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Effect of Nonlinear Intensity on Orbital Stability in the Generalized Camassa-Holm Equation

Lishuang Bian, Jiuli Yin *, Yanmin Wu , Mengjiao Tian, Wenxia Chen
Faculty of Science, Jiangsu University, Zhenjiang, Jiangsu, 212013, China.
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Abstract: In this paper, we study the existence and stability of solitary waves in generalized Camassa-Holm equation. Firstly, using the directly numerical integration method and the theory of dynamic we prove the existence of solitary wave. Then, we apply the theorem proving and numerical simulation to obtain the stability of solitary waves. We find the nonlinear intensity has important influence on the shape and stability of solitary waves. When the power of nonlinear term is odd, the equation admits positive solitary waves which are also proved to be orbitally stable when the wave velocity exceeds a critical value. When the power of nonlinear term is even, the equation admits positive and negative solitary waves which are proved to orbitally stable for any wave velocity.

Keywords: Camassa-Holm equation; Stability; Solitary waves; nonlinear intensity

1 Introduction

The generalized Camassa-Holm equation

$$u_t - u_{xxt} + au^n u_x = 2u_x u_{xx} + uu_{xxx} \tag{1}$$

is a famous shallow water equation[1]. Here n denotes nonlinear intensity and $a(> 0)$ is the parameter of nonlinear term. It was first derived by Fokas and Fuchssteiner for studying completely integrable generalization of the KdV equation with bi-Hamiltonian structure in [2]and later proposed physically by Camassa and Holm as a model for unidirectional propagation of shallow water waves over a flat bottom[3]. The generalized Camassa-Holm equation has three important models and it has been also widely studied by many researchers due to its many intriguing properties. Yu et al. explored the long-time solution behavior of the Camassa- Holm equation with a three-step solution scheme [4]. Yu studied the wave-breaking phenomenon for a generalized spatially periodic Camassa-Holm system [5]. Elboree deduced the soliton solutions for the Camassa-Holm equations by using semi-inverse variational principle [6]. Zheng and Yin investigated blow-up phenomena and global existence of strong solutions to a two-component Camassa-Holm systems with an arbitrary smooth function [7]. By argument of regularity and conservation, Ding constructed admitted traveling solutions of the second order and the third order Camassa Holm equations [8]. When $n = 1$ and $a = 3$, Eq. (1) becomes the famous Camassa-Holm equation [9–11], which has a nonsmooth solitary wave $ce^{-|x-ct|}$ (where c is the wave speed) which is called a peakon. This special solution is proved to be orbitally stable for any wave speed [12].

Eq.(1) is reduced to the modified Camassa-Holm equation as $n = 2$ [13–17], which admits negative and positive smooth solitary waves. This negative smooth solution is proved to be orbitally stable for any wave speed [14], Eq. (1) becomes the Camassa-Holm equation with quartic nonlinearity when $n = 3$ [18, 19], which admits a positive smooth solitary wave. This positive smooth solution is proved to be orbitally stable when the speed exceeds a critical value [19]. More property of generalized Camassa-Holm equation has been extensively studied [20, 21].

It is obviously that the nonlinear intensity n has a great influence on the shape and stability of the solitary wave as the above facts shown. However, previous researches mainly consider the nonlinear intensity when $n \leq 3$. It is more

*Corresponding author. *E-mail address*: yjl@ujs.edu.cn

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interesting to inquiry the more situations of n. Hence our first objective is to carry out a further study on the existence and stability of solitary waves as the nonlinear strength $n > 3$. Moreover, the propagation of solitary waves is not in a pure environment, and it is easy to be affected by external perturbation. It is nature that the second objective of this paper is to study evolution progress and control problem of solitary wave under the external perturbation.

The rest of the paper is organized as follows. In Section 2, existence of solitary wave and homoclinic orbits of Eq. (1) are given. In Section 3, stability of solitary waves is considered. Last is conclusions.

2 Existence of solitary wave

The main purpose of this section is to prove the existence of solitary wave by using directly numerical integration method and the theory of dynamic. The solitary wave of (1) has the form as

$$u(x,t) = \phi_c(x - ct) \tag{2}$$

and the profile ϕ_c propagates at speed $c > 0$ .Then Eq.(1) becoms

$$-c\phi_{cx} + c\phi_{cxxx} + a\phi_c^n\phi_{cx} = 2\phi_{cx}\phi_{cxx} + \phi_c\phi_{cxxx}. \tag{3}$$

Using the decay of ϕ_c at infinity, we obtain

$$c\phi_c - c\phi_{cxx} - \frac{a}{n+1}\phi_c^{n+1} + \phi_c\phi_{cxx} + \frac{\phi_{cx}^2}{2} = 0. \tag{4}$$

