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Weak and Strong Convergence Theorems for Finite Family of Nonself Asymptotically Nonexpansive Mappings in Banach Space

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Abstract: In this paper, several weak and strong convergence theorems are established for a new modified iterations with errors for finite family of nonself asymptotically nonexpansive mappings in Banach spaces. Mann-type, Ishikawa-type and Noor-type iterations are covered by the new iteration scheme. Our convergence theorems improve, unify and generalize many important results in the current literatures.

Keywords: asymptotically nonexpansive mappings; Kadec-Klee property; Opial's condition; Fréchet differentiable norm; uniformly convex Banach spaces; common fixed points

AMS classification: 47H10; 47H09; 46B20

1 Introduction

Let C be a nonempty convex subset of Banach space X. In 1972, Goeble and Kirk introduced the notion of an asymptotically nonexpansive map[11]. A map $T : C \to C$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, +\infty)$ with $\lim_{n\to+\infty} k_n = 1$, such that

$$\|T^n x - T^n y\| \le k_n \|x - y\|$$

for all $x, y \in C$ and for all $n \ge 1$. In particular, if $k_n \equiv 1$ for all $n \ge 1$, it becomes nonexpansive.

Fixed-point iteration processes for nonexpansive and asymptotically nonexpansive mappings including Mann-type and Ishikawa-type iterations have been studied extensively by many authors (see, for example, [1,4-10,12-13,15-20,22-32] and the references cited therein). A classical iteration for a nonexpansive mapping was introduced in 1953 by Mann, which is well-known as Manns iteration process and is defined as follows[18]:

$$x_1 \in C, \qquad x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \qquad n \ge 1,$$

where the sequence $\{\alpha_n\}$ is chosen in [0, 1]. Later, Ishikawa[12] enlarged and improved Manns iteration to the new iteration method, it is often cited as Ishikawas iteration process and is defined recursively by $x_1 \in C$ and

$$\begin{aligned} x_{n+1} &= \alpha_n x_n + (1 - \alpha_n) T y_n; \\ y_n &= \beta_n x_n + (1 - \beta_n) T x_n, \end{aligned}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in the interval [0, 1]. The weak convergence of the Ishikawa sequence for a nonexpansive map in a uniformly convex Banach space with the Opial's condition or with a Fréchet differentiable norm has been studied extensively by various authors[7,13,16,24,28].

Bose, in 1978, initiated the study of iterative construction of asymptotically nonexpansive maps[1]. In 1991, Schu[23] considered the following modified Mann iteration process for an asymptotically nonexpansive map T on C and $\{\alpha_n\}$ a sequence in $[0, 1]$:

$$x_1 \in C, \qquad x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T^n x_n, \qquad n \ge 1.$$

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In 1994, Tan and Xu[27] studied the modified Ishikawa iteration process for an asymptotically nonexpansive map T on C, $\{\alpha_n\}$ in $[0,1]$, $\{\beta_n\}$ bounded away from 1 and the scheme described as: $x_1 \in C$;

$$\begin{aligned} x_{n+1} &= \alpha_n x_n + (1-\alpha_n)T^n y_n; \\ y_n &= \beta_n x_n + (1-\beta_n)T^n x_n, \end{aligned}$$

Noor, in 2000, introduced a three-step iterative scheme and studied the approximate solutions of variational inclusion in Hilbert spaces[20]. Later, Xu and Noor[31], Cho, Zhou and Guo[8], Liu and Kang[17] studied weak and strong convergence theorems for the three-step Noor iterations with errors for asymptotically nonexpansive mappings in a uniformly convex Banach space which satisfies Opial's condition or whose norm is Fréchet differentiable.

A generalization of Mann and Ishikawa iterative schemes was given by Das and Debata[9], they gave the strong convergence theorem for a pair of quasi-nonexpansive mappings in a strictly convex Banach space. Takahashi and Tamura[26], Shahzad[24] dealt with the iterative scheme for a pair of nonexpansive and asymptotically nonexpansive mappings in a uniformly convex Banach space. In 2006, Plubtieng, Wangkeeree and Punpaeng[22] studied a class of three-step iterative scheme, for three asymptotically nonexpansive mappings, in uniformly convex Banach space with the Opial's condition. In 2007, Fukhar-ud-dina and Khan[10] studied the scheme for three nonexpansive mappings in uniformly convex Banach space which has the Opial's condition or which has a Fréchet differentiable norm or whose dual space has the Kadec-Klee property. Also in 2007, Chidume and Bashir Ali[4] introduced the iterative scheme for a family of finite asymptotically nonexpansive mappings and obtained the weak and strong convergence theorems in Banach space whose dual space satisfies the Kadec-Klee property.

In most of these papers, the map T has been assumed to map C into itself. If, however, C is a proper subset of Banach space X and T maps C into X (as is the case in many applications), then $\{x_n\}$ may not be well defined. One method that has been used to overcome this is to introduce a retraction $P : X \to C$. Recent results on approximation of fixed points of nonexpansive and asymptotically nonexpansive nonself mappings can be found in ([5-7,24,27,29,30] and references contained therein).

