval $[t, t+\varepsilon]$. Let $X^{t,\varepsilon,\nu}(\cdot)$ denote the process of the total asset under the perturbation strategy $\boldsymbol{w}^{t,\varepsilon,\nu}(\cdot)$. According to the definition in [16], we present a detailed definition of the equilibrium strategy as follows:

Definition 1 For any $t \in [0, T)$ and a tiny value $\varepsilon > 0$, the given strategy $\bar{\boldsymbol{w}}(\cdot)$ is called the equilibrium strategy if

$$\liminf_{\varepsilon \to 0^+} \frac{J(\boldsymbol{w}^{t,\varepsilon,\nu}(\cdot); X^{t,\varepsilon,\nu}(t)) - J(\bar{\boldsymbol{w}}(\cdot); \bar{X}(t))}{\varepsilon} \geq 0. \tag{5}$$

Definition 1 means that any perturbation strategy will deteriorate the objective function. For further analysis, we introduce the high-dimensional Itô's Lemma.

Lemma 2 (High-dimensional Itô's Lemma): For any two stochastic differential equations:

$$\begin{aligned} dX(t) &= \mu_1 dt + \sigma_1 dB_1(t) + \sigma_2 dB_2(t) + \cdots + \sigma_n dB_n(t), \\ dY(t) &= \mu_2 dt + \gamma_1 dB_1(t) + \gamma_2 dB_2(t) + \cdots + \gamma_2 dB_n(t), \end{aligned}$$

where $B_1(t), \cdots, B_n(t)$ are independent Brownian motions, then

$$d[X(t)Y(t)] = X(t)dY(t) + Y(t)dX(t) + \sum_{i=1}^{n} \sigma_i \gamma_i dt.$$

Then, we show that the solution of the following FBSDEs satisfies the Equation 5 in Theorem 1.

Theorem 1 The strategy $\bar{\boldsymbol{w}}(\cdot)$ is the equilibrium strategy if for any $t \in [0, T)$, there exist stochastic processes $\{Y(\cdot; t), \boldsymbol{Z}(\cdot; t)\}$ on the interval $[t, T]$ that satisfy the following FBSDEs:

$$\begin{cases} d\bar{X}(s) = \{\bar{X}(s) + \bar{\boldsymbol{w}}'(s)[\boldsymbol{P}(s) \circ (\boldsymbol{\mu} - \mathbf{1})]\}ds + \bar{\boldsymbol{w}}'(s)[\boldsymbol{P}(s) \circ \boldsymbol{\Sigma} d\boldsymbol{B}(s)], \quad s \in [0, T], \\ \bar{X}(0) = x_0, \\ dY(s; t) = -Y(s; t)ds + \boldsymbol{Z}(s; t)' d\boldsymbol{B}(s), \quad s \in [t, T], \\ Y(T; t) = \bar{X}(T) - E_t[\bar{X}(T)] - \lambda, \end{cases} \tag{6}$$

where $\boldsymbol{Z}(s; t) = (Z_1(s; t), Z_2(s; t), \cdots, Z_n(s; t))'$, and

$$\liminf_{\varepsilon \to 0^+} \frac{1}{\varepsilon} E_t \left[\int_t^{t+\varepsilon} \boldsymbol{\Lambda}(s; t) ds \right] = 0, \tag{7}$$

where $\boldsymbol{\Lambda}(s; t) = Y(s; t)[\boldsymbol{P}(s) \circ (\boldsymbol{\mu} - \mathbf{1})] + \boldsymbol{P}(s) \circ \boldsymbol{\Sigma} \boldsymbol{Z}(s; t)$.

3.2 Equilibrium strategy

Theorem 1 provides a sufficient condition for the existence of the equilibrium strategy. This section presents the closed form of the equilibrium strategy $\bar{\boldsymbol{w}}(\cdot)$. From Theorem 1, it is sufficient to find the stochastic processes $\{Y(\cdot; t), \boldsymbol{Z}(\cdot; t)\}$ satisfying Equation 6 and 7.

Theorem 2 The following stochastic processes $\{Y(\cdot; t), \boldsymbol{Z}(\cdot; t)\}$ satisfy Equation 6 and 7 for any $t \in [0, T)$:

$$\begin{cases} Y(s; t) = Q(s)\{Q(s)\bar{X}(s) - E_t[Q(s)\bar{X}(s)] - \lambda\}, \\ \boldsymbol{Z}(s; t)' = Q^2(s)\bar{\boldsymbol{w}}(s)'(\boldsymbol{P}(s)\mathbf{1}' \circ \boldsymbol{\Sigma}), \end{cases} \tag{8}$$

where $Q(s) = e^{T-s}$, $s \in [t, T]$, and

$$\bar{\boldsymbol{w}}(t) = \frac{\lambda}{Q(t)} (\mathbf{1}\boldsymbol{P}(t)' \circ \boldsymbol{\Sigma})^{-1} \boldsymbol{\Sigma}^{-1} (\boldsymbol{\mu} - \mathbf{1}) \tag{9}$$

is the equilibrium strategy.

Under the equilibrium strategy, we calculate the objective function.

