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Some Families of Meromorphic Functions with Positive Coefficients Defined by Dziok-Srivastava Operator

T.Rosy[1], K. Muthunagai[2] and G.Murugusundaramoorthy[3] *
[1] Department of Mathematics, Madras Christian College, Chennai - 600 059, India.
E-mail:thomasrosy67@yahoo.in
[2] VIT University-Chennai Campus,Vandalur, Kelambakam Road ,Chennai -600 048
E-mail:muthunagaik@yahoo.com
[3]School of Advanced Sciences, VIT University,Vellore - 632014, India.
E-mail:gmsmoorthy@yahoo.com
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Abstract: In the present investigation, the authors define a new class of meromorphic functions defined in the punctured unit disk $\Delta^* := \{z \in \mathbb{C} : 0 < |z| < 1\}$ by making use of the Dziok-Srivastava operator . Coefficient inequalities, growth and distortion inequalities, as well as closure results are obtained. We also establish some results concerning the partial sums of meromorphic functions and neighbourhood results for functions in the new class.

Keywords: Meromorphic functions; starlike function; convolution; positive coefficients; coefficient inequalities; integral operator

1 Introduction

Let Σ denote the class of normalized meromorphic functions f of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \tag{1}$$

defined on the punctured unit disk $\Delta^* := \{z \in \mathbb{C} : 0 < |z| < 1\}$. A function $f \in \Sigma$ is *meromorphic starlike of order* α $(0 \le \alpha < 1)$ if $-\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \Delta := \Delta^* \cup \{0\})$. The class of all such functions is denoted by $\Sigma^*(\alpha)$. A function $f \in \Sigma$ is *meromorphic convex of order* α $(0 \le \alpha < 1)$ if $-\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \Delta := \Delta^* \cup \{0\})$. Let Σ_P be the class of functions $f \in \Sigma$ with $a_n \ge 0$. The subclass of Σ_P consisting of starlike functions of order α is denoted by $\Sigma_P^*(\alpha)$ and convex functions functions of order α by $\Sigma_P^K(\alpha)$. For functions $f(z)$ given by (1) and $g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n$ we define the Hadamard product or convolution of f and g by $(f * g)(z) := \frac{1}{z} + \sum_{n=1}^{\infty} a_n b_n z^n$.

For complex parameters $\alpha_1, \ldots, \alpha_l$ and $\beta_1, \ldots, \beta_m$ $(\beta_j \ne 0, -1, \ldots; j = 1, 2, \ldots, m)$ the *generalized hypergeometric function* ${}_lF_m(z)$ is defined by

$$\begin{aligned} {}_lF_m(z) \equiv {}_lF_m(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m; z) \quad &:= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_l)_n}{(\beta_1)_n \ldots (\beta_m)_n} \frac{z^n}{n!} \\ (l \le m+1;\ l, m \in N_0 \quad &:= N \cup \{0\}; z \in \Delta) \end{aligned} \tag{2}$$

where N denotes the set of all positive integers and $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & n = 0 \\ a(a+1)(a+2)\ldots(a+n-1), & n \in N; a \in C \end{cases} \tag{3}$$

*Corresponding author. *E-mail address*: gmsmoorthy@yahoo.com

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For positive real values of $\alpha_1, \ldots, \alpha_l$ and $\beta_1, \ldots, \beta_m$ $(\beta_j \neq 0, -1, \ldots; j = 1, 2, \ldots, m)$, let

$$H(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m) : \Sigma \to \Sigma$$

be a linear operator defined by

$$\mathcal{H}(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m)f(z) = \mathcal{H}^l_m f(z) = z^{-1}\ {}_lF_m(\alpha_1, \alpha_2, \ldots \alpha_l; \beta_1, \beta_2 \ldots, \beta_m; z) * f(z)$$

$$\mathcal{H}^l_m f(z) = z^{-1} + \sum_{n=1}^{\infty} \Gamma_n\ a_n\ z^n \tag{4}$$

where

$$\Gamma_n = |\frac{(\alpha_1)_n \ldots (\alpha_l)_n}{(\beta_1)_n \ldots (\beta_m)_n} \frac{1}{n!}| \tag{5}$$

$\alpha_i > 0, (i = 1, 2, ...l), \beta_j > 0, (j = 1, 2, ...m), l \leq m + 1;\ l, m \in N_0 = N \cup \{0\}$. For notational simplicity, we use a shorter notations $H^l_m[\alpha_1]$ for $H(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m)$, in the sequel.