The concept of nonself asymptotically nonexpansive mappings was introduced by Chidume et al.[6] as an important generalization of asymptotically nonexpansive self mappings.

Definition 1.1 *Let C be a nonempty subset of a Banach space X. Let $P : X \to C$ be a nonexpansive retraction of X onto C. A nonself mapping $T : C \to X$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1,+\infty)$ with $\lim_{n\to+\infty} k_n = 1$, such that for all $x, y \in C$ and for all $n \geq 1$,*

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq k_n\|x-y\|.$$

In 2003, Chidume, Ofoedu and Zegeye[6] introduced the following modified Mann iteration process and got the strong and weak convergence theorems for asymptotically nonexpansive nonself mapping:

$$x_1 \in C, \qquad x_{n+1} = P[\alpha_n x_n + (1-\alpha_n)T(PT)^{n-1}x_n], \qquad n \geq 1.$$

Recently, Wang[30] generalized the above iteration process as follows: $x_1 \in C$,

$$\begin{aligned} x_{n+1} &= P[\alpha_n x_n + (1-\alpha_n)T_1(PT_1)^{n-1}y_n]; \\ y_n &= P[\beta_n x_n + (1-\beta_n)T_2(PT_2)^{n-1}x_n]. \end{aligned}$$

In 2007, Chidume, Bashir Ali[5] introduced the following iteration process for finite families of nonself asymptotically nonexpansive mappings:

$$\begin{aligned} & x_1 \in C \\ x_{n+1} &= P[\alpha_{1n}x_n + (1-\alpha_{1n})T_1(PT_1)^{n-1}y_{n+m-2}]; \\ y_{n+m-2} &= P[\alpha_{2n}x_n + (1-\alpha_{2n})T_2(PT_2)^{n-1}y_{n+m-3}]; \\ \cdots &\cdots \cdots \\ y_n &= P[\alpha_{mn}x_n + (1-\alpha_{mn})T_m(PT_m)^{n-1}x_n]. \end{aligned}$$

They proved the strong convergence theorems in uniformly convex Banach spaces and gave the weak convergence theorem in uniformly convex Banach spaces that satisfy Opial's condition or have Fréchet differentiable norm. They also gave the weak convergence theorem for nonslef nonexpansive mappings in uniformly convex Banach spaces whose dual space have the Kadec-Klee property (see [5]).

It is our purpose in this paper to study the following iteration process with errors for approximating common fixed points of finite family of nonself asymptotically nonexpansive mappings:

$$\begin{array}{rcl} & & x_1 \in C \\ x_{n+1} & = & P[\alpha_n^{(1)} x_n + \beta_n^{(1)} T_1 (PT_1)^{n-1} y_n^{(N-2)} + \gamma_n^{(1)} u_n^{(1)}]; \\ y_n^{(N-2)} & = & P[\alpha_n^{(2)} x_n + \beta_n^{(2)} T_2 (PT_2)^{n-1} y_n^{(N-3)} + \gamma_n^{(2)} u_n^{(2)}]; \\ \cdots & \cdots & \cdots \\ y_n^{(1)} & = & P[\alpha_n^{(N-1)} x_n + \beta_n^{(N-1)} T_{N-1} (PT_{N-1})^{n-1} y_n^{(0)} + \gamma_n^{(N-1)} u_n^{(N-1)}]; \\ y_n^{(0)} & = & P[\alpha_n^{(N)} x_n + \beta_n^{(N)} T_N (PT_N)^{n-1} x_n + \gamma_n^{(N)} u_n^{(N)}]. \end{array}$$

In section 3, we first establish some weak convergence theorems of the above iterative scheme for finite families of nonself asymptotically nonexpansive mappings in a uniformly convex Banach space that satisfies Opial's condition, or has Fréchet differentiable norms, or whose dual space has the Kadec-Klee property. We also prove several strong convergence theorems in uniformly convex Banach spaces if one member of the finite family of asymptotically nonexpansive mappings $\{T_i\}_{i=1}^N$ satisfies a condition weaker than semicompactness. Our results extend and improve the recently announced ones [4-8,10,15,16,22,24,26,27,30,31] and many others.

2 Preliminaries

Let C be a nonempty bounded closed convex subset of a Banach space X. Let X^* be the dual of X, then the value of $x^* \in X^*$ at $x \in X$ will be denoted by $\langle x, x^* \rangle$ and we associate the set

$$J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}.$$

Using the Hahn-Banach theorem, it is immediately clear that $J(x) \neq \emptyset$ for any $x \in X$. Then the multi-valued operator $J : X \mapsto X^*$ is called the normalized duality mapping of X. It is well known that if X^* is strictly convex, then J is a single valued mapping. We say that X has a Fréchet differentiable norm, i.e., for each $x \neq 0$, $\lim_{t\to 0}(\|x+ty\| - \|x\|)/t$ exists uniformly in $y \in B_r$, where $B_r = \{z \in X : \|z\| \leq r\}, r > 0$. It is easy to verify that X has Fréchet differentiable norm if and only if for any B_r and $x \in X$,