Theorem 3 For any $t \in [0, T)$, the objective function under the equilibrium strategy $\bar{\boldsymbol{w}}(\cdot)$ is

$$J(\bar{\boldsymbol{w}}(\cdot); \bar{X}(t)) = \frac{1}{2} \int_t^T \|Q(s)\bar{\boldsymbol{w}}(s)'[\boldsymbol{P}(s)\mathbf{1}' \circ \boldsymbol{\Sigma}]\|_2^2 ds - \lambda\{Q(t)\bar{X}(t) + \int_t^T Q(s)\{\bar{\boldsymbol{w}}(s)'[\boldsymbol{P}(s) \circ (\boldsymbol{\mu} - \mathbf{1})]\}ds\}. \tag{10}$$

3.3 Uniqueness

This section shows the uniqueness of the equilibrium strategy. We use the approach similar to [16]. Suppose there is another equilibrium strategy $\tilde{\boldsymbol{w}}(\cdot)$, and $\tilde{X}(\cdot)$ is the corresponding total assets process. According to Equation 6, we can similarly construct following equations:

$$\begin{cases} d\tilde{X}(s) = \{\tilde{X}(s) + \tilde{\boldsymbol{w}}(s)'[\boldsymbol{P}(s) \circ (\boldsymbol{\mu} - \mathbf{1})]\}ds + \tilde{\boldsymbol{w}}(s)'[\boldsymbol{P}(s) \circ \boldsymbol{\Sigma} d\boldsymbol{B}(s)], \ \ s \in [0, T], \\ \tilde{X}(0) = x_0, \\ d\tilde{Y}(s;t) = -\tilde{Y}(s;t)ds + \tilde{\boldsymbol{Z}}(s;t)'d\boldsymbol{B}(s), \ \ s \in [t, T], \\ \tilde{Y}(T;t) = \tilde{X}(T) - E_t[\tilde{X}(T)] - \lambda. \end{cases} \tag{11}$$

Let

$$\begin{cases} \hat{Y}(s;t) = \tilde{Y}(s;t) - Q(s)\{Q(s)\tilde{X}(s) - E_t[Q(s)\tilde{X}(s)] - \lambda\}, \\ \hat{\boldsymbol{Z}}(s;t) = \tilde{\boldsymbol{Z}}(s;t) - Q^2(s)\tilde{\boldsymbol{w}}(s)'[\boldsymbol{P}(s) \circ \boldsymbol{\Sigma}]. \end{cases} \tag{12}$$

Then, we have

$$d\hat{Y}(s;t) = d\tilde{Y}(s;t) - d\{Q(s)\{Q(s)\tilde{X}(s) - E_t[Q(s)\tilde{X}(s)] - \lambda\}\},$$

where

$$\begin{aligned} d\tilde{Y}(s;t) &= -\tilde{Y}(s;t)ds + \tilde{\boldsymbol{Z}}(s;t)'d\boldsymbol{B}(s) \\ &= -\hat{Y}(s;t)ds - Q(s)\{Q(s)\tilde{X}(s) - E_t[Q(s)\tilde{X}(s)] - \lambda\}ds \\ &\quad + \hat{\boldsymbol{Z}}(s;t)'d\boldsymbol{B}(s) + Q^2(s)\tilde{\boldsymbol{w}}(s)'[\boldsymbol{P}(s) \circ \boldsymbol{\Sigma} d\boldsymbol{B}(s)], \end{aligned}$$

and

$$\begin{aligned} & d\{Q(s)\{Q(s)\tilde{X}(s) - E_t[Q(s)\tilde{X}(s)] - \lambda\}\} \\ &= \{Q(s)\tilde{X}(s) - E_t[Q(s)\tilde{X}(s)] - \lambda\}dQ(s) + Q(s)d\{Q(s)\tilde{X}(s) - E_t[Q(s)\tilde{X}(s)] - \lambda\} \\ &= \{Q(s)\tilde{X}(s) - E_t[Q(s)\tilde{X}(s)] - \lambda\}[-Q(s)]ds + Q^2(s)\tilde{\boldsymbol{w}}(s)'[\boldsymbol{P}(s) \circ \boldsymbol{\Sigma} d\boldsymbol{B}(s)]. \end{aligned}$$

For the terminal value $s = T$, since $Q(T) = 1$, we have

$$\begin{aligned} \hat{Y}(T;t) &= \tilde{Y}(T;t) - Q(T)\{Q(T)\tilde{X}(T) - E_t[Q(T)\tilde{X}(T)] - \lambda\} \\ &= \tilde{X}(T) - E_t[\tilde{X}(T)] - \lambda - \{\tilde{X}(T) - E_t[\tilde{X}(T)] - \lambda\} = 0. \end{aligned}$$

Hence, the stochastic process $\hat{Y}(s;t)$ satisfies the following equation:

$$\begin{cases} d\hat{Y}(s;t) = -\hat{Y}(s;t)ds + \hat{\boldsymbol{Z}}(s;t)'d\boldsymbol{B}(s), \quad s \in [t, T], \\ \hat{Y}(T;t) = 0. \end{cases} \tag{13}$$

Then, we propose a necessary condition for the equilibrium strategy $\tilde{\boldsymbol{w}}(\cdot)$.

Theorem 4 Let $\tilde{\boldsymbol{w}}(\cdot)$ be equilibrium strategy. The stochastic processes $\{\tilde{Y}(\cdot;t), \tilde{\boldsymbol{Z}}(\cdot;t)\}$ must satisfy the following equation at the initial time t:

$$\tilde{Y}(t;t)[\boldsymbol{P}(t) \circ (\boldsymbol{\mu} - \mathbf{1})] + \boldsymbol{P}(t) \circ \boldsymbol{\Sigma}\tilde{\boldsymbol{Z}}(t;t) = \mathbf{0}. \tag{14}$$

Thus, we can show that the uniqueness of the equilibrium strategy.

Theorem 5 The equilibrium strategy according to Definition 1 is unique.