Some interesting subclasses of analytic functions associated with the generalized hypergeometric function , were considered recently by Srivastava etal., [23](also see [7–9]). The class $\Sigma^*_P(\alpha)$ and various other subclasses of Σ have been studied rather extensively in[1, 3, 5, 13, 15, 17–19, 26, 27] see also Duren [[6], pages 29 and 137], and Srivastava and Owa [[22], pages 86 and 429]) . Now by making use of the generalized Dziok-Srivastava operator $\mathcal{H}^l_m$, we define a new subclass of functions in Σ_P as follows.

For $0 \leq \alpha < 1$ and $0 \leq \lambda < 1$, we let $\mathcal{M}(\lambda, \alpha)$ denote a subclass of Σ_p consisting functions of the form (1) satisfying the condition that

$$\Re\left(\frac{z(\mathcal{H}^l_m f(z))'}{(\lambda - 1)\mathcal{H}^l_m f(z) + \lambda z(\mathcal{H}^l_m f(z))'}\right) > \alpha, \tag{6}$$

where $\mathcal{H}^l_m$ is given by (5).In this paper, we obtain the coefficient inequalities, growth and distortion inequalities, as well as closure results for the class $\mathcal{M}(\lambda, \alpha)$. Properties of a certain integral operator and its inverse defined on the new class $\mathcal{M}(\lambda, \alpha)$ are also discussed.

2 Coefficients Inequalities

Our first theorem gives a necessary and sufficient condition for a function f to be in the class $\mathcal{M}(\lambda, \alpha)$.

Theorem 1 *Let $f(z) \in \Sigma_P$ be given by (1). Then $f \in \mathcal{M}(\lambda, \alpha)$ if and only if*

$$\sum_{n=1}^{\infty} \{n + \alpha - \alpha\lambda(1 + n)\}\ \Gamma_n a_n \leq 1 - \alpha. \tag{7}$$

Proof. If $f \in \mathcal{M}(\lambda, \alpha)$, then

$$\Re\left(\frac{z(\mathcal{H}^l_m f(z))'}{(\lambda - 1)\mathcal{H}^l_m f(z) + \lambda z(\mathcal{H}^l_m f(z))'}\right) = \Re\left\{\frac{-1 + \sum_{n=1}^{\infty} n\Gamma_n\ a_n z^{n+1}}{-1 + \sum_{n=1}^{\infty}(\lambda - 1 + \lambda n)\Gamma_n\ a_n z^{n+1}}\right\} > \alpha.$$

By letting $z \to 1^-$, we have

$$\left\{\frac{-1 + \sum_{n=1}^{\infty} n\Gamma_n a_n}{-1 + \sum_{n=1}^{\infty}(\lambda - 1 + \lambda n)\Gamma_n a_n}\right\} > \alpha.$$

This shows that (7) holds.Conversely assume that (7) holds. Since

$$\Re(w) > \alpha \quad \text{if and only if} \quad |w - 1| < |w + 1 - 2\alpha|,$$

it is sufficient to show that

$$\left|\frac{z(\mathcal{H}^l_m f(z))' - [(\lambda - 1)\mathcal{H}^l_m f(z) + \lambda z(\mathcal{H}^l_m f(z))']}{z(\mathcal{H}^l_m f(z))' + (1 - 2\alpha)[(\lambda - 1)\mathcal{H}^l_m f(z) + \lambda z(\mathcal{H}^l_m f(z))']}\right| < 1 \quad (z \in \Delta).$$

Using (7), we see that

$$\left|\frac{z(\mathcal{H}_m^l f(z))' - [(\lambda-1)\mathcal{H}_m^l f(z) + \lambda z(\mathcal{H}_m^l f(z))']}{z(\mathcal{H}_m^l f(z))' + (1-2\alpha)[(\lambda-1)\mathcal{H}_m^l f(z) + \lambda z(\mathcal{H}_m^l f(z))']}\right|$$

$$= \left|\frac{\sum_{n=1}^{\infty}(1-\lambda)(n+1)\Gamma_n a_n z^{n+1}}{-2(1-\alpha) + \sum_{n=1}^{\infty}\left[n(1+(1-2\alpha)\lambda) + (1-2\alpha)(\lambda-1)\right]\Gamma_n a_n z^{n+1}}\right|$$

$$\leq \frac{\sum_{n=1}^{\infty}(1-\lambda)(n+1)\Gamma_n a_n}{2(1-\alpha) - \sum_{n=1}^{\infty}\left[n(1+(1-2\alpha)\lambda) + (1-2\alpha)(\lambda-1)\right]\Gamma_n a_n} \leq 1.$$

Thus we have $f \in \mathcal{M}(\lambda, \alpha)$. ■

For the sake of brevity throughout this paper we let

$$d_n(\lambda, \alpha) := n + \alpha - \alpha\lambda(1+n) \tag{8}$$

$$d_1(\lambda, \alpha) = 1 + (1 - 2\,\lambda)\alpha$$

unless otherwise stated. Our next result gives the coefficient estimates for functions in $\mathcal{M}(\lambda, \alpha)$.

