ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.12(2011) No.2,pp.162-168

An analytic study for a Family of Nonlinear Diffusion Equations of the Fisher Type

Yinghui Zhang[1] *, Huixu Li[2], Demin Yu[1], Zhengsu Wan[1]
[1]Department of Mathematics, Hunan Institute of Science and Technology Yueyang, Hunan 414006, China
[2]Department of Mathematical and Physical Science , Henan Institute of Engineering, Zhengzhou, China, 451191
(Received 15 March 2011 , accepted 20 July 2011)

Abstract: In this letter, we investigate the explicit and exact traveling wave solutions for a family of nonlinear diffusion equations of Fisher type. New solutions are obtained by combining the first integral method with the trial function method.

Keywords: Fisher type; first integral method; trial function method; nonlinear diffusion

1 Introduction

It is well recognized that a great number of physical, chemical and biological phenomena can be described by nonlinear differential equations (NDEs). Therefore, solving NDEs and seeking explicit and exact solutions of NDEs has become one of the most exciting and extremely active domains in nonlinear science. In recent years, a vast variety of new and powerful approaches have been developed to construct explicit and exact solutions to NDEs. Among them are the hyperbolic tangent function expansion method [1, 2], the variational iteration method [3, 4], the trial function method [5, 6], the Jacobi elliptic function expansion method [7], the modified trigonometric function series method [8, 9], the extended mapping method [10], the bifurcation method [11], the modified G'/G-expansion method [12], the superposition method [13], and the first integral method [14–16], and so on.

The well-known Fisher's equation combines diffusion with logistic nonlinearity. Fisher proposed the following equation

$$u_t - u_{xx} = u(1-u),$$

as a model for the propagation of a mutant gene, with u denoting the density of an advantageous. This equation is encountered in chemical kinetics and population dynamics which includes problems such as nonlinear evolution of a population in a one-dimensional habitat, neutron population in a nuclear reaction. Moreover, the same equation occurs in logistic population growth models, flame propagation, neurophysiology, autocatalytic chemical reactions, and branching Brownian motion processes.

In this paper, following Kawahara and Tanaka [17], Wazwaz and Gorguis [18], and Matinfar and Ghanbari [19], we consider a family of nonlinear diffusion equations of Fisher type as follows

$$u_t - u_{xx} = ku(1-u^m)(u^m-\alpha), \tag{1}$$

where $k > 0$ and α are given constants and $m = 1, 2, 3, \cdots$.

When $k = m = 1$ in (1), Kawahara and Tanaka in [17] shown the equation (1) has a traveling wave of the following type

$$u(x,t) = \frac{\alpha}{2}\left\{1 + tanh\left[\pm\frac{\sqrt{2}\alpha x}{4} + \frac{(\alpha^2-2\alpha)t}{4}\right]\right\}.$$

Moreover, Wazwaz and Gorguis in [18] used the adomian decomposition to get the exact solution of (1) which has the following form

$$u(x,t) = \frac{1}{1+\xi_0 e^{-\frac{\sqrt{2}}{2}x+(\alpha-\frac{1}{2})t}}.$$

*Corresponding author. *E-mail address*: zhangyinghui1009@yahoo.com.cn

IJNS.2011.10.15/527

Recently, Matinfar and Ghanbari in [19] applied the homotopy perturbation method to get the same result as in [18].

The motivation of this article is to generalize the results of [17–19] to a family of nonlinear diffusion equations of Fisher type (1). Compared with [17–19], our method is brief and efficient. More precisely, for the case $m = 1, 2$, we use the first integral method to construct explicit traveling wave solutions for the equations (1). For the case $m > 2$, we combine the obtained results for the case $m = 1, 2$ with the trial function method to get new exact traveling wave solutions for the equations (1).

The rest of this paper is organized as follows: Section 2 is a brief introduction to the first integral method. In section 3, by using the first integral method and the trial function method we get new exact traveling wave solutions for the equations (1).