$$\lim_{t\to 0}(2t)^{-1}(\|x+ty\|^2 - \|x\|^2) = \langle y, J(x) \rangle$$

uniformly in $y \in B_r$. Recall that a Banach space X is said to be uniformly convex if for each $\varepsilon \in [0, 2]$, the modulus of convexity of X given by:

$$\delta(\varepsilon) = \inf\{1 - \frac{1}{2}\|x+y\| : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \varepsilon\},$$

satisfies the inequality $\delta(\varepsilon) > 0$ for all $\varepsilon > 0$. We say that X has the Kadec-Klee property if for every sequence $\{x_n\} \subset X$, whenever $x_n \rightharpoonup x$ with $\|x_n\| \to \|x\|$, it follows that $x_n \to x$. We would like to remark that a reflexive Banach space X with a Fréchet differentiable norm implies that its dual X^* has Kadec-Klee property, while the converse implication fails [14].

A subset C of X is said to be retract if there exists a continuous mapping $P : X \to C$ such that $Px = x$ for all $x \in C$. Every closed convex subset of a uniformly convex Banach space is a retract. A mapping $P : X \to X$ is said to be a retraction if $P^2 = P$. It follows that if a map P is a retraction, then $Py = y$ for all y in the range of P.

Recall that a Banach space X is said to satisfy Opial's condition[21] if $x_n \rightharpoonup x$ and $x \neq y$ implies that

$$\limsup_{n\to+\infty} \|x_n - x\| < \limsup_{n\to+\infty} \|x_n - y\|.$$

The following lemmas are needed to prove our main results in next section.

Lemma 2.1 *Let $\{a_n\}$ be a nonnegative number sequence, then*

$$\sum_{n=1}^{+\infty} a_n < +\infty \Longleftrightarrow \prod_{n=1}^{+\infty}(1+a_n) < +\infty.$$

In this case,

$$\lim_{n\to+\infty}\lim_{m\to+\infty}\sum_{i=n}^{n+m-1} a_i = \lim_{n\to+\infty}\sum_{i=n}^{+\infty} a_i = 0$$

and

$$\lim_{n\to+\infty}\lim_{m\to+\infty}\prod_{i=n}^{n+m-1}(1+a_i) = \lim_{n\to+\infty}\prod_{i=n}^{+\infty}(1+a_i) = 1.$$

Proof. This is easily checked, once the conclusion is given. ■

Lemma 2.2 [28] *Let the nonnegative number sequences $\{a_n\}$, $\{c_n\}$ and $\{w_n\}$ satisfy*

$$c_{n+1} \le (1+a_n)c_n + w_n, \qquad n = 1, 2...$$

If $\sum_{n=1}^{+\infty} a_n < +\infty$, $\sum_{n=1}^{+\infty} w_n < +\infty$, then $\lim_{n\to+\infty} c_n$ exists.

Lemma 2.3 [3] *Let X be a Banach space and J be the normalized duality mapping. Then for given $x, y \in X$ and $j(x+y) \in J(x+y)$, we have*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x+y)\rangle.$$

Lemma 2.4 [23] *Suppose that X is a uniformly convex Banach space and for all positive integers n, $0 < p \le t_n \le q < 1$. If $\{x_n\}$ and $\{y_n\}$ are two sequences of X such that $\limsup_{n\to+\infty}\|x_n\| \le r$, $\limsup_{n\to+\infty}\|y_n\| \le r$ and*

$$\lim_{n\to+\infty}\|t_n x_n + (1-t_n)y_n\| = r$$

hold for some $r \ge 0$. Then $\lim_{n\to+\infty}\|x_n - y_n\| = 0$.

Lemma 2.5 [2] *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space X. Then there exists a strictly increasing continuous convex function $\gamma : [0, +\infty) \mapsto [0, +\infty)$ with $\gamma(0) = 0$ such that*

$$\|T[\lambda x + (1-\lambda y)] - [\lambda Tx + (1-\lambda)Ty]\| \le \gamma^{-1}(\|x-y\| - \|Tx - Ty\|),$$

for all $\lambda \in [0,1]$, $x, y \in C$ and all nonexpansive mapping $T : C \mapsto C$.