Theorem 2 *If $f \in \mathcal{M}(\lambda, \alpha)$, then*

$$a_n \leq \frac{1-\alpha}{d_n(\lambda, \alpha)\Gamma_n}, \qquad n = 1, 2, 3, \ldots.$$

The result is sharp for the functions $F_n(z)$ given by

$$F_n(z) = \frac{1}{z} + \frac{1-\alpha}{d_n(\lambda, \alpha)\Gamma_n} z^n, \qquad n = 1, 2, 3, \ldots.$$

Proof. If $f \in \mathcal{M}(\lambda, \alpha)$, then we have, for each n,

$$d_n(\lambda, \alpha) a_n \leq \sum_{n=1}^{\infty} d_n(\lambda, \alpha)\Gamma_n a_n \leq 1 - \alpha.$$

Therefore we have

$$a_n \leq \frac{1-\alpha}{d_n(\lambda, \alpha)\Gamma_n}.$$

Since

$$F_n(z) = \frac{1}{z} + \frac{1-\alpha}{d_n(\lambda, \alpha)\Gamma_n} z^n$$

satisfies the conditions of Theorem 1, $F_n(z) \in \mathcal{M}(\lambda, \alpha)$ and the equality is attained for this function. ■

Theorem 3 *If $f \in \mathcal{M}(\lambda, \alpha)$, then*

$$\frac{1}{r} - \frac{1-\alpha}{(1+\alpha-2\alpha\lambda)\Gamma_1} r \leq |f(z)| \leq \frac{1}{r} + \frac{1-\alpha}{(1+\alpha-2\alpha\lambda)\Gamma_1} r \quad (|z| = r).$$

The result is sharp for

$$f(z) = \frac{1}{z} + \frac{1-\alpha}{(1+\alpha-2\alpha\lambda)\Gamma_1} z. \tag{9}$$

Proof. Since $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$, we have

$$|f(z)| \leq \frac{1}{r} + \sum_{n=1}^{\infty} a_n r^n \leq \frac{1}{r} + r\sum_{n=1}^{\infty} a_n.$$

Since,

$$\sum_{n=1}^{\infty} a_n \leq \frac{1-\alpha}{(1+\alpha-2\alpha\lambda)\Gamma_1}.$$

Using this, we have

$$|f(z)| \leq \frac{1}{r} + \frac{1-\alpha}{(1+\alpha-2\alpha\lambda)\Gamma_1} r.$$

Similarly

$$|f(z)| \geq \frac{1}{r} - \frac{1-\alpha}{(1+\alpha-2\alpha\lambda)\Gamma_1} r.$$

The result is sharp for $f(z) = \frac{1}{z} + \frac{1-\alpha}{(1+\alpha-2\alpha\lambda)\Gamma_1} z$. ■

Similarly we have the following:

Theorem 4 *If $f \in \mathcal{M}(\lambda, \alpha)$, then*

$$\frac{1}{r^2} - \frac{1-\alpha}{(1+\alpha-2\alpha\lambda)\Gamma_1} \leq |f'(z)| \leq \frac{1}{r^2} + \frac{1-\alpha}{(1+\alpha-2\alpha\lambda)\Gamma_1} \quad (|z| = r).$$

The result is sharp for the function given by (9).

3 Closure Theorems

Let the functions $F_k(z)$ be given by

$$F_k(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,k} z^n, \quad k = 1, 2, ..., m. \tag{10}$$

We shall prove the following closure theorems for the class $\mathcal{M}(\lambda, \alpha)$.

Theorem 5 *Let the function $F_k(z)$ defined by (10) be in the class $\mathcal{M}(\lambda, \alpha)$ for every $k = 1, 2, ..., m$. Then the function $f(z)$ defined by $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$, $(a_n \geq 0)$ belongs to the class $\mathcal{M}(\lambda, \alpha)$, where $a_n = \frac{1}{m} \sum_{k=1}^{m} a_{n,k}$ $(n = 1, 2, ..)$.*

Proof. Since $F_k(z) \in \mathcal{M}(\lambda, \alpha)$, it follows from Theorem 1, that

$$\sum_{n=1}^{\infty} d_n(\lambda, \alpha)\Gamma_n a_{n,k} \leq 1 - \alpha, \forall k = 1, 2, .., m. \tag{11}$$

