Estimation of Viscosity and Controller Strengthening in Boundary Stabilization of Korteweg-de Vries-Burgers’ Equation

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Abstract: In this paper, we propose a fortified boundary control law and an adaptation law for KdV-Burgers’ equation with unknown viscosity, where no a priori knowledge of a lower bound on viscosity is needed. Using the Lyapunov method, we prove that the closed loop system, including the parameter estimator as a dynamic component, is globally $H^1$ stable and well posed.

Keywords: KdV-Burgers’ equation; boundary control; stabilization

1 Introduction

Recently, extensive attention has been paid to the problem of controllability and stabilization for the Burgers’ and KdV-Burgers’ equation [1][2][3][4][5][6][7]. In this paper, we are concerned with the problem of the boundary control of the KdV-Burgers’ equation

$$
\begin{align*}
&u_t - \varepsilon u_{xx} + \delta u_{xxx} + uu_x = 0, \quad 0 < x < 1, t > 0, \\
&u_x(0, t) = u_x(1, t) = 0, \quad t > 0, \\
&u_{xx}(0, t) = \varphi_0, \quad t > 0, \\
&u_{xx}(1, t) = \varphi_1, \quad t > 0, \\
&u(x, 0) = u^0(x), \quad 0 < x < 1.
\end{align*}
$$

(1.1)

where $\varepsilon$ and $\delta$ are positive parameters. In this problem, $\varphi_0$ and $\varphi_1$ are control inputs and $u^0(x)$ is an initial state in an appropriate function space.

Our objective is to find feedback functions $\varphi_0$ and $\varphi_1$ such that the equilibrium $u(x) \equiv 0$ is globally stable and $u(x, t) \to 0$ as $t \to \infty$ for all $x \in (0, 1)$. Since $\varepsilon$ and $\delta$ take arbitrary positive values, this objective cannot be achieved by static feedback, hence, we need to design an adaptive controller which incorporates a parameter estimator as a dynamic component of the control law.

The paper is organized as follows: main result is given in section 2. In section 3, using the Lyapunov method, we prove that the closed-loop system, including the parameter estimator as a dynamic component, is globally $H^1$ stable and well posed.

Throughout this paper, we denote by $H^s(0, 1)$ the usual Sobolev space [8] for any $s \in \mathbb{R}$. Set

$$
H^1_0(0, 1) = \{ \varphi \in H^1(0, 1) : \varphi(0) = 0 \}
$$

(1.2)

$$
H^2_0(0, 1) = \{ \varphi \in H^2(0, 1) : \varphi_x(0) = \varphi_x(1) = 0 \}
$$

(1.3)

Let $X$ be a Banach space and $T > 0$. We denote by $C^n([0, T]; X)$ the space of $n$ times continuously differentiable functions defined on $[0, T]$ with values in $X$, and write $C([0, T]; X)$ for $C^0([0, T]; X)$. The norm on $L^2(0, 1)$ is denoted by $\| \cdot \|_2$.

In what follows, for simplicity, we omit the indication of the varying rang of $x$ and $t$ in equations and we understand that $x$ varies from 0 to 1 and $t$ from 0 to $\infty$.

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2 Main result

For notational convenience, we denote
\[ \omega_0 = u_{|x=0}, \quad \omega_1 = u_{|x=1} \] (2.1)
and
\[ \tilde{\eta}_0 = \eta_0 - \frac{1}{6\delta}, \quad \tilde{\eta}_1 = \eta_1 - \frac{1}{6\delta} \] (2.2)
where \( \eta_0 \) and \( \eta_1 \) shall be used as estimates of \( \frac{1}{6\delta} \).