Lemma 2.6 (Demiclosed principle for nonself-map[6]) *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and $T : C \to X$ an asymptotically nonexpansive nonself mapping. Then $I - T$ is demiclosed at 0, i.e., if $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$, then $x \in F(T)$, where $F(T)$ is the set of fixed points of T.*

3 Main Results

In this section, let X be a uniformly convex Banach space and C be a nonempty bounded closed convex subset of X. Let $P : X \to C$ be a nonexpansive retraction from X onto C. Let $T_1, T_2, ..., T_N : C \to X$ be finite family of nonself asymptotically nonexpansive mappings ($N \ge 2$), then we can suppose that

$$\|T_i(PT_i)^{n-1}x - T_i(PT_i)^{n-1}y\| \le k_n\|x-y\|, \qquad \forall\, x, y \in C, \quad i = 1, 2, \cdots, N.$$

where $k_n \ge 1$, and $\lim_{n\to+\infty} k_n = 1$. For a given $x_1 \in C$, we can define the sequence $\{x_n\} \subset C$ by

$$\begin{array}{lll} x_{n+1} & = & P[\alpha_n^{(1)}x_n + \beta_n^{(1)}T_1(PT_1)^{n-1}y_n^{(N-2)} + \gamma_n^{(1)}u_n^{(1)}]; \\ y_n^{(N-2)} & = & P[\alpha_n^{(2)}x_n + \beta_n^{(2)}T_2(PT_2)^{n-1}y_n^{(N-3)} + \gamma_n^{(2)}u_n^{(2)}]; \\ \cdots & \cdots & \cdots \\ y_n^{(1)} & = & P[\alpha_n^{(N-1)}x_n + \beta_n^{(N-1)}T_{N-1}(PT_{N-1})^{n-1}y_n^{(0)} + \gamma_n^{(N-1)}u_n^{(N-1)}]; \\ y_n^{(0)} & = & P[\alpha_n^{(N)}x_n + \beta_n^{(N)}T_N(PT_N)^{n-1}x_n + \gamma_n^{(N)}u_n^{(N)}]. \end{array} \qquad (3.\ 1)$$

where $\{\alpha_n^{(i)}\}, \{\beta_n^{(i)}\}, \{\gamma_n^{(i)}\}$ are in $[0,1]$ with $0 < p \le \alpha_n^{(i)}, \beta_n^{(i)} \le q < 1$, $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$, $\sum_{n=1}^{+\infty} \gamma_n^{(i)} < +\infty$ and $\{u_n^{(i)}\}$ are bounded sequences in X, $i = 1, 2, \cdots, N$.

We start our investigation with the following lemmas, which are preparation for the proofs of the main results of this section. In the following, we assume that $\sum_{n=1}^{+\infty}(k_n - 1) < +\infty$ and the set of common fixed points of $\{T_1, T_2, ..., T_N\}$ is nonempty, i.e.,

$$\cap_{i=1}^N F(T_i) = \cap_{i=1}^N \{x \in C : T_i x = x\} \neq \emptyset.$$

Lemma 3.1

$$\lim_{n\to+\infty} \|x_n - f\| = r$$

exists for all $f \in \cap_{i=1}^N F(T_i)$.

Proof. Set $L = \sup\{k_n : n \ge 1\}$ and

$$M = diamC + \sup\{\|u_n^{(i)} - x\| : x \in C, i = 1, 2, ..., N, n \ge 1\},$$

we have the following estimates:

$$\begin{aligned}
& \|y_n^{(0)} - f\| \\
= \ & \|P[\alpha_n^{(N)} x_n + \beta_n^{(N)} T_N (PT_N)^{n-1} x_n + \gamma_n^{(N)} u_n^{(N)}] - f\| \\
\le \ & \|[\alpha_n^{(N)} x_n + \beta_n^{(N)} T_N (PT_N)^{n-1} x_n + \gamma_n^{(N)} u_n^{(N)}] - f\| \\
\le \ & \alpha_n^{(N)} \|x_n - f\| + \beta_n^{(N)} \|T_N (PT_N)^{n-1} x_n - f\| + \gamma_n^{(N)} \|u_n^{(N)} - f\| \\
\le \ & (\alpha_n^{(N)} + \beta_n^{(N)} k_n) \|x_n - f\| + M\gamma_n^{(N)} \\
\le \ & k_n \|x_n - f\| + M\gamma_n^{(N)}.
\end{aligned}$$

Hence

$$\begin{aligned}
& \|y_n^{(1)} - f\| = \|P[\alpha_n^{(N-1)} x_n + \beta_n^{(N-1)} T_{N-1}(PT_{N-1})^{n-1} y_n^{(0)} + \gamma_n^{(N-1)} u_n^{(N-1)}] - f\| \\
\le \ & \alpha_n^{(N-1)} \|x_n - f\| + \beta_n^{(N-1)} \|T_{N-1}(PT_{N-1})^{n-1} y_n^{(0)} - f\| + \gamma_n^{(N-1)} \|u_n^{(N-1)} - f\| \\
\le \ & \alpha_n^{(N-1)} \|x_n - f\| + \beta_n^{(N-1)} k_n \|y_n^{(0)} - f\| + M\gamma_n^{(N-1)} \\
\le \ & \alpha_n^{(N-1)} \|x_n - f\| + \beta_n^{(N-1)} k_n (k_n \|x_n - f\| + M\gamma_n^{(N)}) + M\gamma_n^{(N-1)} \\
= \ & (\alpha_n^{(N-1)} + \beta_n^{(N-1)} k_n^2) \|x_n - f\| + \beta_n^{(N-1)} k_n M \gamma_n^{(N)} + M\gamma_n^{(N-1)} \\
\le \ & k_n^2 \|x_n - f\| + LM(\gamma_n^{(N)} + \gamma_n^{(N-1)})
\end{aligned}$$