Hence

$$\begin{aligned} \sum_{n=1}^{\infty} d_n(\lambda, \alpha)\Gamma_n a_n &= \sum_{n=1}^{\infty} d_n(\lambda, \alpha)\Gamma_n \left(\frac{1}{m} \sum_{k=1}^{m} a_{n,k} \right) \\ &= \frac{1}{m} \sum_{k=1}^{m} \left(\sum_{n=1}^{\infty} d_n(\lambda, \alpha)\Gamma_n a_{n,k} \right) \\ &\leq 1 - \alpha. \end{aligned}$$

By Theorem 1, we have $f(z) \in \mathcal{M}(\lambda, \alpha)$. ■

Theorem 6 *The class $\mathcal{M}(\lambda, \alpha)$ is closed under convex linear combination.*

Proof. Let the function $F_k(z)$ given by (10) be in the class $\mathcal{M}(\lambda, \alpha)$. Then it is enough to show that the function

$$H(z) = \lambda F_1(z) + (1 - \lambda)F_2(z) \quad (0 \leq \lambda \leq 1)$$

is also in the class $\mathcal{M}(\lambda, \alpha)$. Since for $0 \leq \mu \leq 1$,

$$H(z) = \frac{1}{z} + \sum_{n=1}^{\infty} [\mu a_{n,1} + (1 - \mu)a_{n,2}]z^n,$$

we observe that

$$\sum_{n=1}^{\infty}[\mu a_{n,1}+(1-\mu)a_{n,2}]d_n(\lambda,\alpha)\Gamma_n$$

$$\begin{aligned} &= \mu\sum_{n=1}^{\infty}d_n(\lambda,\alpha)\Gamma_n a_{n,1}+(1-\mu)\sum_{n=1}^{\infty}d_n(\lambda,\alpha)\Gamma_n a_{n,2} \\ &\leq 1-\alpha. \end{aligned}$$

By Theorem 1, we have $H(z)\in\mathcal{M}(\lambda,\alpha)$. ■

Theorem 7 *Let $F_0(z)=\frac{1}{z}$ and $F_n(z)=\frac{1}{z}+\frac{1-\alpha}{d_n(\lambda,\alpha)\Gamma_n}z^n$ for $n=1,2,\ldots$. Then $f(z)\in\mathcal{M}(\lambda,\alpha)$ if and only if $f(z)$ can be expressed in the form $f(z)=\sum_{n=0}^{\infty}\nu_n F_n(z)$ where $\nu_n\geq 0$ and $\sum_{n=0}^{\infty}\nu_n=1$.*

Proof. Let

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty}\nu_n F_n(z) \\ &= \frac{1}{z}+\sum_{n=1}^{\infty}\frac{\nu_n(1-\alpha)}{d_n(\lambda,\alpha)\Gamma_n}z^n. \end{aligned}$$

Then

$$\sum_{n=1}^{\infty}\nu_n\frac{1-\alpha}{d_n(\lambda,\alpha)\Gamma_n}\frac{d_n(\lambda,\alpha)\Gamma_n}{(1-\alpha)}=\sum_{n=1}^{\infty}\nu_n=1-\nu_0\leq 1.$$

By Theorem 2, we have $f(z)\in\mathcal{M}(\lambda,\alpha)$. Conversely, let $f(z)\in\mathcal{M}(\lambda,\alpha)$. From Theorem 2, we have $a_n\leq\frac{1-\alpha}{d_n(\lambda,\alpha)\Gamma_n}$; for $n=1,2,..$ we may take $\nu_n=\frac{d_n(\lambda,\alpha)\Gamma_n}{1-\alpha}a_n$, for $n=1,2,...$ and $\lambda_0=1-\sum_{n=1}^{\infty}\nu_n$. Then $f(z)=\sum_{n=0}^{\infty}\nu_n F_n(z)$. ■

4 Integral Operators

In this section, we consider integral transforms of functions in the class $\mathcal{M}(\alpha,\lambda)$.

Theorem 8 *Let the function $f(z)$ given by (1) be in $\mathcal{M}(\lambda,\alpha)$. Then the integral operator*

$$F(z)=c\int_0^1 u^c f(uz)du \qquad (0<u\leq 1, 0<c<\infty)$$

is in $\mathcal{M}(\lambda,\alpha)$, where

$$\delta\leq\frac{(c+n+1)d_n(\lambda,\alpha)-cn(1-\alpha)}{c(1-\alpha)\{1-\lambda(1+n)\}+d_n(\lambda,\alpha)(c+n+1)}.$$

The result is sharp for the function $f(z)=\frac{1}{z}+\frac{1-\alpha}{(1+\alpha-2\alpha\lambda)\Gamma_1}z$.