Multiplying by ϕ_{cx} and integrating (4), we have

$$\phi_{cx}^2 = \frac{\phi_c^2(c - \frac{a\phi_c^n}{2(n+2)})}{c - \phi_c}. \tag{5}$$

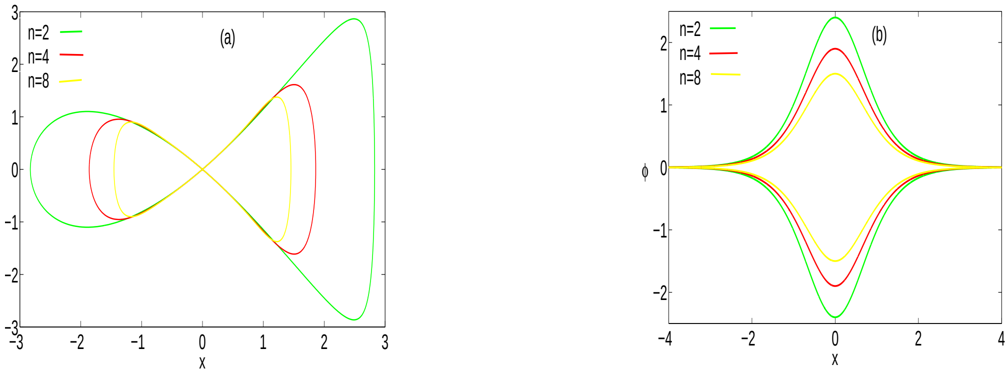


Figure 1: (a) phase portrait of system (7) with n is even ; (b)The negative and positive solitary wave system (7) with n is even .

From (5), we can deduce that solitary waves exist for $c > \sqrt[n-1]{\frac{2(n+2)}{a}}$.

Next, we prove the existence of solitary wave through Theorem1 and give its different existence form by numerical simulation.

Theorem 1 *For any $n\epsilon Z^+$, Eq. (1) admits homoclinic orbits associated with the solitary waves as $a > 0$.*

Proof. Letting $y = \phi^{'}$, Eq (4) becomes

$$\begin{cases} \phi' = y, \\ y' = \frac{-c\varphi + \frac{a}{n+1}\varphi^{n+1} - \frac{y^2}{2}}{\varphi - c}. \end{cases} \tag{6}$$

Taking the commutation $d\xi = (\phi - c)d\tau$, Eq. (6) is rewritten as

$$\begin{cases} \frac{d\phi}{d\tau} = (\phi - c)y, \\ \frac{dy}{d\tau} = -c\phi + \frac{a}{n+1}\phi^{n+1} - \frac{y^2}{2}. \end{cases} \tag{7}$$

Hence Eq.(7) has the Hamiltonian function as follows

$$H(\phi, y) = (c - \phi)y^2 + \frac{a}{2(n+2)}\phi^{n+2} - c\phi^2 \tag{8}$$

The following facts are observed:

Case I. When n is an even number, Eq.(7) has three equilibrium points: $A(0, 0), B(\sqrt[n]{\frac{(n+1)c}{a}}, 0)$.

Case II. When n is an odd Eq.(7) has two equilibrium points: $D(0, 0), E(\sqrt[n]{\frac{(n+1)c}{a}}, 0)$.

According to the theory of dynamics, we know the system (7) admits homoclinic orbits (see Figs.1(a) and 2(c)). Furthermore, system(7) also has the solitary waves (see Figs. 1(b) and 2(b)).

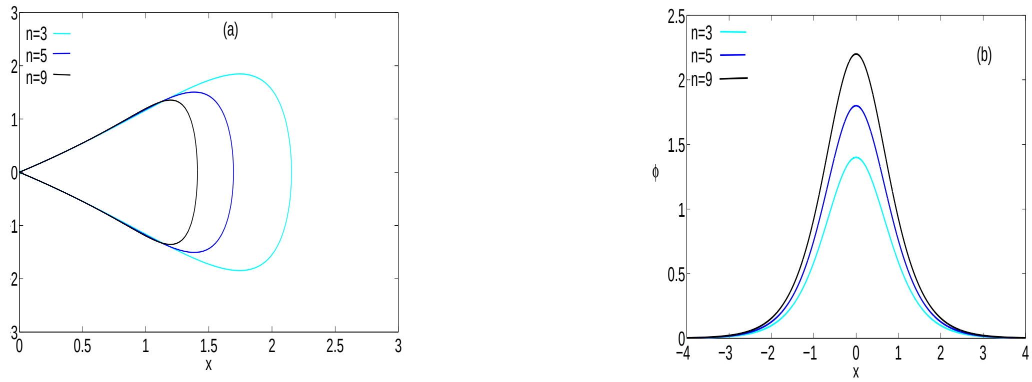


Figure 2: (a) phase portrait of system (7) when n is odd; (b)The positive solitary wave of system (7) when n is odd.