and

$$\begin{aligned}
& \|y_n^{(2)} - f\| \\
\le \ & \alpha_n^{(N-2)} \|x_n - f\| + \beta_n^{(N-2)} \|T_{N-2}(PT_{N-2})^{n-1} y_n^{(1)} - f\| + \gamma_n^{(N-2)} \|u_n^{(N-2)} - f\| \\
\le \ & \alpha_n^{(N-2)} \|x_n - f\| + \beta_n^{(N-2)} k_n \|y_n^{(1)} - f\| + \gamma_n^{(N-2)} \|u_n^{(N-2)} - f\| \\
\le \ & \alpha_n^{(N-2)} \|x_n - f\| + \beta_n^{(N-2)} k_n [k_n^2 \|x_n - f\| + LM(\gamma_n^{(N)} + \gamma_n^{(N-1)})] + M\gamma_n^{(N-2)} \\
\le \ & (\alpha_n^{(N-2)} + \beta_n^{(N-2)} k_n^3) \|x_n - f\| + L^2 M(\gamma_n^{(N)} + \gamma_n^{(N-1)}) + M\gamma_n^{(N-2)} \\
\le \ & k_n^3 \|x_n - f\| + L^2 M(\gamma_n^{(N)} + \gamma_n^{(N-1)} + \gamma_n^{(N-2)}), \\
\cdots \ & \cdots \\
& \|y_n^{(N-2)} - f\| = \|P[\alpha_n^{(2)} x_n + \beta_n^{(2)} T_2(PT_2)^{n-1} y_n^{(N-3)} + \gamma_n^{(2)} u_n^{(2)}] - f\| \\
\le \ & \|[\alpha_n^{(2)} x_n + \beta_n^{(2)} T_2(PT_2)^{n-1} y_n^{(N-3)} + \gamma_n^{(2)} u_n^{(2)}] - f\| \\
\le \ & \alpha_n^{(2)} \|x_n - f\| + \beta_n^{(2)} \|T_2(PT_2)^{n-1} y_n^{(N-3)} - f\| + \gamma_n^{(2)} \|u_n^{(2)} - f\| \\
\le \ & \alpha_n^{(2)} \|x_n - f\| + \beta_n^{(2)} k_n \|y_n^{(N-3)} - f\| + M\gamma_n^{(2)} \\
\le \ & \alpha_n^{(2)} \|x_n - f\| + \beta_n^{(2)} k_n [k_n^{N-2} \|x_n - f\| + L^{N-3} M(\gamma_n^{(N)} + \cdots + \gamma_n^{(3)})] + M\gamma_n^{(2)} \\
\le \ & k_n^{N-1} \|x_n - f\| + L^{N-2} M(\gamma_n^{(N)} + \cdots + \gamma_n^{(3)} + \gamma_n^{(2)}).
\end{aligned} \tag{3. 2}$$

Thus we have

$$\begin{aligned}
& \|x_{n+1} - f\| \\
\le \ & \|[\alpha_n^{(1)} x_n + \beta_n^{(1)} T_1(PT_1)^{n-1} y_n^{(N-2)} + \gamma_n^{(1)} u_n^{(1)}] - f\| \\
\le \ & \alpha_n^{(1)} \|x_n - f\| + \beta_n^{(1)} \|T_1(PT_1)^{n-1} y_n^{(N-2)} - f\| + \gamma_n^{(1)} \|u_n^{(1)} - f\| \\
\le \ & \alpha_n^{(1)} \|x_n - f\| + \beta_n^{(1)} k_n \|y_n^{(N-2)} - f\| + M\gamma_n^{(1)} \\
\le \ & k_n^N \|x_n - f\| + L^N M(\gamma_n^{(N)} + \cdots + \gamma_n^{(2)} + \gamma_n^{(1)})
\end{aligned} \tag{3. 3}$$

Set $a_n = k_n^N - 1 \geq 0$ and $w_n = \gamma_n^{(N)} + \gamma_n^{(N-1)} + \cdots + \gamma_n^{(1)}$, then $\sum_{n=1}^{+\infty} a_n < +\infty$, $\sum_{n=1}^{+\infty} w_n < +\infty$ and

$$\|x_{n+1} - f\| \leq (1 + a_n)\|x_n - f\| + L^N M w_n. \tag{3. 4}$$

By Lemma 2.2, we get that

$$\lim_{n\to+\infty} \|x_n - f\| = r$$

exists. This completes the proof. ■

Remark 3.1 *If $\gamma_n^{(i)} \equiv 0$ for all $i = 1, 2, ..., N$ and $n \geq 1$, then the boundedness of C is unnecessary. Because, in that case, we can use the same argument to prove that $\lim_{n\to+\infty} \|x_n - f\| = r$ exists. Hence the sequence $\{x_n\}$ is bounded.*