We define the energy function by
\[ E = \int_0^1 u^2 \, dx \] (2.3)
and the Lyapunov function \( V \) as the energy function \( E \) augmented by the estimation error \( \tilde{\eta}_0^2 + \tilde{\eta}_1^2 \), that is
\[ V = E + \frac{\delta}{\gamma} (\tilde{\eta}_0^2 + \tilde{\eta}_1^2) \] (2.4)
where \( \gamma \) is a positive constant. Let us calculate the time derivative of \( V \). Using the equation (1.1) and integrating by parts, we obtain
\[ \dot{V} \leq -2\varepsilon \int_0^1 u_x^2 \, dx - 2\delta \omega_1 \varphi_1 + 2\delta \omega_0 \varphi_0 + \frac{1}{3} (\omega_0^2 + \omega_0^4 + \omega_1^2 + \omega_1^4) + \frac{2\delta}{\gamma} (\eta_0 - \frac{1}{6\delta}) \eta_0 + \frac{2\delta}{\gamma} (\eta_1 - \frac{1}{6\delta}) \eta_1 \]
\[ = -2\varepsilon \int_0^1 u_x^2 \, dx - 2\delta \omega_1 \varphi_1 - \eta_1 (\omega_1^3 + \omega_1) + 2\delta \omega_0 \varphi_0 + \eta_0 (\omega_0^3 + \omega_0) + 2\delta (\frac{1}{6\delta} - \eta_0) (\omega_0^4 + \omega_0^4 - \frac{\tilde{\eta}_0}{\gamma}) + 2\delta (\frac{1}{6\delta} - \eta_1) (\omega_1^4 + \omega_1^2 - \frac{\tilde{\eta}_1}{\gamma}) \] (2.5)
This leads us to select the adaptive feedback control
\[ \dot{\eta}_0 = \gamma (\omega_0^2 + \omega_0^4) \] (2.6)
\[ \dot{\eta}_1 = \gamma (\omega_1^2 + \omega_1^4) \] (2.7)
\[ \varphi_0 = -k(\omega_0 + \omega_0^3) - \eta_0 (\omega_0 + \omega_0^3) \] (2.8)
\[ \varphi_1 = k(\omega_1 + \omega_1^3) + \eta_1 (\omega_1 + \omega_1^3) \] (2.9)
where \( k \) is any positive constant, then we get
\[ \dot{V} \leq -2\varepsilon \int_0^1 u_x^2 \, dx - 2\delta k(\omega_0^2 + \omega_0^4 + \omega_1^2 + \omega_1^4) \] (2.10)
which implies the \( L^2 \) stability. Furthermore, the closed-loop system
\[
\begin{cases}
  u_t - \varepsilon u_{xx} + \delta u_{xxxx} + uu_x = 0, \\
  u_x(0, t) = u_x(1, t) = 0, \\
  u_{xx}(0, t) = -k(\omega_0 + \omega_0^3) - \eta_0 (\omega_0 + \omega_0^3), \\
  u_{xx}(1, t) = k(\omega_1 + \omega_1^3) + \eta_1 (\omega_1 + \omega_1^3), \\
  \eta_0 = \gamma (\omega_0^2 + \omega_0^4), \\
  \eta_1 = \gamma (\omega_1^2 + \omega_1^4), \\
  u(x, 0) = u_0(x), \quad \eta_0(0) = \tilde{\eta}_0^0, \quad \eta_1(0) = \tilde{\eta}_1^0
\end{cases}
\] (2.11)
satisfies the following theorem.
2. the equilibrium $u(x) \equiv 0, \tilde{\eta}_0 = \tilde{\eta}_1 = 0$ is globally $L^2$-stable, i.e.
\[
\|u(t)\|^2 + \frac{\delta}{\gamma} \tilde{\eta}_0(t)^2 + \frac{\delta}{\gamma} \tilde{\eta}_1(t)^2 \leq \|u^0\|^2 + \frac{\delta}{\gamma} \tilde{\eta}_0(0)^2 + \frac{\delta}{\gamma} \tilde{\eta}_1(0)^2
\]  
for all $t \geq 0$, and $u$ is regulated to zero in $L^2$ sense:
\[
\lim_{t \to \infty} \|u(t)\| = 0
\]

3. the equilibrium $u(x) \equiv 0, \tilde{\eta}_0 = \tilde{\eta}_1 = 0$ is globally $H^1$-stable, i.e.
\[
\|u(t)\|^2_{H^1} \leq C(\|u^0\|^2_{H^1} + \|u^0\|^2_{H^1} + \tilde{\eta}_0(0)^2 + \tilde{\eta}_0(0)^2 + \tilde{\eta}_1(0)^2)
\]  
\[
\times \exp(C(\|u^0\|^2 + \tilde{\eta}_0(0)^2 + \tilde{\eta}_1(0)^2))
\]
for all $t \geq 0$, where $C = C(k, \varepsilon, \delta, \gamma)$ is a positive constant, and $u$ is regulated to zero for all $x \in (0, 1)$:
\[
\lim_{t \to \infty} \max_{x \in [0, 1]} |u(x, t)| = \lim_{t \to \infty} \|u(t)\|_{H^1} = 0
\]

for any $s < 3$.