We find that Eq. (1) only admits positive solitary waves when n is an odd, while both positive and negative solitary waves appear when n is an even. It is noted that the height of the solitary wave is smaller and tends to $\sqrt[n]{\frac{(n+1)c}{a}}$ with the increasing n. ■

3 Stability of solitary waves

In this section, we study the stability of solitary waves when the nonlinear intensity n is an odd and an even number respectively. Eq. (1) has two invariants as follows

$$\begin{cases} E(u) = \frac{1}{2}\int_R (u^2 + u_x^2)dx, \\ F(u) = -\frac{1}{2}\int_R \frac{au^{n+2}}{2(n+2)} + uu_x^2 dx. \end{cases} \tag{9}$$

Eq. (4) can be also described as

$$F'(\phi_c) + cE'(\phi_c) = 0, \tag{10}$$

where where represents the Frechet derivatives of and the linearized operator is given as

$$H_c = F''(\phi_c) + cE''(\phi_c) = -\partial_x(c - \phi_c)\partial_x - a\phi_c^n + \phi_{cxx} + c \tag{11}$$

It is noted that the functions ϕ_c, ϕ_{cx} and ϕ_{cxx} all tend to zero when $|x| \to 0$. The Liouville transformation admits the following expression

$$\psi(y) = (2c - 2\phi_c(z))^{\frac{1}{n+1}}\nu(x), y = \int_0^x \frac{1}{\sqrt{c - \phi_c(z)}} dz.$$

Taking advantage of spectral equation $H_c\nu = \lambda\nu$ and the Liouville transformation, we have

$$F_c\psi(y) = (-\partial_y^n + P_c(y) + c)\psi(y) = \lambda\psi(y), \tag{12}$$

where

$$P_c(y) = -a\phi_c^n(x) + \frac{3}{4}\phi_c'' - \frac{(\phi_c'(x))^2}{8(c - \phi_c(x))}.$$

The stability depends on the convexity properties of the function $d(c) = F(\phi_c)+cE(\phi_c)$. Then we have the following Theorem.

Theorem 2 *According to the literature, the stability we know that the solitary wave ϕ_c is unstable if $d^{''} < 0$ and stable if $d^{''} > 0$.*

Proof. By differentiating $d(c) < 0$ once, we get

$$d'(c) = \langle F'(\phi_c) + cE'(\phi_c), \frac{\partial\phi_c}{\partial c}\rangle + E(\phi_c) = E(\phi_c). \tag{13}$$

Case 1. Positive solitary wave

In this case, the solitary wave $\phi_{(c)}\epsilon[0, \sqrt[n]{\frac{(n+1)c}{a}}]$ and is an even function. So we have

$$\begin{aligned} d''(c) &= \frac{d}{dc}\int_R \frac{1}{2}(\phi_c^2 + \phi_{cx}^2)dx \\ &= \frac{d}{dc}\int_0^{+\infty} \phi_c^2 \frac{2c - \phi_c - \frac{a\phi_c^n}{2(n+2)}}{c - \phi_c} dx \\ &= \frac{d}{dc}\int_0^{+\infty} \phi_c(-\phi_{cx})\sqrt{\frac{c - \phi_c}{c - \frac{a\phi_c^n}{2(n+2)}}} \cdot \frac{2c - \phi_c - \frac{a\phi_c^n}{2(n+2)}}{c - \phi_c} dx \\ &= \frac{d}{dc}\int_{\sqrt[n]{\frac{2(n+2)c}{a}}}^0 y\frac{-4(n+2)c + 2(n+2)y + ay^n}{\sqrt{2(n+2)}\sqrt{2(n+2)c - ay^n}\sqrt{c - y}} dy. \end{aligned} \tag{14}$$

Letting $y = \sqrt[n]{\frac{2(n+2)c}{a}}s$, we can get a simpler formas follows

$$\begin{aligned} d''(c) &= \frac{d}{dc}\int_0^1 \frac{(\frac{2(n+2)c}{a})^{\frac{2}{n}}s[4(n+2)c - 2(n+2)\sqrt[n]{\frac{2(n+2)c}{a}}s - 2(n+2)cs^n]}{\sqrt{2(n+2)}\sqrt{2(n+2)c - 2(n+2)cs^n}\sqrt{c - (\frac{2(n+2)c}{a})^{\frac{1}{n}}s}} ds \\ &= \frac{d}{dc}\int_0^1 \frac{(\frac{2(n+2)c}{a})^{\frac{2}{n}}s(2c - \sqrt[n]{\frac{2(n+2)c}{a}}s - cs^n)}{\sqrt{c - cs^n}\sqrt{c - (\frac{2(n+2)c}{a})^{\frac{1}{n}}s}} ds. \end{aligned} \tag{15}$$