Lemma 3.2

$$\lim_{n\to+\infty} \|x_n - T_i x_n\| = 0, \qquad i = 1, 2, \cdots, N.$$

Proof. By the Lemma 3.1, we get that $\lim_{n\to+\infty} \|x_n - f\| = r$ exists. If $r = 0$, then it is obvious to see that the conclusion holds. In the following, we assume that $r > 0$. According to the proof of Lemma 3.1, we can get

$$\limsup_{n\to+\infty} \|y_n^{(j)} - f\| \leq r, \qquad j = 0, 1, \cdots, N-2.$$

Then for any $j = 0, 1, ..., N-2$,

$$\limsup_{n\to+\infty} \|T_{N-1-j}(PT_{N-1-j})^{n-1} y_n^{(j)} - f\| \leq r,$$

and hence the sequences $\{T_{N-1-j}(PT_{N-1-j})^{n-1} y_n^{(j)}\}_{n=1}^{+\infty}$ is bounded. By (3.3), we can get

$$\lim_{n\to+\infty} \|[\alpha_n^{(1)} x_n + \beta_n^{(1)} T_1(PT_1)^{n-1} y_n^{(N-2)} + \gamma_n^{(1)} u_n^{(1)}] - f\| = r.$$

We also can see

$$\limsup_{n\to+\infty} \|T_1(PT_1)^{n-1} y_n^{(N-2)} - f + \gamma_n^{(1)}(u_n^{(1)} - x_n)\| \leq r$$

and

$$\limsup_{n\to+\infty} \|x_n - f + \gamma_n^{(1)}(u_n^{(1)} - x_n)\| \leq r.$$

It follows from Lemma 2.4 and

$$\begin{aligned} r &= \lim_{n\to+\infty} \|[\alpha_n^{(1)} x_n + \beta_n^{(1)} T_1(PT_1)^{n-1} y_n^{(N-2)} + \gamma_n^{(1)} u_n^{(1)}] - f\| \\ &= \lim_{n\to+\infty} \|(1-\alpha_n^{(1)})[T_1(PT_1)^{n-1} y_n^{(N-2)} - f + \gamma_n^{(1)}(u_n^{(1)} - x_n)] + \\ &\quad \alpha_n^{(1)}[x_n - f + \gamma_n^{(1)}(u_n^{(1)} - x_n)] + \gamma_n^{(1)}[x_n - T_1(PT_1)^{n-1} y_n^{(N-2)}]\| \\ &= \lim_{n\to+\infty} \|(1-\alpha_n^{(1)})[T_1(PT_1)^{n-1} y_n^{(N-2)} - f + \gamma_n^{(1)}(u_n^{(1)} - x_n)] + \\ &\quad \alpha_n^{(1)}[x_n - f + \gamma_n^{(1)}(u_n^{(1)} - x_n)]\| \end{aligned}$$

that

$$\lim_{n\to+\infty} \|T_1(PT_1)^{n-1} y_n^{(N-2)} - x_n\| = 0. \tag{3. 5}$$

Combining it with

$$\begin{aligned} \|x_n - f\| &\leq \|x_n - T_1(PT_1)^{n-1} y_n^{(N-2)}\| + \|T_1(PT_1)^{n-1} y_n^{(N-2)} - f\| \\ &\leq \|x_n - T_1(PT_1)^{n-1} y_n^{(N-2)}\| + k_n \|y_n^{(N-2)} - f\|, \end{aligned}$$

we obtain

$$\liminf_{n\to+\infty} \|y_n^{(N-2)} - f\| \geq r.$$

Thus $\lim_{n\to+\infty}\|y_n^{(N-2)}-f\|=r$, according to (3.2), we have

$$\begin{aligned} r &= \lim_{n\to+\infty}\|[\alpha_n^{(2)}x_n+\beta_n^{(2)}T_2(PT_2)^{n-1}y_n^{(N-3)}+\gamma_n^{(2)}u_n^{(2)}]-f\| \\ &= \lim_{n\to+\infty}\|(1-\alpha_n^{(2)})[T_2(PT_2)^{n-1}y_n^{(N-3)}-f+\gamma_n^{(2)}(u_n^{(2)}-x_n)]+ \\ &\quad \alpha_n^{(2)}[x_n-f+\gamma_n^{(2)}(u_n^{(2)}-x_n)]+\gamma_n^{(2)}[x_n-T_2(PT_2)^{n-1}y_n^{(N-3)}]\| \\ &= \lim_{n\to+\infty}\|(1-\alpha_n^{(2)})[T_2(PT_2)^{n-1}y_n^{(N-3)}-f+\gamma_n^{(2)}(u_n^{(2)}-x_n)]+ \\ &\quad \alpha_n^{(2)}[x_n-f+\gamma_n^{(2)}(u_n^{(2)}-x_n)]\| \end{aligned}$$