Proof. Let $f(z)\in\mathcal{M}(\lambda,\alpha)$. Then

$$F(z) = c\int_0^1 u^c f(uz)du=\frac{1}{z}+\sum_{n=1}^{\infty}\frac{c}{c+n+1}f_n z^n.$$

It is sufficient to show that

$$\sum_{n=1}^{\infty}\frac{c\, d_n(\lambda,\delta)\Gamma_n}{(c+n+1)(1-\delta)}a_n\leq 1. \tag{12}$$

Since $f\in\mathcal{M}(\lambda,\alpha)$, we have

$$\sum_{n=1}^{\infty}\frac{d_n(\lambda,\alpha)\Gamma_n}{(1-\alpha)}a_n\leq 1.$$

Note that (12) is satisfied if

$$\frac{c\,d_n(\lambda,\delta)\Gamma_n}{(c+n+1)(1-\delta)} \le \frac{d_n(\lambda,\alpha)\Gamma_n}{(1-\alpha)}.$$

Solving for δ, we have

$$\delta \le \frac{(c+n+1)d_n(\lambda,\alpha) - cn(1-\alpha)}{c(1-\alpha)\{1-\lambda(1+n)\} + d_n(\lambda,\alpha)(c+n+1)} = \Phi(n).$$

A simple computation will show that $\Phi(n)$ is increasing and $\Phi(n) \ge \Phi(1)$. Using this, the results follows. ■

Theorem 9 *Let $f(z)$, given by (1), be in $\mathcal{M}(\lambda,\alpha)$,*

$$F(z) = \frac{1}{c}[(c+1)f(z) + zf'(z)] = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{c+n+1}{c} f_n z^n, \qquad c > 0. \tag{13}$$

Then $F(z)$ is in $\mathcal{M}(\lambda,\alpha)$ for $|z| \le r(\alpha,\lambda,\beta)$ where

$$r(\alpha,\lambda,\beta) = \inf_n \left(\frac{c(1-\beta)d_n(\lambda,\alpha)}{(1-\alpha)(c+n+1)d_n(\lambda,\beta)} \right)^{1/(n+1)}, \quad n = 1,2,3,\ldots.$$

The result is sharp for the function $f_n(z) = \frac{1}{z} + \frac{1-\alpha}{d_n(\lambda,\alpha)\Gamma_n} z^n, \quad n = 1,2,3,\ldots.$

Proof. Let $w = \frac{zf'(z)}{(\lambda-1)f(z)+\lambda zf'(z)}$. Then it is sufficient to show that $\left|\frac{w-1}{w+1-2\beta}\right| < 1$. A simple computation shows that this is satisfied if

$$\sum_{n=1}^{\infty} \frac{(c+n+1)d_n(\lambda,\beta)\Gamma_n}{c\,(1-\beta)} a_n |z|^{n+1} \le 1. \tag{14}$$

Since $f \in \mathcal{M}(\lambda,\alpha)$, by Theorem 1, we have

$$\sum_{n=1}^{\infty} d_n(\lambda,\alpha)\Gamma_n a_n \le 1-\alpha.$$

The equation (14) is satisfied if

$$\frac{(c+n+1)d_n(\lambda,\beta)\Gamma_n}{c\,(1-\beta)} a_n |z|^{n+1} \le \frac{d_n(\lambda,\alpha)\Gamma_n a_n}{1-\alpha}.$$

Solving for $|z|$, we get the desired result. ■

5 Partial Sums

Silverman [21] determined sharp lower bounds on the real part of the quotients between the normalized starlike or convex functions and their sequences of partial sums. As a natural extension, one is interested to search results analogous to those of Silverman for meromorphic univalent functions. In this section, motivated essentially by the work of silverman [21] and Cho and Owa [4] we will investigate the ratio of a function of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \tag{15}$$

to its sequence of partial sums

$$f_1(z) = \frac{1}{z} \text{ and } f_k(z) = \frac{1}{z} + \sum_{n=1}^{k} a_n z^n \tag{16}$$

when the coefficients are sufficiently small to satisfy the condition analogous to

$$\sum_{n=1}^{\infty} d_n(\lambda,\alpha)\Gamma_n a_n \le 1-\alpha.$$

More precisely we will determine sharp lower bounds for $\Re\{f(z)/f_k(z)\}$ and $\Re\{f_k(z)/f(z)\}$. In this connection we make use of the well known results that$\Re\left\{\frac{1+w(z)}{1-w(z)}\right\} > 0 \quad (z \in \Delta)$ if and only if $\omega(z) = \sum\limits_{n=1}^{\infty} c_n z^n$ satisfies the inequality $|\omega(z)| \le |z|$. Unless otherwise stated, we will assume that f is of the form (1) and its sequence of partial sums is denoted by $f_k(z) = \frac{1}{z} + \sum_{n=1}^{k} a_n z^n$.