4 Conclusions

This paper focuses on the mean-variance asset management problem. The objective is to find an equilibrium strategy of assets weights in the financial market consisting of mult-dimensional risky assets. We use the covariance matrix to represent the relationship between n assets. By solving a system of forward-backward stochastic differential equations, we derive a sufficient condition to attain the equilibrium strategy, which is shown to be unique in the system. In the future work, the liability term could be considered in our framework. Besides, we will consider to extend our model with other risk aversion.

Appendix: Proofs of Lemmas and Theorems

Lemma 1: According to Equation 1 and $X(t) = \boldsymbol{w}(t)'\boldsymbol{P}(t)$, we have

$$\begin{aligned} dX(t) &= \boldsymbol{w}(t)'d\boldsymbol{P}(t) = \boldsymbol{w}(t)'[\boldsymbol{P}(t)\circ\boldsymbol{\mu}]dt + \boldsymbol{w}(t)'[\boldsymbol{P}(t)\circ\boldsymbol{\Sigma}d\boldsymbol{B}(t)] \\ &= \{X(t) + \boldsymbol{w}(t)'[\boldsymbol{P}(t)\circ(\boldsymbol{\mu}-\mathbf{1})]\}dt + \boldsymbol{w}(t)'[\boldsymbol{P}(t)\circ\boldsymbol{\Sigma}d\boldsymbol{B}(t)]. \end{aligned}$$

Proven.

Lemma 2: Note that

$$d[X(t)Y(t)] = \frac{1}{4}\{d[X(t)+Y(t)]^2 - d[X(t)-Y(t)]^2\},$$

and

$$\begin{aligned} d[X(t)+Y(t)] &= (\mu_1+\mu_2)dt + (\sigma_1+\gamma_1)dB_1(t) + (\sigma_2+\gamma_2)dB_2(t) + \cdots + (\sigma_n+\gamma_n)dB_n(t), \\ d[X(t)-Y(t)] &= (\mu_1-\mu_2)dt + (\sigma_1-\gamma_1)dB_1(t) + (\sigma_2-\gamma_2)dB_2(t) + \cdots + (\sigma_n-\gamma_n)dB_n(t). \end{aligned}$$

According to Itô's Lemma, we have

$$\begin{aligned} d[X(t)+Y(t)]^2 &= 2[X(t)+Y(t)]d[X(t)+Y(t)] + \sum_{i=1}^{n}(\sigma_i+\gamma_i)^2dt, \\ d[X(t)-Y(t)]^2 &= 2[X(t)-Y(t)]d[X(t)-Y(t)] + \sum_{i=1}^{n}(\sigma_i-\gamma_i)^2dt. \end{aligned}$$

Thus,

$$d[X(t)Y(t)] = X(t)dY(t) + Y(t)dX(t) + \sum_{i=1}^{n}\sigma_i\gamma_i dt.$$

Proven.

Theorem 1: It is sufficient to show that the conditions in Equation 6 and 7 lead to Equation 5 in Definition 1. Define $X_1^{t,\varepsilon,\nu}(\cdot) = X^{t,\varepsilon,\nu}(\cdot) - \bar{X}(\cdot)$, i.e.,

$$X_1^{t,\varepsilon,\nu}(s) = \begin{cases} 0, & s\in[0,t), \\ X^{t,\varepsilon,\nu}(s) - \bar{X}(s), & s\in[t,T]. \end{cases}$$

The processes of total assets under the given strategy and the perturbation strategy differ from time t. Then, we have

$$\begin{aligned} & J(\boldsymbol{w}^{t,\varepsilon,\nu}(\cdot);X^{t,\varepsilon,\nu}(t)) - J(\bar{\boldsymbol{w}}(\cdot);\bar{X}(t)) \\ &= \frac{1}{2}\text{Var}_t[X^{t,\varepsilon,\nu}(T)] - \lambda E_t[X^{t,\varepsilon,\nu}(T)] - \frac{1}{2}\text{Var}_t[\bar{X}(T)] + \lambda E_t[\bar{X}(T)] \\ &= \frac{1}{2}\text{Var}_t[X_1^{t,\varepsilon,\nu}(T)+\bar{X}(T)] - \lambda E_t[X_1^{t,\varepsilon,\nu}(T)+\bar{X}(T)] - \frac{1}{2}\text{Var}_t[\bar{X}(T)] + \lambda E_t[\bar{X}(T)] \\ &= \frac{1}{2}\text{Var}_t[X_1^{t,\varepsilon,\nu}(T)] + \frac{1}{2}\text{Var}_t[\bar{X}(T)] + \text{Cov}(X_1^{t,\varepsilon,\nu}(T),\bar{X}(T)) \\ &\qquad - \lambda E_t[X_1^{t,\varepsilon,\nu}(T)] - \lambda E_t[\bar{X}(T)] - \frac{1}{2}\text{Var}_t[\bar{X}(T)] + \lambda E_t[\bar{X}(T)] \\ &= \frac{1}{2}\text{Var}_t[X_1^{t,\varepsilon,\nu}(T)] + \text{Cov}(X_1^{t,\varepsilon,\nu}(T),\bar{X}(T)) - \lambda E_t[X_1^{t,\varepsilon,\nu}(T)] \\ &= \frac{1}{2}\text{Var}_t[X_1^{t,\varepsilon,\nu}(T)] + E_t[X_1^{t,\varepsilon,\nu}(T)\bar{X}(T)] - E_t[X_1^{t,\varepsilon,\nu}(T)]E_t[\bar{X}(T)] - \lambda E_t[X_1^{t,\varepsilon,\nu}(T)] \\ &= \frac{1}{2}\text{Var}_t[X_1^{t,\varepsilon,\nu}(T)] + E_t[(\bar{X}(T) - E_t[\bar{X}(T)] - \lambda)X_1^{t,\varepsilon,\nu}(T)] \\ &= \frac{1}{2}\text{Var}_t[X_1^{t,\varepsilon,\nu}(T)] + E_t[Y(T;t)X_1^{t,\varepsilon,\nu}(T)]. \end{aligned}$$