2 The first integral method

Consider the nonlinear partial differential equation (NPDE):

$$F(u, u_t, u_x, u_{xx}, u_{xt}, ...) = 0, \tag{2}$$

where $u(x, t)$ is the solution of equation (2). We use the transformations

$$u(x, t) = f(\xi), \quad \xi = x - st, \tag{3}$$

where s is a constant. By using the chain rule, we obtain

$$\frac{\partial}{\partial t}(.) = -s\frac{d}{d\xi}(.), \;\; \frac{\partial}{\partial x}(.) = \frac{d}{d\xi}(.), \;\; \frac{\partial^2}{\partial x^2}(.) = \frac{d^2}{d\xi^2}(.), \;\; \cdots . \tag{4}$$

We use (4) to change NPDE (2) to the following ordinary differential equation (ODE):

$$G(f, f_\xi, f_{\xi\xi}, ...) = 0. \tag{5}$$

Next, we introduce new independent variables

$$X(\xi) = f(\xi), \qquad Y(\xi) = f_\xi(\xi), \tag{6}$$

which change (4) to a system of ODEs

$$\begin{cases} X_\xi(\xi) = Y(\xi), \\ Y_\xi(\xi) = F(X(\xi), Y(\xi)). \end{cases} \tag{7}$$

According to the qualitative theory of ordinary differential equations [20], if we can find the integrals to (7) under the same conditions, then the general solutions to (7) can be solved directly. However, in general, it is really difficult for us to realize this even for one first integral, because for a given plane autonomous system, there is no systematic theory that can tell us how to find its first integrals, nor is there a logical way for telling us what these first integrals are. A key idea of this paper to find the first integral is to utilize the Division Theorem. For readers' convenience, we give the Division Theorem for two variables in the complex domain $\mathbb{C}$ [14–16].

Division Theorem: Suppose that $P(x, y)$ and $Q(x, y)$ are polynomials of two variables x and y in $\mathbb{C}[x, y]$ and $P(x, y)$ is irreducible in $\mathbb{C}[x, y]$. If $Q(x, y)$ vanishes at all zero points of $P(x, y)$, then there exists a polynomial $G(x, y)$ in $\mathbb{C}[x, y]$ such that

$$Q[x, y] = P(x, y)G(x, y).$$

3 Exact solutions for the nonlinear equations of Fisher type (1)

In this section, we use the first integral method and the trial function method to find the explicit solutions for the equations (1). Firstly, we set

$$v(\xi) = f^m(\xi), \tag{8}$$

which together with (1), (3)-(4) imply

$$v_{\xi\xi}(\xi) = -sv_\xi(\xi) + \frac{m-1}{m}\frac{v_\xi^2(\xi)}{v(\xi)} - kmv(\xi)(1 - v(\xi))(v(\xi) - \alpha). \tag{9}$$

If we let $v_\xi(\xi) = y(\xi)$, then equation (9) can be rewritten as the two-dimensional autonomous system

$$\begin{cases} v_\xi = y, \\ y_\xi = -sy + \dfrac{m-1}{m}\dfrac{y^2}{v} - kmv(1-v)(v-\alpha). \end{cases} \tag{10}$$

Making the following transformation

$$d\xi = vd\tau, \tag{11}$$

then (10) becomes

$$\begin{cases} v_\tau = vy, \\ y_\tau = -svy + \dfrac{m-1}{m}y^2 - kmv^2(1-v)(v-\alpha). \end{cases} \tag{12}$$