Theorem 10 *Let $f(z) \in \Sigma_P$ be given by (1)satisfies condition, (7)*

$$Re\left\{\frac{f(z)}{f_k(z)}\right\} \ge \frac{d_{k+1}(\lambda,\alpha)\Gamma_{k+1} - 1 + \alpha}{d_{k+1}(\lambda,\alpha)\Gamma_{k+1}} \qquad (z \in U) \tag{17}$$

where

$$d_n(\lambda,\alpha) \ge \begin{cases} 1-\alpha, & if \quad n = 1,2,3,\dots,k \\ d_{k+1}(\lambda,\alpha)\Gamma_{k+1}, & if \quad n = k+1, k+2, \dots. \end{cases} \tag{18}$$

The result (17) is sharp with the function given by

$$f(z) = \frac{1}{z} + \frac{1-\alpha}{d_{k+1}(\lambda,\alpha)\Gamma_{k+1}} z^{k+1}. \tag{19}$$

Proof.

Define the function $w(z)$ by

$$\frac{1+w(z)}{1-w(z)} = \frac{d_{k+1}(\lambda,\alpha)\Gamma_{k+1}}{1-\alpha}\left[\frac{f(z)}{f_k(z)} - \frac{d_{k+1}(\lambda,\alpha)\Gamma_{k+1} - 1 + \alpha}{d_{k+1}(\lambda,\alpha)\Gamma_{k+1}}\right]$$

$$= \frac{1 + \sum\limits_{n=1}^{k} a_n z^{n+1} + \left(\frac{d_{k+1}(\lambda,\alpha)\Gamma_{k+1}}{1-\alpha}\right) \sum\limits_{n=k+1}^{\infty} a_n z^{n+1}}{1 + \sum\limits_{n=1}^{k} a_n z^{n+1}} \tag{20}$$

It suffices to show that $|w(z)| \le 1$. Now, from (20) we can write

$$w(z) = \frac{\left(\frac{d_{k+1}(\lambda,\alpha)\Gamma_{k+1}}{1-\alpha}\right) \sum\limits_{n=k+1}^{\infty} a_n z^{n+1}}{2 + 2\sum\limits_{n=1}^{k} a_n z^{n+1} + \left(\frac{d_{k+1}(\lambda,\alpha)\Gamma_{k+1}}{1-\alpha}\right) \sum\limits_{k=n+1}^{\infty} a_n z^{n+1}}.$$

Hence we obtain

$$|w(z)| \le \frac{\left(\frac{d_{k+1}(\lambda,\alpha)\Gamma_{k+1}}{1-\alpha}\right) \sum\limits_{k=n+1}^{\infty} |a_n|}{2 - 2\sum\limits_{n=1}^{k} |a_n| - \left(\frac{d_{k+1}(\lambda,\alpha)\Gamma_{k+1}}{1-\alpha}\right) \sum\limits_{n=k+1}^{\infty} |a_n|}$$

Now $|w(z)| \le 1$ if

$$2\left(\frac{d_{k+1}(\lambda,\alpha)\Gamma_{k+1}}{1-\alpha}\right) \sum_{n=k+1}^{\infty} |a_n| \le 2 - 2\sum_{n=1}^{k} |a_n|$$

or, equivalently,

$$\sum_{n=1}^{k} |a_n| + \frac{d_{k+1}(\lambda,\alpha)\Gamma_{k+1}}{1-\alpha} \sum_{n=k+1}^{\infty} |a_n| \le 1.$$

From the condition (7), it is sufficient to show that

$$\sum_{n=1}^{k} |a_n| + \frac{d_{k+1}(\lambda,\alpha)\Gamma_{k+1}}{1-\alpha} \sum_{n=k+1}^{\infty} |a_n| \le \sum_{n=1}^{\infty} \frac{d_n(\lambda,\alpha)\Gamma_n}{1-\alpha} |a_n|$$

which is equivalent to

$$\begin{aligned}
&\sum_{n=1}^{k}\left(\frac{d_n(\lambda,\alpha)\Gamma_n-1+\alpha}{1-\alpha}\right)|a_n| \\
&+\sum_{n=k+1}^{\infty}\left(\frac{d_n(\lambda,\alpha)\Gamma_n-d_{k+1}(\lambda,\alpha)\Gamma_{k+1}}{1-\alpha}\right)|a_n| \\
\geq\ & 0 \qquad (21)
\end{aligned}$$