Let $J_1 = \frac{1}{2}\mathrm{Var}_t[X_1^{t,\varepsilon,\nu}(T)]$. Since $J_1 \geq 0$, we have

$$J(\boldsymbol{w}^{t,\varepsilon,\nu}(\cdot); X^{t,\varepsilon,\nu}(t)) - J(\bar{\boldsymbol{w}}(\cdot); \bar{X}(t)) \geq E_t[Y(T;t)X_1^{t,\varepsilon,\nu}(T)].$$

Let $J_2 = E_t[Y(T;t)X_1^{t,\varepsilon,\nu}(T)]$. Similar with Equation 2, we have

$$\begin{aligned} d\bar{X}(s) &= \{\bar{X}(s) + \bar{\boldsymbol{w}}(s)'[\boldsymbol{P}(s) \circ (\boldsymbol{\mu} - \boldsymbol{1})]\}ds + \bar{\boldsymbol{w}}(s)'[\boldsymbol{P}(s) \circ \boldsymbol{\Sigma} d\boldsymbol{B}(s)], \quad s \in [0,T], \\ dX^{t,\varepsilon,\nu}(s) &= \{X^{t,\varepsilon,\nu}(s) + \boldsymbol{w}^{t,\varepsilon,\nu}(s)'[\boldsymbol{P}(s) \circ (\boldsymbol{\mu} - \boldsymbol{1})]\}ds + \boldsymbol{w}^{t,\varepsilon,\nu}(s)'[\boldsymbol{P}(s) \circ \boldsymbol{\Sigma} d\boldsymbol{B}(s)], \quad s \in [0,T]. \end{aligned}$$

Note that $X_1^{t,\varepsilon,\nu}(s) = X^{t,\varepsilon,\nu}(s) - \bar{X}(s)$, and $\boldsymbol{w}^{t,\varepsilon,\nu}(s) = \bar{\boldsymbol{w}}(s) + \boldsymbol{\nu}(s)I_{[t,t+\varepsilon]}(s)$. Then

$$dX_1^{t,\varepsilon,\nu}(s) = \begin{cases} 0, \quad s \in [0,t), \\ \{X_1^{t,\varepsilon,\nu}(s) + \boldsymbol{\nu}(s)'I_{[t,t+\varepsilon]}(s)[\boldsymbol{P}(s) \circ (\boldsymbol{\mu} - \boldsymbol{1})]\}ds + \boldsymbol{\nu}(s)'I_{[t,t+\varepsilon]}(s)[\boldsymbol{P}(s) \circ \boldsymbol{\Sigma} d\boldsymbol{B}(s)], \quad s \in [t,T]. \end{cases}$$

Thus, we only consider the stochastic differential equation on $[t,T]$, which can be written as

$$\begin{cases} dX_1^{t,\varepsilon,\nu}(s) = \{X_1^{t,\varepsilon,\nu}(s) + \boldsymbol{\nu}(s)'I_{[t,t+\varepsilon]}(s)[\boldsymbol{P}(s) \circ (\boldsymbol{\mu} - \boldsymbol{1})]\}ds + \boldsymbol{\nu}(s)'I_{[t,t+\varepsilon]}(s)[\boldsymbol{P}(s) \circ \boldsymbol{\Sigma} d\boldsymbol{B}(s)], \quad s \in [t,T], \\ X_1^{t,\varepsilon,\nu}(t) = 0. \end{cases}$$

According to Lemma 2, we have

$$\begin{aligned} d[Y(s;t)X_1^{t,\varepsilon,\nu}(s)] &= X_1^{t,\varepsilon,\nu}(s)dY(s;t) + Y(s;t)dX_1^{t,\varepsilon,\nu}(s) + \boldsymbol{\nu}(s)'I_{[t,t+\varepsilon]}(s)[\boldsymbol{P}(s) \circ \boldsymbol{\Sigma}\boldsymbol{Z}(s;t)]ds \\ &= X_1^{t,\varepsilon,\nu}(s)\{-Y(s;t)ds + \boldsymbol{Z}(s;t)'d\boldsymbol{B}(s)\} \\ &\quad + Y(s;t)\{[X_1^{t,\varepsilon,\nu}(s) + \boldsymbol{\nu}(s)'I_{[t,t+\varepsilon]}(s)[\boldsymbol{P}(s) \circ (\boldsymbol{\mu} - \boldsymbol{1})]]ds + \boldsymbol{\nu}(s)'I_{[t,t+\varepsilon]}(s)[\boldsymbol{P}(s) \circ \boldsymbol{\Sigma} d\boldsymbol{B}(s)]\} \\ &\quad + \boldsymbol{\nu}(s)'I_{[t,t+\varepsilon]}(s)[\boldsymbol{P}(s) \circ \boldsymbol{\Sigma}\boldsymbol{Z}(s;t)]ds \\ &= X_1^{t,\varepsilon,\nu}(s)\boldsymbol{Z}(s;t)'d\boldsymbol{B}(s) \\ &\quad + Y(s;t)\{\boldsymbol{\nu}(s)'I_{[t,t+\varepsilon]}(s)[\boldsymbol{P}(s) \circ (\boldsymbol{\mu} - \boldsymbol{1})]ds + \boldsymbol{\nu}(s)'I_{[t,t+\varepsilon]}(s)[\boldsymbol{P}(s) \circ \boldsymbol{\Sigma} d\boldsymbol{B}(s)]\} \\ &\quad + \boldsymbol{\nu}(s)'I_{[t,t+\varepsilon]}(s)[\boldsymbol{P}(s) \circ \boldsymbol{\Sigma}\boldsymbol{Z}(s;t)]ds \end{aligned}$$

for $s \in [t,T]$.