Next, we apply the above Division Theorem to look for the first integral of system (12). Suppose that $v = v(\tau)$ and $y = y(\tau)$ are the nontrivial solutions to (12), and $\Omega(v, y) = \sum_{i=0}^{l} a_i(v)y^i$ is an irreducible polynomial in $\mathbb{C}[v, y]$ such that

$$\Omega(v(\xi), y(\xi)) = \sum_{i=0}^{l} a_i(v)y^i = 0, \tag{13}$$

where $a_i(v)(i = 1, 2, 3, \cdots, l)$ are polynomials of v and $a_l(v) \neq 0$. For simplicity, we only consider the case $l = 1$. Note that $\frac{d\Omega}{d\tau}$ is a polynomial of v and y, and $\Omega(v(\tau), y(\tau)) = 0$ implies that $\frac{d\Omega}{d\tau}|_{(12)} = 0$. According to the Divisor Theorem, there exists a polynomial $H(v, y) = p_0(v) + q_0(v)y$ such that

$$\frac{d\Omega}{d\tau}|_{(12)} = H(v, y)\Omega(v, y), \tag{14}$$

that is

$$\begin{aligned} &\sum_{i=0}^{1} a_i'(v)vy^{i+1} + \sum_{i=0}^{1} a_i(v)iy^{i-1}(-svy + \frac{m-1}{m}y^2 - kmv^2(1-v)(v-\alpha)) \\ &= [a_0(v) + a_1(v)y][p_0(v) + q_0(v)y] \end{aligned} \tag{15}$$

Equating the coefficients of $y_i(i = 2, 1, 0)$ on both sides of (15), we obtain that

$$a_1'(v)v + \frac{m-1}{m}a_1(v) = q_0(v)a_1(v), \tag{16a}$$

$$a_0'(v)v - sa_1(v)v = a_0(v)q_0(v) + a_1(v)p_0(v), \tag{16b}$$

$$-kma_1(v)v^2(1-v)(v-\alpha) = a_0(v)p_0(v). \tag{16c}$$

Next, we will deal with the case $m = 1$, $m = 2$, and $m > 2$ respectively.

Case I: $m = 1$. Taking $m = 1$ in $(16a)$ and noting that $a_1(v)$ and $q_0(v)$ are polynomials, we deduce that $q_0(v)$ is a constant. For simplicity, we take $q_0(v) = 0$, and hence $a_1(v)$ is a constant. Again for simplicity, we take $a_1(v) = 1$. Therefore, (16) can be rewritten as

$$a_1(v) = 1, \tag{17a}$$

$$a_0'(v)v - sv = p_0(v), \tag{17b}$$

$$-kv^2(1-v)(v-\alpha) = a_0(v)p_0(v). \tag{17c}$$

Balancing the degrees of $a_0(v)$ and $p(v)$, we deduce that deg $a_0(v) =$ deg $p_0(v) = 2$. This together with $(17b)$ imply that

$$p_0(v) = A_0v^2 + B_0v, \tag{18}$$

where A_0 and B_0 are constants determined later.

Substituting (18) into $(17b)$, we get

$$a_0(v) = \frac{A_0}{2}v^2 + (B_0 + s)v. \tag{19}$$

Substituting (18) and (19) into $(17c)$ and setting all the coefficients of v to be zero, then we obtain a system of nonlinear equations and by solving it, we obtain

$$A_0 = \sqrt{2k}, \quad B_0 = -\sqrt{2k}, \quad s = (1 - \frac{\alpha}{2})\sqrt{2k}, \tag{20a}$$

$$A_0 = \sqrt{2k}, \quad B_0 = -\alpha\sqrt{2k}, \quad s = (\alpha - \frac{1}{2})\sqrt{2k}, \tag{20b}$$

$$A_0 = -\sqrt{2k}, \quad B_0 = \sqrt{2k}, \quad s = (\frac{\alpha}{2} - 1)\sqrt{2k}, \tag{20c}$$

$$A_0 = -\sqrt{2k}, \quad B_0 = \alpha\sqrt{2k}, \quad s = (\frac{1}{2} - \alpha)\sqrt{2k}. \tag{20d}$$

Using $(20a)$ in (13), we obtain

$$y = -\frac{\sqrt{2k}}{2}v^2 + \frac{\alpha\sqrt{2k}}{2}v. \tag{21}$$

Using this first integral method, the second-order ordinary differential equation (10) reduces to

$$\frac{dv}{d\xi} = -\frac{\sqrt{2k}}{2}v^2 + \frac{\alpha\sqrt{2k}}{2}v. \tag{22}$$

From this, we easily obtain the exact traveling wave solutions to (1) as follows:

$$u(x,t) = \frac{\alpha}{1 + \xi_0 e^{-\frac{\alpha\sqrt{2k}}{2}x + k\alpha(1-\frac{\alpha}{2})t}}, \tag{23a}$$

where ξ_0 is an arbitrary constant.