3 Proof

To prove the regulation result (2.13) independent of (2.15), we establish the following alternative to Barbalat’s lemma[9].

Lemma 2 Suppose that the function $f(t)$ defined on $[0, \infty)$ satisfies the following conditions:
(i) $f(t) \geq 0$ for all $t \in [0, \infty)$,
(ii) $f(t)$ is differentiable on $[0, \infty)$ and there exists a constant $M$ such that
\[f'(t) \leq M, \quad \forall t \geq 0,
\]
(iii) $\int_0^\infty f(t)dt < \infty$ Then we have
\[
\lim_{t \to \infty} f(t) = 0
\]

Proof. See [1]

We are now in the position to prove our main result.

Step1: Well-posedness. By (2.11), we first have
\[
\eta_0 = \eta_0^0 + \gamma \int_0^1 [\omega_0(s)^2 + \omega_0(s)^2]ds
\]
\[
\eta_1 = \eta_1^0 + \gamma \int_0^1 [\omega_1(s)^2 + \omega_1(s)^2]ds
\]

Substituting $\eta_0, \eta_1$ into the boundary condition of (2.11), it becomes the following problem
\[
\begin{aligned}
&u_t - \varepsilon u_{xx} + \delta u_{xxx} + uu_x = 0, \\
&u_x(0, t) = u_x(1, t) = 0, \\
&u_{xx}(0, t) = -g(\omega_0, \eta_0^0)(t), \\
&u_{xx}(1, t) = g(\omega_1, \eta_1^0)(t), \\
&u(x, 0) = u_0(x).
\end{aligned}
\]
where

$$g(\omega, r)(t) = k[\omega(t) + \omega(t)^9] + (r + \gamma \int_0^1 [\omega(s)^4 + \omega(s)^2]ds)[\omega(t)^3 + \omega(t)]$$

for any function $\omega = \omega(t)$ and $r \in R$. Thus, it suffices to prove that problem (3.3) has an unique solution. Let the theta function be defined by [10]

$$\theta(x, t) = \frac{1}{\sqrt{4\pi \delta t}} \sum_{n=\infty}^0 \exp(-\frac{(x+2n)^2}{4\delta t}) = \frac{1}{2} + \sum_{n=1}^\infty \cos(n\pi x)e^{-\delta n^2\pi^2 t}$$

It is well known that the problem (3.3) is equivalent to the following integral equation

$$u(x, t) = G(u(x, t)) = \int_0^1 [\theta(x - y, t) + \theta(x + y, t)]u^0(y)dy + \frac{1}{2} \int_0^1 \int_0^1 [\theta_2(x + y, t - \tau) - \theta_2(x - y, t - \tau)]u(y, \tau)^2dyd\tau$$

and

$$+ \int_0^1 \theta(x - 1, t - \tau)[2g(\omega_1, \eta^0_1)(\tau) - \omega_1(\tau)^2]d\tau$$

$$+ \int_0^1 \theta(x, t - \tau)[2g(\omega_0, \eta^0_0)(\tau) + \omega_0(\tau)^2]d\tau$$

We introduce the function space $C = C([0, T]; C[0, 1])$ with the norm

$$\| u \|_C = \max_{0 \leq x \leq 1, 0 \leq t \leq T} | u(x, t) |$$

In order to prove that (3.3) has an unique solution solution, it suffices to prove that the mapping $G$ defined by (3.4) has an unique fixed point in $C$. By using the standard properties of the theta function[10], we can prove that $G$ is contractive if $T$ is small enough. By Banach contraction fixed point theorem, $G$ has an unique fixed point $u \in C([0, T]; C[0, 1])$ if $T$ is small enough. Furthermore, in view of the following prior estimates which we are going to establish, we conclude that the local solution is an actually global classical solution.