Since $X_1^{t,\varepsilon,\nu}(t) = 0$, then

$$Y(T;t)X_1^{t,\varepsilon,\nu}(T) = Y(t;t)X_1^{t,\varepsilon,\nu}(t) + \int_t^T d[Y(s;t)X_1^{t,\varepsilon,\nu}(s)]ds = \int_t^T d[Y(s;t)X_1^{t,\varepsilon,\nu}(s)]ds.$$

Hence

$$\begin{aligned} J_2 &= E_t[Y(T;t)X_1^{t,\varepsilon,\nu}(T)] \\ &= E_t[\int_t^T \{Y(s;t)\boldsymbol{\nu}(s)'I_{[t,t+\varepsilon]}(s)[\boldsymbol{P}(s) \circ (\boldsymbol{\mu} - \boldsymbol{1})] + \boldsymbol{\nu}(s)'I_{[t,t+\varepsilon]}(s)[\boldsymbol{P}(s) \circ \boldsymbol{\Sigma}\boldsymbol{Z}(s;t)]\}ds] \\ &= E_t[\int_t^{t+\varepsilon} \boldsymbol{\nu}(s)'\boldsymbol{\Lambda}(s;t)ds]. \end{aligned}$$

According to Equation 7, we have,

$$\liminf_{\varepsilon \to 0^+} \frac{J_2}{\varepsilon} = \liminf_{\varepsilon \to 0^+} \frac{1}{\varepsilon} E_t[\int_t^{t+\varepsilon} \boldsymbol{\nu}(s)'\boldsymbol{\Lambda}(s;t)ds] = 0.$$

Thus

$$\liminf_{\varepsilon \to 0^+} \frac{J(\boldsymbol{w}^{t,\varepsilon,\nu}(\cdot); X^{t,\varepsilon,\nu}(t)) - J(\bar{\boldsymbol{w}}(\cdot); \bar{X}(t))}{\varepsilon} = \liminf_{\varepsilon \to 0^+} \frac{J_1 + J_2}{\varepsilon} \geq \liminf_{\varepsilon \to 0^+} \frac{J_2}{\varepsilon} = 0.$$

Proven.

Theorem 2: Suppose that $Y(s;t)$ has the following form for any $t \in [0,T)$ [19]:

$$Y(s;t) = Q(s)\{Q(s)\bar{X}(s) - E_t[Q(s)\bar{X}(s)] - \lambda\}, \;\; s \in [t,T],$$

where $Q(s) = e^{T-s}$. For the terminal value, we have $Q(T) = 1$. Thus,

$$Y(T;t) = \bar{X}(T) - E_t[\bar{X}(T)] - \lambda,$$

which is corresponding to that in Equation 6. According to Lemma 2, we have

$$\begin{aligned} d[Q(s)\bar{X}(s)] &= Q(s)d\bar{X}(s) + \bar{X}(s)dQ(s) \\ &= Q(s)\{[\bar{X}(s) + \bar{\boldsymbol{w}}(s)'[\boldsymbol{P}(s) \circ (\boldsymbol{\mu} - \mathbf{1})]]ds + \bar{\boldsymbol{w}}(s)'[\boldsymbol{P}(s) \circ \boldsymbol{\Sigma} d\boldsymbol{B}(s)]\} - \bar{X}(s)Q(s)ds \\ &= Q(s)\{\bar{\boldsymbol{w}}(s)'[\boldsymbol{P}(s) \circ (\boldsymbol{\mu} - \mathbf{1})]ds + \bar{\boldsymbol{w}}(s)'[\boldsymbol{P}(s) \circ \boldsymbol{\Sigma} d\boldsymbol{B}(s)]\}. \end{aligned}$$

Then

$$dE_t[Q(s)\bar{X}(s)] = E_t[d[Q(s)\bar{X}(s)]] = Q(s)\bar{\boldsymbol{w}}(t)'[\boldsymbol{P}(s) \circ (\boldsymbol{\mu} - \mathbf{1})]ds.$$

Let $H(s;t) = Q(s)\bar{X}(s) - E_t[Q(s)\bar{X}(s)] - \lambda$. Thus, we have $Y(s;t) = Q(s)H(s;t)$. So, according to the Itô's lemma, we have

$$\begin{aligned} dY(s;t) &= d[Q(s)H(s;t)] = H(s;t)dQ(s) + Q(s)dH(s;t) \\ &= -H(s;t)Q(s)ds + Q^2(s)\bar{\boldsymbol{w}}(s)'[\boldsymbol{P}(s) \circ \boldsymbol{\Sigma} d\boldsymbol{B}(s)]. \end{aligned}$$

Compared with Equation 6, we have

$$\boldsymbol{Z}(s;t)' = Q^2(s)\bar{\boldsymbol{w}}(s)'(\boldsymbol{P}(s)\mathbf{1}' \circ \boldsymbol{\Sigma}).$$

It is difficult to construct an equilibrium strategy through Equation 7, which is involved in a limit. Therefore, here we consider a stronger version that

$$\boldsymbol{\Lambda}(t;t) = Y(t;t)[\boldsymbol{P}(t) \circ (\boldsymbol{\mu} - \mathbf{1})] + \boldsymbol{P}(t) \circ \boldsymbol{\Sigma}\boldsymbol{Z}(t;t) = \mathbf{0}.$$