Case 2. Negative solitary wave

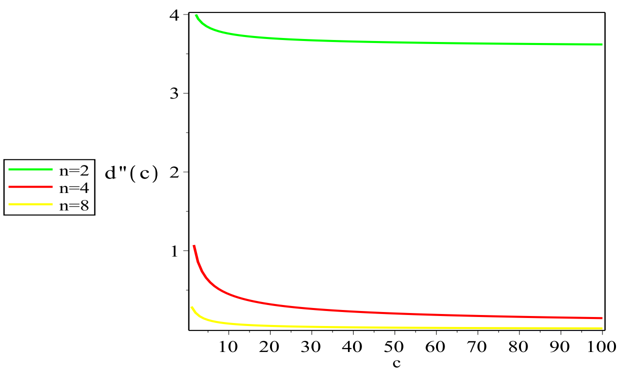
In this case, the solitary wave $\phi_{(c)}\epsilon[-\sqrt[n]{\frac{(n+1)c}{a}}, 0]$ and $\phi_{(c)}$ is an even function. So we have the following fact

$$
\begin{aligned}
d''(c) &= \frac{d}{dc}\int_R \frac{1}{2}(\phi_c^2+\phi_{cx}^2)dx \\
&= \frac{d}{dc}\int_0^{+\infty} \phi_c^2 \frac{2c-\phi_c-\frac{a\phi_c^n}{2(n+2)}}{c-\phi_c}dx \\
&= \frac{d}{dc}\int_0^{+\infty} \phi_c(-\phi_{cx})\frac{4(n+2)c-2(n+2)\phi_c-a\phi_c^n}{\sqrt{2(n+2)}\sqrt{2(n+2)c-a\phi_c^n}\sqrt{c-\phi_c}}dx \\
&= \frac{d}{dc}\int_{-\sqrt[n]{\frac{2(n+2)c}{a}}}^{0} y\frac{-4(n+2)c+2(n+2)y+ay^n}{\sqrt{2(n+2)}\sqrt{2(n+2)c-ay^n}\sqrt{c-y}}dy.
\end{aligned}
\tag{16}
$$

Letting $y=-\sqrt[n]{\frac{2(n+2)c}{a}}s$, we can also get a simpler form as follows

$$
d''(c)=\frac{d}{dc}\int_0^1 \frac{(\frac{2(n+2)c}{a})^{\frac{2}{n}}s(2c+\sqrt[n]{\frac{2(n+2)c}{a}}s-cs^n)}{\sqrt{c-cs^n}\sqrt{c+(\frac{2(n+2)c}{a})^{\frac{1}{n}}s}}ds. \tag{17}
$$

Next, we analysis the integrable value of $d''(c)$ by means of Matlab (see Figs. 3(a)and (b)).



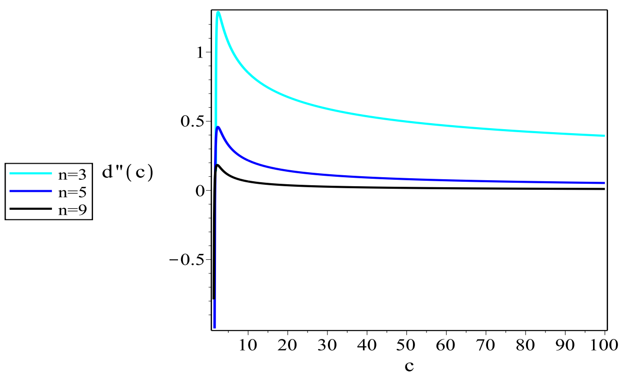


Figure 3: (a) The integration value of Eq (15); (b) The integration value of Eq (16).

Fig.3(a) shows that the positive solitary wave is unstable when the speed c is close to the critical value $\sqrt[n-1]{\frac{2(n+2)}{a}}$, which connects with the bifurcation condition contributes. And the positive solitary wave is stable when the speed is slightly greater than the critical value. Meanwhile, Fig.3(b) shows that the negative solitary wave is stable for any wave speed. ■

4 Conclusions

Based on the generalized Camassa-Holm equation, we study the existence and stability of solitary waves. We find the nonlinear intensity has important influence on the shape and stability of solitary waves. Different from previous studies, focusing on stability of solitary waves with the nonlinear intensity $n \le 3$, we study the stability of solitary waves with the nonlinear intensity $n > 3$ and in cases of odd and even numbers respectively. Following conclusions could be obtained.

(1)When the power of nonlinear term is odd, the equation admits positive solitary waves which are also proved to be orbitally stable when the wave velocity exceeds a critical value.

(2)When the power of nonlinear term is even, the equation admits positive and negative solitary waves which are proved to orbitally stable for any wave velocity.

(3)All solitary waves turn to chaos under the external periodic perturbation with arbitrary nonlinear intensity.

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