**Step2:** Stability Estimate(2.12): By (2.10), we obtain

$$\| u(t) \|^2 + \frac{\delta}{\gamma} \tilde{\eta}_0(t)^2 + \frac{\delta}{\gamma} \tilde{\eta}_1(t)^2 \leq \| u^0 \|^2 + \frac{\delta}{\gamma} \tilde{\eta}_0(0)^2 + \frac{\delta}{\gamma} \tilde{\eta}_1(0)^2$$

and

$$2\epsilon \int_0^\infty \int_0^1 u^2 dxdt + \delta k \int_0^\infty [\omega_0(t)^2 + \omega_1(t)^10 + \omega_1(t)^2 + \omega_1(t)^10]dt$$

$$\leq \| u^0 \|^2 + \frac{\delta}{\gamma} \tilde{\eta}_0(0)^2 + \frac{\delta}{\gamma} \tilde{\eta}_1(0)^2$$

Hence (2.12) is established.

**Step3:** Regulation(2.13). To prove (2.13), it suffices to verify conditions (ii) and (iii) of Lemma 3.1. By (3.6), we obtain

$$\int_0^\infty \| u(t) \|^2 dt \leq C(\| u_0 \|^2 + \tilde{\eta}_0(0)^2 + \tilde{\eta}_1(0)^2) < \infty$$

Here and in the sequel, $C = C(\epsilon, \delta, \gamma, k)$ denotes various positive constants.

On the other hand, we have

$$\frac{d}{dt}(\| u(t) \|^2) = 2 \int_0^1 u(\varepsilon u_{xx} - \delta u_{xxx} - u_x) dx$$

$$\leq -2\delta [\omega^0_0 + \omega^1_0 + \omega^2_1 + \omega^1_0] - 2\delta [\eta_0(\omega^0_0 + \omega^0_0) + \eta_1(\omega^2_1 + \omega^1_1)] + \frac{1}{3}(\omega^0_0 + \omega^0_0 + \omega^2_1 + \omega^1_0)$$

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Since we have assumed that By (3.6) and Gronwall’s inequality, we obtain that for \(t\)

\[
\int_0^t \frac{\partial}{\partial t} u_x^2 dx \leq 2 \int_0^1 u_x^2 dx \leq 2 \int_0^1 (\varepsilon u_{xx} - \delta u_{xxx} - u_x) dx
\]

\[
= -2\varepsilon \int_0^1 u_x^2 dx + \delta [k(\omega_1 + \omega_1^0) + \eta_1 (\omega_1 + \omega_1^0)^2] - \delta [k(\omega_0 + \omega_0^0) + \eta_0 (\omega_0 + \omega_1^0)^2]
\]

\[
+ 2 \int_0^1 u_x u_{xx} dx
\]

(3.9)

Therefore we have

\[
\int_0^1 u_{xx} u_{xx} dx \leq \frac{1}{\varepsilon} (\omega_0^2 + \int_0^1 u_x^2 dx) \int_0^1 u_x^2 dx + \frac{1}{\varepsilon} \left( \int_0^1 u_x^2 dx \right)^2
\]

(3.11)

Integrating with respect to \(t\), we deduce from (2.11) that

\[
\| u_x(t) \|^2 \leq C(\| u^0 \|^2_{H^1} + \| u^0 \|^2_{H^2} + \int_0^t (\omega_0^2 + \int_0^1 u_x^2 dx) ds)
\]

(3.12)

By (3.6) and Gronwall’s inequality, we obtain that for \(t \geq 0\)

\[
\| u_x(t) \|^2 \leq C(\| u^0 \|^2_{H^1} + \| u^0 \|^2_{H^2} + \int_0^t (\omega_0^2 + \int_0^1 u_x^2 dx) ds)
\]

(3.13)

Thus (2.14) gets from (3.5) and (3.14).

**Step 4**: Stability Estimation (2.14). We first note that the solution \(u\) of (3.3) is infinitely differentiable in \((0, 1) \times (0, T)\) since it satisfies the integral equation (3.4) and the theta function \(\theta\) is infinitely differentiable. This guarantees that the calculations performed below are valid.