Noting

$$\limsup_{n\to+\infty}\|T_2(PT_2)^{n-1}y_n^{(N-3)}-f+\gamma_n^{(2)}(u_n^{(2)}-x_n)\|\le r$$

and

$$\limsup_{n\to+\infty}\|x_n-f+\gamma_n^{(2)}(u_n^{(2)}-x_n)\|\le r,$$

by Lemma 2.4 again, we have

$$\lim_{n\to+\infty}\|T_2(PT_2)^{n-1}y_n^{(N-3)}-x_n\|=0. \tag{3. 6}$$

Similarly, we can get

$$\begin{aligned} \lim_{n\to+\infty}\|T_3(PT_3)^{n-1}y_n^{(N-4)}-x_n\| &= \cdots = \lim_{n\to+\infty}\|T_{N-1}(PT_{N-1})^{n-1}y_n^{(0)}-x_n\| \\ &= \lim_{n\to+\infty}\|T_N(PT_N)^{n-1}x_n-x_n\|=0. \end{aligned}$$

Therefore, by (3.1) and (3.5),

$$\lim_{n\to+\infty}\|x_{n+1}-x_n\|\le\lim_{n\to+\infty}\|\beta_n^{(1)}[T_1(PT_1)^{n-1}y_n^{(N-2)}-x_n]+\gamma_n^{(1)}(u_n^{(1)}-x_n)\|=0$$

and similarly,

$$\lim_{n\to+\infty}\|y_n^{(N-2)}-x_n\|=\cdots=\lim_{n\to+\infty}\|y_n^{(1)}-x_n\|=\lim_{n\to+\infty}\|y_n^{(0)}-x_n\|=0. \tag{3. 7}$$

It follows from the inequality (3.5) and

$$\begin{aligned} &\|T_1(PT_1)^{n-1}x_n-x_n\| \\ \le\ &\|T_1(PT_1)^{n-1}x_n-T_1(PT_1)^{n-1}y_n^{(N-2)}\|+\|T_1(PT_1)^{n-1}y_n^{(N-2)}-x_n\| \\ \le\ &k_n\|x_n-y_n^{(N-2)}\|+\|T_1(PT_1)^{n-1}y_n^{(N-2)}-x_n\| \end{aligned}$$

that

$$\lim_{n\to+\infty}\|T_1(PT_1)^{n-1}x_n-x_n\|=0.$$

Thus

$$\begin{aligned} &\|x_n-T_1x_n\| \\ \le\ &\|x_n-x_{n+1}\|+\|x_{n+1}-T_1(PT_1)^nx_{n+1}\| \\ &+\|T_1(PT_1)^nx_{n+1}-T_1(PT_1)^nx_n\|+\|T_1(PT_1)^nx_n-T_1x_n\| \\ \le\ &\|x_n-x_{n+1}\|+\|x_{n+1}-T_1(PT_1)^nx_{n+1}\|+k_{n+1}\|x_{n+1}-x_n\|+k_1\|(PT_1)^nx_n-x_n\| \\ \le\ &\|x_n-x_{n+1}\|+\|x_{n+1}-T_1(PT_1)^nx_{n+1}\| \\ &+k_{n+1}\|x_{n+1}-x_n\|+k_1\|T_1(PT_1)^{n-1}x_n-x_n\| \end{aligned}$$

which implies that

$$\lim_{n\to+\infty}\|T_1x_n-x_n\|=0.$$

Noting (3.6), (3.7) and

$$\begin{aligned} &\|T_2(PT_2)^{n-1}x_n-x_n\| \\ \le\ &\|T_2(PT_2)^{n-1}x_n-T_2(PT_2)^{n-1}y_n^{(N-3)}\|+\|T_2(PT_2)^{n-1}y_n^{(N-3)}-x_n\| \\ \le\ &k_n\|x_n-y_n^{(N-3)}\|+\|T_2(PT_2)^{n-1}y_n^{(N-3)}-x_n\|, \end{aligned}$$

we can see

$$\lim_{n\to+\infty}\|T_2(PT_2)^{n-1}x_n - x_n\| = 0.$$

Thus

$$\begin{aligned}
& \|x_n - T_2x_n\| \\
\le\ & \|x_n - x_{n+1}\| + \|x_{n+1} - T_2(PT_2)^n x_{n+1}\| \\
& +\|T_2(PT_2)^n x_{n+1} - T_2(PT_2)^n x_n\| + \|T_2(PT_2)^n x_n - T_2x_n\| \\
\le\ & (1+k_{n+1})\|x_n - x_{n+1}\| + \|x_{n+1} - T_2(PT_2)^n x_{n+1}\| + k_1\|T_2(PT_2)^{n-1}x_n - x_n\|
\end{aligned}$$

which implies that

$$\lim_{n\to+\infty}\|T_2x_n - x_n\| = 0.$$

Also, by the same argument, we can get

$$\lim_{n\to+\infty}\|T_3x_n - x_n\| = \cdots = \lim_{n\to+\infty}\|T_{N-1}x_n - x_n\| = 0$$