Note that $Y(t;t) = -\lambda Q(t)$, $\boldsymbol{Z}(t;t)' = Q^2(t)\bar{\boldsymbol{w}}(t)'(\boldsymbol{P}(t)\mathbf{1}' \circ \boldsymbol{\Sigma})$. So,

$$-\lambda Q(t)[\boldsymbol{P}(t) \circ (\boldsymbol{\mu} - \mathbf{1})] + \boldsymbol{P}(t) \circ \boldsymbol{\Sigma}[Q^2(t)\bar{\boldsymbol{w}}(t)'(\boldsymbol{P}(t)\mathbf{1}' \circ \boldsymbol{\Sigma})]' = \mathbf{0}.$$

So we can obtain

$$\bar{\boldsymbol{w}}(t) = \frac{\lambda}{Q(t)}(\mathbf{1}\boldsymbol{P}(t)' \circ \boldsymbol{\Sigma})^{-1}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{1}).$$

Proven.

Theorem 3: Note that

$$d[Q(s)\bar{X}(s)] = Q(s)\{\bar{\boldsymbol{w}}(s)'[\boldsymbol{P}(s) \circ (\boldsymbol{\mu} - \mathbf{1})]ds + \bar{\boldsymbol{w}}(s)'[\boldsymbol{P}(s) \circ \boldsymbol{\Sigma} d\boldsymbol{B}(s)]\}.$$

Since $Q(T) = 1$, we have

$$\begin{aligned} \bar{X}(T) &= Q(T)\bar{X}(T) \\ &= Q(t)\bar{X}(t) + \int_t^T Q(s)\bar{\boldsymbol{w}}(s)'[\boldsymbol{P}(s) \circ (\boldsymbol{\mu} - \mathbf{1})]ds + \int_t^T Q(s)\bar{\boldsymbol{w}}(s)'[\boldsymbol{P}(s) \circ \boldsymbol{\Sigma} d\boldsymbol{B}(s)]. \end{aligned}$$

Thus,

$$E_t[\bar{X}(T)] = Q(t)\bar{X}(t) + \int_t^T Q(s)\bar{\boldsymbol{w}}(s)'[\boldsymbol{P}(s) \circ (\boldsymbol{\mu} - \mathbf{1})]ds.$$

According to Itô's lemma and Itô's isometry [13], we have

$$
\begin{aligned}
&E_t[\bar{X}^2(T)] = E_t[(Q(T)\bar{X}(T))^2] \\
&= E_t[Q(t)\bar{X}(t) + \int_t^T Q(s)\bar{\boldsymbol{w}}(s)'[\boldsymbol{P}(s)\circ(\boldsymbol{\mu}-\boldsymbol{1})]ds + \int_t^T Q(s)\bar{\boldsymbol{w}}(s)'[\boldsymbol{P}(s)\circ\boldsymbol{\Sigma}d\boldsymbol{B}(s)]]^2 \\
&= E_t[Q(t)\bar{X}(t) + \int_t^T Q(s)\bar{\boldsymbol{w}}(s)'[\boldsymbol{P}(s)\circ(\boldsymbol{\mu}-\boldsymbol{1})]ds]^2 + E_t[\int_t^T Q(s)\bar{\boldsymbol{w}}(s)'[\boldsymbol{P}(s)\circ\boldsymbol{\Sigma}d\boldsymbol{B}(s)]]^2 \\
&\quad + 2E_t\left[\left(Q(t)\bar{X}(t) + \int_t^T Q(s)\bar{\boldsymbol{w}}(s)'[\boldsymbol{P}(s)\circ(\boldsymbol{\mu}-\boldsymbol{1})]ds\right)\left(\int_t^T Q(s)\bar{\boldsymbol{w}}(s)'[\boldsymbol{P}(s)\circ\boldsymbol{\Sigma}d\boldsymbol{B}(s)]\right)\right] \\
&= [Q(t)\bar{X}(t) + \int_t^T Q(s)\{\bar{\boldsymbol{w}}(s)'[\boldsymbol{P}(s)\circ(\boldsymbol{\mu}-\boldsymbol{1})]\}ds]^2 + E_t[\int_t^T Q(s)\bar{\boldsymbol{w}}(s)'[\boldsymbol{P}(s)\circ\boldsymbol{\Sigma}d\boldsymbol{B}(s)]]^2 \\
&= [Q(t)\bar{X}(t) + \int_t^T Q(s)\{\bar{\boldsymbol{w}}(s)'[\boldsymbol{P}(s)\circ(\boldsymbol{\mu}-\boldsymbol{1})]\}ds]^2 + \int_t^T \|Q(s)\bar{\boldsymbol{w}}(s)'[\boldsymbol{P}(s)\boldsymbol{1}'\circ\boldsymbol{\Sigma}]\|_2^2 ds.
\end{aligned}
$$

So,

$$
\begin{aligned}
J(\bar{\boldsymbol{w}}(\cdot);\bar{X}(t)) &= \frac{1}{2}\mathrm{Var}_t[\bar{X}(T)] - \lambda E_t[\bar{X}(T)] \\
&= \frac{1}{2}(E_t[\bar{X}(T)^2] - E_t^2[\bar{X}(T)]) - \lambda E_t[\bar{X}(T)] \\
&= \frac{1}{2}\int_t^T \|Q(s)\bar{\boldsymbol{w}}(s)'[\boldsymbol{P}(s)\boldsymbol{1}'\circ\boldsymbol{\Sigma}]\|_2^2 ds - \lambda\{Q(t)\bar{X}(t) + \int_t^T Q(s)\{\bar{\boldsymbol{w}}(s)'[\boldsymbol{P}(s)\circ(\boldsymbol{\mu}-\boldsymbol{1})]\}ds\}.
\end{aligned}
$$

Proven.