Using the equation (2.11), we obtain

\[
\frac{d}{dt} \int_0^1 u_x^2 dx = -2 \int_0^1 u_x (\varepsilon u_{xx} - \delta u_{xxx} - u_x) dx
\]

\[
= -2\varepsilon \int_0^1 u_x^2 dx + \delta [k(\omega_1 + \omega_1^0) + \eta_1 (\omega_1 + \omega_1^0)^2] - \delta [k(\omega_0 + \omega_0^0) + \eta_0 (\omega_0 + \omega_1^0)^2]
\]

\[
+ 2 \int_0^1 u_x u_{xx} dx
\]

(3.14)

Since

\[
\int_0^1 u_{xx} u_{xx} dx \leq \frac{1}{\varepsilon} (\omega_0^2 + \int_0^1 u_x^2 dx) \int_0^1 u_x^2 dx + \frac{1}{\varepsilon} \left( \int_0^1 u_x^2 dx \right)^2
\]

(3.15)

Since we have assumed that \(\eta_0^0, \eta_1^0 \geq 0\), we have \(\eta_0, \eta_1 \geq 0\) and then

\[
\frac{d}{dt} \int_0^1 u_x^2 dx \leq -2\varepsilon \int_0^1 u_x^2 dx - 2\delta k \omega_{1T} (1 + 9 \omega_1^0) - 2\delta \omega_{1T} \eta_1 (1 + 3 \omega_1^0) - 2\delta \omega_{1T} \eta_1 (1 + 9 \omega_1^0)
\]

\[
- 2\delta \omega_{0T} \eta_0 (1 + 3 \omega_1^0) - 2\delta \omega_{0T} \eta_0 (\omega_0 + \omega_1^0) + 2 \int_0^1 u_x (u_x u_{xx} + u_{xx} dx)
\]

(3.16)

\[
\int_0^1 u_x^2 dx \leq C(\| u^0 \|^2_{H^1} + \| u^0 \|^2_{H^2} + \int_0^t (\omega_0^2 + \int_0^1 u_x^2 dx) ds)
\]

(3.17)

**Step 5**: Regulation (2.15). In order to prove (2.15), we first estimate \(\| u \|_{H^3}\). Integrating by parts, we have

\[
\frac{d}{dt} \int_0^1 u_x^2 dx = 2 \int_0^1 u_x (\varepsilon u_{xx} - \delta u_{xxx} - u_x) dx
\]

\[
= -2\varepsilon \int_0^1 u_x^2 dx - 2\delta k \omega_{1T} (1 + 9 \omega_1^0) - 2\delta \omega_{1T} \eta_1 (1 + 3 \omega_1^0) - 2\delta \omega_{1T} \eta_1 (1 + 9 \omega_1^0)
\]

\[
- 2\delta \omega_{0T} \eta_0 (1 + 3 \omega_1^0) - 2\delta \omega_{0T} \eta_0 (\omega_0 + \omega_1^0) + 2 \int_0^1 u_x (u_x u_{xx} + u_{xx} dx)
\]

(3.18)

Since we have assumed that \(\eta_0^0, \eta_1^0 \geq 0\), we have \(\eta_0, \eta_1 \geq 0\) and then

\[
\frac{d}{dt} \int_0^1 u_x^2 dx \leq -2\varepsilon \int_0^1 u_x^2 dx - 2\delta k \omega_{1T} (1 + 9 \omega_1^0) - 2\delta \omega_{1T} \eta_1 (1 + 3 \omega_1^0) - 2\delta \omega_{0T} \eta_0 (1 + 9 \omega_1^0)
\]