Similarly, as for the case of $(20b) - (20d)$, the exact solutions are expressed respectively

$$u(x,t) = \frac{1}{1 + \xi_0 e^{-\frac{\sqrt{2k}}{2}x + k(\alpha-\frac{1}{2})t}}, \tag{23b}$$

$$u(x,t) = \frac{\alpha}{1 + \xi_0 e^{\frac{\alpha\sqrt{2k}}{2}x + k\alpha(1-\frac{\alpha}{2})t}}, \tag{23c}$$

$$u(x,t) = \frac{1}{1 + \xi_0 e^{\frac{\sqrt{2k}}{2}x + k(\alpha-\frac{1}{2})t}}, \tag{23d}$$

We remark that if we take $\xi_0 = 0$ in $(23a) - (23d)$, we easily find the two trivial equilibrium solutions $u = 1$ and $u = \alpha$ to (1). Moreover, by taking $\xi_0 = 1$ and $k = 1$ in $(23b)$, it is to see that the solution is exactly the same as obtained by adomain decomposition method [18] and the homotopy perturbation method [19]. These solutions are all new exact solutions.

Case II: $m = 2$. Taking $m = 2$ in $(16a)$ and noting that $a_1(v)$ and $q_0(v)$ are polynomials, we deduce that $q_0(v)$ is a constant. For simplicity, we take $q_0(v) = \frac{1}{2}$, and hence $a_1(v)$ is a constant. Again for simplicity, we take $a_1(v) = 1$. Therefore, (16) can be rewritten as

$$a_1(v) = 1, \tag{24a}$$

$$a_0'(v)v - sv = \frac{1}{2}a_0(v) + p_0(v), \tag{24b}$$

$$-2kv^2(1-v)(v-\alpha) = a_0(v)p_0(v). \tag{24c}$$

Balancing the degrees of $a_0(v)$ and $p(v)$, we deduce that deg $a_0(v) =$ deg $p_0(v) = 2$. This together with $(24c)$ imply that

$$a_0(v) = A_1v^2 + B_1v, \tag{25}$$

where A_1 and A_2 are constants determined later. Substituting (25) into $(24b) - (24c)$ and setting all the coefficients of v to be zero, then we obtain a system of nonlinear equations and by solving it, we obtain

$$A_1 = \sqrt{\frac{4k}{3}}, \quad B_1 = -\sqrt{\frac{4k}{3}}, \quad s = (\alpha - \frac{1}{3})\sqrt{3k}, \tag{26a}$$

$$A_1=\sqrt{\frac{4k}{3}},\quad B_1=-\alpha\sqrt{\frac{4k}{3}},\quad s=(1-\frac{\alpha}{3})\sqrt{3k}, \tag{26b}$$

$$A_1=-\sqrt{\frac{4k}{3}},\quad B_1=\sqrt{\frac{4k}{3}},\quad s=(\frac{1}{3}-\alpha)\sqrt{3k}, \tag{26c}$$

$$A_1=-\sqrt{\frac{4k}{3}},\quad B_1=\alpha\sqrt{\frac{4k}{3}},\quad s=(\frac{\alpha}{3}-1)\sqrt{3k}. \tag{26d}$$

By virtue of $(26a)-(26d)$, (8) and (10), we easily get the exact traveling wave solutions to (1).