To see that the function given by (19) gives the sharp result, we observe that for $z=re^{i\pi/k}$

$$\begin{aligned}
\frac{f(z)}{f_k(z)} &= 1+\frac{1-\alpha}{d_{k+1}(\lambda,\alpha)\Gamma_{k+1}}z^n \to 1-\frac{1-\alpha}{d_{k+1}(\lambda,\alpha)\Gamma_{k+1}} \\
&= \frac{d_{k+1}(\lambda,\alpha)\Gamma_{k+1}-1+\alpha}{d_{k+1}(\lambda,\alpha)\Gamma_{k+1}} \quad \text{when } r\to 1^-
\end{aligned}$$

which shows the bound (17) is the best possible for each $k\in\mathbb{N}$. ■

We next determine bounds for $f_k(z)/f(z)$.

Theorem 11 *If f of the form (1) satisfies the condition (7), then*

$$Re\left\{\frac{f_k(z)}{f(z)}\right\}\geq\frac{d_{k+1}(\lambda,\alpha)\Gamma_{k+1}}{d_{k+1}(\lambda,\alpha)\Gamma_{k+1}+1-\alpha} \qquad (z\in U), \tag{22}$$

where $d_{k+1}(\lambda,\alpha)\Gamma_{k+1}\geq 1-\alpha$ and

$$d_k(\lambda,\alpha)\omega_k^{\tau}(\alpha_1;\eta;l;m)\geq\begin{cases}1-\alpha, & if \quad k=1,2,3,\ldots,n\\ d_{k+1}(\lambda,\alpha)\Gamma_{k+1}, & if \quad k=n+1,n+2,\ldots.\end{cases}. \tag{23}$$

The result (22) is sharp with the function given by (19).

Proof. We write

$$\frac{1+w(z)}{1-w(z)}=\frac{d_{k+1}(\lambda,\alpha)\Gamma_{k+1}+1-\gamma}{1-\alpha}\left[\frac{f_k(z)}{f(z)}-\frac{d_{k+1}(\lambda,\alpha)\Gamma_{k+1}}{d_{k+1}(\lambda,\alpha)\Gamma_{k+1}+1-\alpha}\right]$$

$$=\frac{1+\sum\limits_{n=1}^{k}a_nz^{n+1}-\left(\frac{d_{k+1}(\lambda,\alpha)\Gamma_{k+1}}{1-\alpha}\right)\sum\limits_{n=k+1}^{\infty}a_nz^{n+1}}{1+\sum\limits_{n=1}^{\infty}a_nz^{n+1}},$$

where

$$|w(z)|\leq\frac{\left(\frac{d_{k+1}(\lambda,\alpha)\Gamma_{k+1}+1-\alpha}{1-\alpha}\right)\sum\limits_{n=k+1}^{\infty}|a_n|}{2-2\sum\limits_{n=1}^{k}|a_n|-\left(\frac{d_{k+1}(\lambda,\alpha)\Gamma_{k+1}-1+\alpha}{1-\alpha}\right)\sum\limits_{n=k+1}^{\infty}|a_n|}\leq 1.$$

This last inequality is equivalent to

$$\sum_{n=1}^{k}|a_n|+\frac{d_{k+1}(\lambda,\alpha)\Gamma_{k+1}}{1-\alpha}\sum_{n=k+1}^{\infty}|a_n|\leq 1.$$

Make use of (7) to get (21). Finally, equality holds in (22) for the extremal function $f(z)$ given by (19). ■

6 Neighborhoods for the class $\mathcal{M}(\alpha,\lambda)$

Following the earlier works on neighborhoods of analytic functions by Goodman [11] and Ruscheweyh [20], we begin by introducing here the δ-neighborhood of a function $f\in\Sigma_p$ of the form(1) by means of the definition below:

$$N_\delta(f) := \left\{ g\in\Sigma_p \ : \ g(z)=\frac{1}{z}+\sum_{n=1}^{\infty} b_n z^n \text{ and } \sum_{n=1}^{\infty} n|a_n-b_n|\le\delta \right\}. \tag{24}$$

Particulary for the identity function $e(z)=\frac{1}{z}$, we have

$$N_\delta(e) := \left\{ g\in\Sigma : g(z)=\frac{1}{z}+\sum_{n=1}^{\infty} b_n z^n \ \text{ and } \ \sum_{n=2}^{\infty} n|b_n|\le\delta \right\}. \tag{25}$$

Theorem 12 *If*

$$\delta := \frac{1-\alpha}{(1+(1-2\lambda)\alpha)\Gamma_1} \tag{26}$$

then $\mathcal{M}(\alpha,\lambda)\subset N_\delta(e)$.