(3.19)
Furthermore, since
\[ 2\eta_i \omega_i (\omega_i + \omega_i^3) \leq \frac{k}{\gamma_i} \omega_i^2 (\omega_i^2 + 1) + \frac{2\gamma_i}{k} (\omega_i^4 + \omega_i^2) \]
\[ = \frac{k}{2} \omega_i^2 \omega_i^2 (\omega_i^2 + 1)^2 + \frac{2\gamma_i}{k} (\omega_i^4 + \omega_i^2)^2 \]
\[ \leq k\omega_i^2 (\omega_i^6 + \omega_i^2) + \frac{4\gamma_i}{k} (\omega_i^8 + \omega_i^4) \]
\[ \leq 2k\omega_i^2 (1 + \omega_i^6) + \frac{8\gamma_i}{k} (\omega_i^2 + \omega_i^{10}), \quad i = 0, 1 \] (3.17)
and by (3.10) we obtain
\[ \int_0^1 |2 \int_0^1 u_t (u_t u_x + uu_{xt}) \|dx \| \leq 2(|\omega_0 | + || u_x ||) \| u_t \| || u_x || + 2(|\omega_0 | + || u_x ||) \| u_t \| || u_x || \]
\[ \leq \delta \omega_0 |^2 + \delta || u_x |^2 + \frac{2}{\delta} || u_t \| || u_x |^2 + \frac{1}{\delta} (|| \omega_0 |^2 + || u_x |^2) \| u_t \| ^2 + \delta || u_x ||^2 \] (3.18)

It follows from (3.16) that
\[ \frac{d}{dt} \int_0^1 u_{xt}^2 dx \leq \frac{8\gamma_i}{k} (\omega_0^2 + \omega_1^{10} + \omega_1^2 + \omega_1^6) + \frac{1}{\delta} (|| \omega_0 |^2 + || u_x |^2) \| u_t \| ^2 \] (3.19)
We then deduce using (3.6) that
\[ \| u_t (t) \|^2 \leq C (|| u_0 ||^2 + \eta_0 (0)^2 + \eta_1 (0)^2 + \| u_t (0) \|^2 + \| u_t(s) \|^2 + \frac{1}{\delta} \int_0^1 (|| \omega_0 (s) |^2 + || u_x (s) |^2) ) \| u_t \| ^2 ds \] (3.20)
which, by (3.6) and Gronwall’s inequality, implies that for \( t \geq 0 \)
\[ \| u_t (t) \|^2 \leq C (\eta_0 (0)^2 + \eta_1 (0)^2 + \| u_0 \|^2_{H^2} + \| u_0 \|^4_{H^3}) \times \exp (C (|| u_0 ||^2 + \eta_0 (0)^2 + \eta_1 (0)^2)) \] (3.21)
To obtain the estimation of \( \| u_{xx} (t) \|^2 \), we set \( y = u_{xx} \). Then by (2.11), we have
\[ y_x = \frac{1}{\delta} (\epsilon y - u_t - uu_x) \]
(3.22)

Solving the equation, we obtain
\[ u_{xx} = \frac{1}{\delta} \int_0^{1-\epsilon} [u_t (1 - \xi, t) + u (1 - \xi, t) u_x (1 - \xi, t)] e^{s (x-1)} ds \] (3.24)
It therefore follows that
\[ \| u_{xx} \| ^2 \leq C (|| u_t \| ^2 + || u_x \| ^2 + || u_x \| ^4 + || u_x \| ^6 + || u_x \| ^8) \] (3.25)
Further, since
\[ \| u_{xxx} \| ^2 \leq C (|| u_{xx} \| ^2 + || u_t \| ^2 + || u_x \| ^4) \] (3.26)
it follows from (3.5), (3.14)and (3.21) that
\[ \| u(t) \|^2_{H^3} \leq C \left( \sum_{i=0}^{1} (\eta_0 (0)^2 + \eta_1 (0)^4 + \eta_3 (0)^2 + \eta_3 (0)^4 + \eta_5 (0)^2 + \eta_5 (0)^4) + \| u_0 \|^2_{H^3} + \| u_0 \|^4_{H^3} \right) \]
Now, we argue by contradiction to prove (2.15). Assume, on the contrary, that there exists a positive $\sigma_0 > 0$ and a sequence $\{t_n\}$ with
\[
\lim_{n \to \infty} t_n = \infty
\]
such that
\[
\| u(t_n) \|_{H^s} \geq \sigma_0, \quad n = 1, 2, \cdots
\] (3.28)
By (3.27) and the imbedding theorem, there exists a subsequence $\{u(x, t_{n_i})\}$ such that $u(x, t_{n_i})$ converges to a function $\omega(x)$ in $H^s(0,1)$ and also in $L^2(0,1)$ as $i \to \infty$. Since by (2.13) we know that $u(x, t_{n_i})$ converges to 0 in $L^2(0,1)$, we have $\omega = 0$, which contradicts (3.28). ■

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References