$$u(x,t)=\pm\left(\frac{1}{1+\xi_0 e^{-\sqrt{\frac{4k}{3}}x+(2\alpha-\frac{2}{3})kt}}\right)^{\frac{1}{2}}, \tag{27a}$$

$$u(x,t)=\pm\left(\frac{\alpha}{1+\xi_0 e^{-\alpha\sqrt{\frac{4k}{3}}x+2\alpha(1-\frac{\alpha}{3})kt}}\right)^{\frac{1}{2}}, \tag{27b}$$

$$u(x,t)=\pm\left(\frac{1}{1+\xi_0 e^{\sqrt{\frac{4k}{3}}x+(2\alpha-\frac{2}{3})kt}}\right)^{\frac{1}{2}}, \tag{27c}$$

$$u(x,t)=\pm\left(\frac{\alpha}{1+\xi_0 e^{\alpha\sqrt{\frac{4k}{3}}x+2\alpha(1-\frac{\alpha}{3})kt}}\right)^{\frac{1}{2}}. \tag{27d}$$

Here we remark that $u(x,t)$ may be the complex traveling wave solutions and all these solutions are new exact solutions.

Case III: $m>2$. From $(23a)-(23d)$ and $(27a)-(27d)$, we guess that the solutions to (1) has the following form:

$$u(x,t)=\left(\frac{1}{A_2+A_3e^{A_4\xi}}\right)^{\frac{1}{m}}, \tag{28}$$

where $A_i(i=2,3,4)$ are constants determined later. In the following, we apply the trial function method to get the exact solutions to (1). Taking $z=e^{A_4\xi}$, substituting it into (1), and setting all the coefficients of z to be zero, then we obtain a system of nonlinear equations and by solving it, we obtain

$$A_2=1,\quad A_4=\sqrt{\frac{km}{2}},\quad s=(\frac{1}{2}-\alpha)\sqrt{2km}, \tag{29a}$$

$$A_2=1,\quad A_4=-\sqrt{\frac{km}{2}},\quad s=-(\frac{1}{2}-\alpha)\sqrt{2km}, \tag{29b}$$

$$A_2=\frac{1}{\alpha},\quad A_4=\alpha\sqrt{\frac{km}{2}},\quad s=\sqrt{2km}(\frac{\alpha}{2}-1), \tag{29c}$$

$$A_2=\frac{1}{\alpha},\quad A_4=-\alpha\sqrt{\frac{km}{2}},\quad s=-\sqrt{2km}(\frac{\alpha}{2}-1), \tag{29d}$$

where A_3 can be an arbitrary constant.

Thus, the exact solutions for (1) are given as follows:

$$u(x,t)=\left(\frac{1}{1+\xi_0 e^{\sqrt{\frac{km}{2}}x+km(\alpha-\frac{1}{2})t}}\right)^{\frac{1}{m}}, \tag{30a}$$

$$u(x,t)=\left(\frac{1}{1+\xi_0 e^{-\sqrt{\frac{km}{2}}x+km(\alpha-\frac{1}{2})t}}\right)^{\frac{1}{m}}, \tag{30b}$$

$$u(x,t)=\left(\frac{\alpha}{1+\xi_0 e^{\alpha\sqrt{\frac{km}{2}}x+km(\alpha-\frac{\alpha^2}{2})t}}\right)^{\frac{1}{m}}, \tag{30c}$$

$$u(x,t)=\left(\frac{\alpha}{1+\xi_0 e^{-\alpha\sqrt{\frac{km}{2}}x+km(\alpha-\frac{\alpha^2}{2})t}}\right)^{\frac{1}{m}}, \tag{30d}$$

where ξ_0 is an arbitrary constant.

4 Conclusion

In this work, the first integral method and the trial function method were applied successfully for solving a family of nonlinear partial differential equations of Fisher type. Thus, we conclude that the proposed method can be extended to solve the nonlinear problems arising in the theory of solitons and other areas.

Acknowledgments

This work was partially supported by NSF (No: 10976026 and No: 10871061) of China, the Research Foundation of Education Bureau of Hunan Province, China (Grant No. 11C0628) and Foundation of Hunan Institute of Science and Technology (Grant No. 2011Y49). Part of this work was done while the first author was visiting School of Mathematics at Georgia Institute of Technology. The first author expresses his gratitude to Prof. Ronghua Pan for his hospitality and help.