Theorem 4: For the equilibrium strategy $\tilde{\boldsymbol{w}}(\cdot)$, according to Definition 1, we have

$$
\begin{aligned}
&\liminf_{\varepsilon\to 0^+} \frac{J(\boldsymbol{w}^{t,\varepsilon,\nu}(\cdot);X^{t,\varepsilon,\nu}(t)) - J(\tilde{\boldsymbol{w}}(\cdot);\tilde{X}(t))}{\varepsilon} \\
=&\liminf_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\{\frac{1}{2}\mathrm{Var}_t[\tilde{X}_1^{t,\varepsilon,\nu}(T)] + E_t[(\tilde{X}(T) - E_t[\tilde{X}(T)] - \lambda)\tilde{X}_1^{t,\varepsilon,\nu}(T)]\} \geq 0,
\end{aligned}
$$

Note that

$$
\begin{aligned}
d[Q(s)\tilde{X}_1^{t,\varepsilon,\nu}(s)] &= \tilde{X}_1^{t,\varepsilon,\nu}(s)dQ(s) + Q(s)d\tilde{X}_1^{t,\varepsilon,\nu}(s) \\
&= Q(s)[\boldsymbol{\nu}(s)'I_{[t,t+\varepsilon]}(s)[\boldsymbol{P}(s)\circ(\boldsymbol{\mu}-\boldsymbol{1})]]ds + Q(s)[\boldsymbol{\nu}(s)'I_{[t,t+\varepsilon]}(s)(\boldsymbol{P}(s)\circ\boldsymbol{\Sigma}d\boldsymbol{B}(s))].
\end{aligned}
$$

Since $Q(T) = 1$ and Itô isometry, we have

$$
\begin{aligned}
&\mathrm{Var}_t[\tilde{X}_1^{t,\varepsilon,\nu}(T)] = \mathrm{Var}_t[Q(T)\tilde{X}_1^{t,\varepsilon,\nu}(T)] = E_t[Q(T)\tilde{X}_1^{t,\varepsilon,\nu}(T)]^2 - \{E_t[Q(T)\tilde{X}_1^{t,\varepsilon,\nu}(T)]\}^2 \\
&= E_t\{\int_t^T Q(s)[\boldsymbol{\nu}(s)'I_{[t,t+\varepsilon]}(s)[\boldsymbol{P}(s)\circ(\boldsymbol{\mu}-\boldsymbol{1})]]ds\}^2 + E_t\{\int_t^T Q(s)\boldsymbol{\nu}(s)'I_{[t,t+\varepsilon]}(s)(\boldsymbol{P}(s)\circ\boldsymbol{\Sigma}d\boldsymbol{B}(s))\}^2 \\
&\quad - E_t\{\int_t^T Q(s)[\boldsymbol{\nu}(s)'I_{[t,t+\varepsilon]}(s)[\boldsymbol{P}(s)\circ(\boldsymbol{\mu}-\boldsymbol{1})]]ds\}^2 \\
&= E_t\{\int_t^T Q(s)\boldsymbol{\nu}(s)'I_{[t,t+\varepsilon]}(s)(\boldsymbol{P}(s)\circ\boldsymbol{\Sigma}d\boldsymbol{B}(s))\}^2 \\
&= \int_t^{t+\varepsilon} \|Q(s)\boldsymbol{\nu}(s)'(\boldsymbol{P}(s)\circ\boldsymbol{\Sigma})\|_2^2 ds.
\end{aligned}
$$

Similarly with the proof in Theorem 1, we have

$$
\begin{aligned}
&E_t[(\tilde{X}(T) - E_t[\tilde{X}(T)] - \lambda)\tilde{X}_1^{t,\varepsilon,\nu}(T)] = E_t[\tilde{Y}(T;t)\tilde{X}_1^{t,\varepsilon,\nu}(T)] \\
&= E_t[\int_t^{t+\varepsilon} \boldsymbol{\nu}(s)'\{\tilde{Y}(s;t)[\boldsymbol{P}(s)\circ(\boldsymbol{\mu}-\boldsymbol{1})] + \boldsymbol{P}(s)\circ\boldsymbol{\Sigma}\tilde{\boldsymbol{Z}}(s;t)\}ds].
\end{aligned}
$$

Thus, we have

$$\begin{aligned}0 &\leq \liminf_{\varepsilon\to 0^+} \frac{J(\boldsymbol{w}^{t,\varepsilon,\nu}(\cdot); X^{t,\varepsilon,\nu}(t)) - J(\tilde{\boldsymbol{w}}(\cdot); \tilde{X}(t))}{\varepsilon}\\ &= \liminf_{\varepsilon\to 0^+} \frac{1}{\varepsilon} E_t[\int_t^{t+\varepsilon} \{\frac{1}{2}\|Q(s)\boldsymbol{\nu}(s)'(\boldsymbol{P}(s)\circ\boldsymbol{\Sigma})\|_2^2 + \boldsymbol{\nu}(s)'\{\tilde{Y}(s;t)[\boldsymbol{P}(s)\circ(\boldsymbol{\mu}-\mathbf{1})] + \boldsymbol{P}(s)\circ\boldsymbol{\Sigma}\tilde{\boldsymbol{Z}}(s;t)\}\}ds].\end{aligned}$$