Proof. For $f\in\mathcal{M}(\alpha,\lambda)$, Theorem 1, immediately yields

$$d_1(\lambda,\alpha)\Gamma_1\sum_{n=1}^{\infty} a_n\le 1-\alpha,$$

so that

$$\sum_{n=1}^{\infty} a_n\le\frac{(1-\alpha)}{d_1(\lambda,\alpha)\Gamma_1} \tag{27}$$

On the other hand, from (7) and (27) that

$$\begin{aligned}\Gamma_1\sum_{n=2}^{\infty} na_n &\le (1-\alpha)-(1-2\lambda)\alpha\,\Gamma_1\sum_{n=1}^{\infty} a_n\\ &\le (1-\alpha)-(1-2\lambda)\alpha\,\Gamma_1\frac{(1-\alpha)}{d_1(\lambda,\alpha)\Gamma_1}\\ &\le \frac{1-\alpha}{1+(1-2\lambda)\alpha},\end{aligned}$$

that is

$$\sum_{n=2}^{\infty} na_n\le\frac{1-\alpha}{(1+(1-2\lambda)\alpha)\Gamma_1}:=\delta \tag{28}$$

which, in view of the definition (25) proves Theorem 12. ■

Definition 1 *A function $f\in\Sigma_P$ is said to be in the class $\mathcal{M}(\lambda,\alpha,\gamma)$ if there exists a function $g\in\mathcal{M}(\lambda,\alpha)$ such that*

$$\left|\frac{f(z)}{g(z)}-1\right|<1-\gamma, \qquad (z\in\Delta, 0\le\gamma<1). \tag{29}$$

Theorem 13 *If $g\in\mathcal{M}(\lambda,\alpha)$ and*

$$\gamma=1-\frac{\delta\, d_1(\lambda,\alpha)\Gamma_1}{d_1(\lambda,\alpha)\Gamma_1-(1-\alpha)}, \tag{30}$$

then

$$N_\delta(g)\subset\mathcal{M}(\lambda,\alpha,\gamma).$$

Proof. Let $f \in N_\delta(g)$. Then we find from (24) that

$$\sum_{n=1}^{\infty} n|a_n - b_n| \le \delta, \tag{31}$$

which implies the coefficient inequality

$$\sum_{n=1}^{\infty} |a_n - b_n| \le \delta, \qquad (n \in \mathbb{N}). \tag{32}$$

Since $g \in \mathcal{M}(\lambda, \alpha)$, we have [cf. equation (7)]

$$\sum_{n=1}^{\infty} b_n \le \frac{1-\alpha}{d_1(\lambda,\alpha)\Gamma_1}, \tag{33}$$

so that

$$\begin{aligned} \left|\frac{f(z)}{g(z)} - 1\right| &< \frac{\sum_{n=1}^{\infty} |a_n - b_n|}{1 - \sum_{n=1}^{\infty} b_n} \\ &= \frac{\delta\, d_1(\lambda,\alpha)\Gamma_1}{d_1(\lambda,\alpha)\Gamma_1 - (1-\alpha)} \\ &= 1 - \gamma, \end{aligned}$$

provided γ is given by (30). Hence, by definition, $f \in \mathcal{M}(\lambda, \alpha, \gamma)$ for γ given by (30), which completes the proof. ■

Concluding Remarks: The results obtained here are generalizes the results obtained in [12] and by choosing $\lambda = 1; \lambda = 0$ and specializing the parameters l, m, λ,the various results presented in this paper would provide interesting extensions and generalizations of those considered earlier by [1–3, 5, 13, 22, 26, 27] .In fact, by appropriately selecting these arbitrary sequences, the results presented in this paper would find further applications for the class of meromorphic functions which would incorporate a generalized form of the DziokSrivastava linear operator involving the Hadamard product (or convolution) of the function in (1) with the Fox-Wright generalization ${}_l\psi_m$ of the hypergeometric function ${}_lF_m$. Theorem 1 to 13 would thus eventually lead us further to new results for the class of functions (defined analogously to the class $\mathcal{M}(\lambda, \alpha)$ by associating instead the Fox-Wright generalized hypergeometric function ${}_l\psi_m$. These considerations can fruitfully be worked out by closely following the recent investigations by Dziok and Raina [8] and Dziok et al. [9]. We choose to skip further details in this regard.

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