References

[1] E. G. Fan. Extended tanh-function method and its applications to nonlinear equations. *Phys. Lett. A.*, 277(2000):212-218.

[2] E. G. Fan. Traveling wave solutions for generalized Hirota-Satsuma coupled KdV systems. *Z Naturforsch A*, 56(2001):312-318.

[3] M. Matinfar, M. Ghanbari. Solving the Fisher's Equation by Means of Variational Iteration Method. *Int. J. Contemp. Math. Sciences*, 4(7)(2009):343-348.

[4] M. Matinfar, M. Ghanbari. The application of the modied variational iteration method on the generalized Fisher's equation. *J. Appl. Math. Comput*, 31 (2009):165-175.

[5] Y. X. Xie, J. S. Tang. New Solitary Wave Solutions to the KdV-Burgers Equation. *Int. J. Theor. Phys.*,44(2005):293-301.

[6] Y. X. Xie, J. S. Tang. A unified trial function method in finding the explicit and exact solutions to three NPDEs. *Physica Scripta*, 74(2006):197-200.

[7] E. Fan, Y. H. Benny. Double periodic solutions with Jacobi elliptic functions for two generalized Hirota-Satsuma coupled KdV systems. *Phys. Lett. A*, 292(2002):335-337.

[8] Z. Y. Zhang. New Exact Traveling Wave Solutions for the Nonlinear Klein-Gordon Equation . *Turkish Journal of Physics*, 32(2008):235-240.

[9] Z. Y. Zhang, Y. X. Li, Z. H. Liu. New exact solutions to the perturbed nonlinear Schrodinger's equation with Kerr law nonlinearity via modified trigonometric function series method . *Communications in Nonlinear Science and Numerical Simulation*, 16(8)(2011):3097-3106.

[10] Z. Y. Zhang, Z. H. Liu, X. J. Miao. New exact solutions to the perturbed nonlinear Schrodinger's equation with Kerr law nonlinearity . A*ppl. Math. Comput.*, 216(2010):3064-3072.

[11] Z. Y. Zhang, Z. H. Liu, X. J. Miao. Qualitative analysis and traveling wave solutions for the perturbed nonlinear Schrodinger's equation with Kerr law nonlinearity . *Phys. Lett. A* ,375(2011):1275-1280.

[12] X. J. Miao, Z. Y. Zhang, The modified G'/G-expansion method and traveling wave solutions of nonlinear the perturbed nonlinear Schrodinger's equation with Kerr law nonlinearity, to appear in Communications in Nonlinear Science and Numerical Simulation (2011).

[13] Y. X. Xie, J. S. Tang. The superposition method in seeking the solitary wave solutions to the KdV-Burgers equation. *Pramana-Journal of Physics*,66(2006):479-483.

[14] Z. S. Feng. On explicit exact solutions to the compound Burgers-KdV equation. *Phys. Lett. A*, 293 (2002):57-66.

[15] Z. S. Feng. The first integer method to study the Burgers- Korteweg-de Vries equation. *Phys. Lett. A: Math Gen*, 35(2002):343-349.

[16] Z. S. Feng, X. H. Wang. The first integral method to the two-dimensional Burgers-Korteweg-de Vries equation. *Phys. Lett. A* ,308(2003):173-178.

[17] T. Kawahara, M. Tanaka. Interactions of traveling fronts: an exact solution of a nonlinear diffusion equation. *Phys. Lett. A* ,97(2002):311-314.

[18] A. Wazwaz, A. Gorguis. An analytic study of Fisher's equation by using Adomian decomposition method . *Appl. Math. Comput.*, 154(2004):609-620.

[19] M. Matinfar, M. Ghanbari. Homotopy Perturbation Method for the Fisher's Equation and Its Generalized . *International Journal of Nonlinear Science*, 4 (2009):448-455.

[20] T. R. Ding, C. Z. Li. Ordinary Differential Equations, Peking University Press, Peking, 1996.