It follows Lemma 3.4 in [13] that

$$0 \leq \frac{1}{2}\|Q(t)\boldsymbol{\nu}(t)'(\boldsymbol{P}(t)\circ\boldsymbol{\Sigma})\|_2^2 + \boldsymbol{\nu}(t)'\{\tilde{Y}(t;t)[\boldsymbol{P}(t)\circ(\boldsymbol{\mu}-\mathbf{1})] + \boldsymbol{P}(t)\circ\boldsymbol{\Sigma}\tilde{\boldsymbol{Z}}(t;t)\}.$$

The necessary condition for the above equation is

$$\boldsymbol{\nu}(t)'\{\tilde{Y}(t;t)[\boldsymbol{P}(t)\circ(\boldsymbol{\mu}-\mathbf{1})] + \boldsymbol{P}(t)\circ\boldsymbol{\Sigma}\tilde{\boldsymbol{Z}}(t;t)\} \geq 0.$$

This inequality holds for any perturbation strategy. Let $\boldsymbol{\nu}(t) = \kappa\mathbf{1}$, where κ is a constant. Thus, we have

$$\kappa\mathbf{1}'\{\tilde{Y}(t;t)[\boldsymbol{P}(t)\circ(\boldsymbol{\mu}-\mathbf{1})] + \boldsymbol{P}(t)\circ\boldsymbol{\Sigma}\tilde{\boldsymbol{Z}}(t;t)\} \geq 0.$$

Let $\kappa \to 0^+$. Then

$$\tilde{Y}(t;t)[\boldsymbol{P}(t)\circ(\boldsymbol{\mu}-\mathbf{1})] + \boldsymbol{P}(t)\circ\boldsymbol{\Sigma}\tilde{\boldsymbol{Z}}(t;t) \geq \mathbf{0}.$$

Similarly, let $\kappa \to 0^-$, we have

$$\tilde{Y}(t;t)[\boldsymbol{P}(t)\circ(\boldsymbol{\mu}-\mathbf{1})] + \boldsymbol{P}(t)\circ\boldsymbol{\Sigma}\tilde{\boldsymbol{Z}}(t;t) \leq \mathbf{0}.$$

Thus, we have

$$\tilde{Y}(t;t)[\boldsymbol{P}(t)\circ(\boldsymbol{\mu}-\mathbf{1})] + \boldsymbol{P}\circ\boldsymbol{\Sigma}\tilde{\boldsymbol{Z}}(t;t) = \mathbf{0}.$$

Proven.

Theorem 5: According to Equation 6, at the initial point $s = t$, we have

$$\begin{cases} \tilde{Y}(t;t) = \hat{Y}(t;t) - \lambda Q(t), \\ \tilde{\boldsymbol{Z}}(t;t) = \hat{\boldsymbol{Z}}(t;t) + [Q^2(t)\tilde{\boldsymbol{w}}(t)'(\boldsymbol{P}(t)\circ\boldsymbol{\Sigma})]'. \end{cases}$$

By substituting $\{\tilde{Y}(t;t), \tilde{\boldsymbol{Z}}(t;t)\}$ into Equation 8, we have

$$[\hat{Y}(t;t) - \lambda Q(t)][\boldsymbol{P}(t)\circ(\boldsymbol{\mu}-\mathbf{1})] + \boldsymbol{P}(t)\circ\boldsymbol{\Sigma}[\hat{\boldsymbol{Z}}(t;t) + (Q^2(t)\tilde{\boldsymbol{w}}(t)'(\boldsymbol{P}(t)\circ\boldsymbol{\Sigma}))'] = \mathbf{0}.$$

Thus,

$$\begin{aligned}\tilde{\boldsymbol{w}}(t) &= (Q^2(t)\boldsymbol{P}(t)\circ\boldsymbol{\Sigma}(\boldsymbol{P}(t)\circ\boldsymbol{\Sigma})')^{-1}[\lambda Q(t)[\boldsymbol{P}(t)\circ(\boldsymbol{\mu}-\mathbf{1})]]\\ &\quad + (Q^2(t)\boldsymbol{P}(t)\circ\boldsymbol{\Sigma}(\boldsymbol{P}(t)\circ\boldsymbol{\Sigma})')^{-1}[-\hat{Y}(t;t)\boldsymbol{P}(t)\circ(\boldsymbol{\mu}-\mathbf{1}) - \boldsymbol{P}(t)\circ\boldsymbol{\Sigma}\hat{\boldsymbol{Z}}(t;t)].\end{aligned}$$

Note that $\tilde{\boldsymbol{w}}(t) = \bar{\boldsymbol{w}}(t)$ if $\hat{Y}(t;t) = 0, \hat{\boldsymbol{Z}}(t;t) = \mathbf{0}$. According to Equation 11,

$$\begin{cases} d\hat{Y}(s;t) = -\hat{Y}(s;t)ds + \hat{\boldsymbol{Z}}(s;t)'d\boldsymbol{B}(s), \\ \hat{Y}(T;t) = 0. \end{cases}$$

Since the coefficient and the final value $\hat{Y}(T;t) = 0$ are not random variables, the equation has a solution if and only if the coefficients $\hat{\boldsymbol{Z}}(s;t) = \mathbf{0}$ by [20]. Then, it reduces to an ordinary differential equation

$$\begin{cases} d\hat{Y}(s;t) = -\hat{Y}(s;t)ds. \\ \hat{Y}(T;t) = 0. \end{cases}$$

Then, this equation has a unique solution that $\hat{Y}(s;t) = 0$ for all $s \in [t,T]$.

Thus, $\{\hat{Y}(t;t), \hat{\boldsymbol{Z}}(t;t)\} = \{0, \mathbf{0}\}$ is the unique solution to Equation 11. So $\tilde{\boldsymbol{w}}(t) = \bar{\boldsymbol{w}}(t)$ is a unique equilibrium strategy.

Proven.

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