and

$$\begin{aligned}
& \|x_n - T_Nx_n\| \\
\le\ & \|x_n - x_{n+1}\| + \|x_{n+1} - T_N(PT_N)^n x_{n+1}\| \\
& +\|T_N(PT_N)^n x_{n+1} - T_N(PT_N)^n x_n\| + \|T_N(PT_N)^n x_n - T_Nx_n\| \\
\le\ & (1+k_{n+1})\|x_n - x_{n+1}\| + \|x_{n+1} - T_N(PT_N)^n x_{n+1}\| + k_1\|T_N(PT_N)^{n-1}x_n - x_n\|,
\end{aligned}$$

which implies

$$\lim_{n\to+\infty}\|T_Nx_n - x_n\| = 0.$$

This completes the proof. ■

Define the operator $W_n : C \to C$ by

$$\begin{aligned}
W_nx &= P[\alpha_n^{(1)}x + \beta_n^{(1)}T_1(PT_1)^{n-1}x^{(N-2)} + \gamma_n^{(1)}u_n^{(1)}]; \\
x^{(N-2)} &= P[\alpha_n^{(2)}x + \beta_n^{(2)}T_2(PT_2)^{n-1}x^{(N-3)} + \gamma_n^{(2)}u_n^{(2)}]; \\
\cdots & \cdots \cdots \\
x^{(1)} &= P[\alpha_n^{(N-1)}x + \beta_n^{(N-1)}T_{N-1}(PT_{N-1})^{n-1}x^{(0)} + \gamma_n^{(N-1)}u_n^{(N-1)}]; \\
x^{(0)} &= P[\alpha_n^{(N)}x + \beta_n^{(N)}T_N(PT_N)^{n-1}x + \gamma_n^{(N)}u_n^{(N)}],
\end{aligned}$$

where $x \in C$. Then $x_{n+1} = W_nx_n$ and for all $x, y \in C$, we have

$$\begin{aligned}
\|x^{(0)} - y^{(0)}\| &\le \alpha_n^{(N)}\|x-y\| + \beta_n^{(N)}\|T_N(PT_N)^{n-1}x - T_N(PT_N)^{n-1}y\| \\
&\le (\alpha_n^{(N)} + \beta_n^{(N)}k_n)\|x-y\| \le k_n\|x-y\|, \\
\|x^{(1)} - y^{(1)}\| &\le \alpha_n^{(N-1)}\|x-y\| + \beta_n^{(N-1)}\|T_{N-1}(PT_{N-1})^{n-1}x^{(0)} - T_{N-1}(PT_{N-1})^{n-1}y^{(0)}\| \\
&\le \alpha_n^{(N-1)}\|x-y\| + \beta_n^{(N-1)}k_n\|x^{(0)} - y^{(0)}\| \\
&\le \alpha_n^{(N-1)}\|x-y\| + \beta_n^{(N-1)}k_n^2\|x-y\| \\
&= (\alpha_n^{(N-1)} + \beta_n^{(N-1)}k_n^2)\|x-y\| \le k_n^2\|x-y\|, \\
\cdots & \cdots \cdots
\end{aligned}$$

and

$$\begin{aligned}
\|W_nx - W_ny\| &\le \alpha_n^{(1)}\|x-y\| + \beta_n^{(1)}\|T_1(PT_1)^{n-1}x^{(N-2)} - T_1(PT_1)^{n-1}y^{(N-2)}\| \\
&\le \alpha_n^{(1)}\|x-y\| + \beta_n^{(1)}k_n\|x^{(N-2)} - y^{(N-2)}\| \\
&\le \alpha_n^{(1)}\|x-y\| + \beta_n^{(1)}k_n^{N-1}\|x-y\| \\
&= (\alpha_n^{(1)} + \beta_n^{(1)}k_n^{N-1})\|x-y\| \\
&\le k_n^N\|x-y\| = (1+a_n)\|x-y\|.
\end{aligned}$$

For any $f \in \cap_{i=1}^N F(T_i)$, we get

$$\begin{array}{rcl} \|f^{(0)} - f\| & \leq & \|\alpha_n^{(N)} f + \beta_n^{(N)} T_N (PT_N)^{n-1} f + \gamma_n^{(N)} u_n^{(N)} - f\| \\ & = & \gamma_n^{(N)} \|u_n^{(N)} - f\| \leq M\gamma_n^{(N)}, \\ \|f^{(1)} - f\| & \leq & \beta_n^{(N-1)} k_n \|f^{(0)} - f\| + \gamma_n^{(N-1)} \|u_n^{(N-1)} - f\| \\ & \leq & k_n M\gamma_n^{(N)} + M\gamma_n^{(N-1)} \leq k_n M(\gamma_n^{(N)} + \gamma_n^{(N-1)}), \\ \cdots & \cdots & \cdots \\ \|f^{(N-2)} - f\| & \leq & k_n^{N-2} M(\gamma_n^{(N)} + \cdots + \gamma_n^{(2)}) \end{array}$$

and

$$\begin{array}{rcl} \|W_n f - f\| & \leq & \beta_n^{(1)} k_n \|f^{(N-2)} - f\| + \gamma_n^{(1)} \|u_n^{(1)} - f\| \\ & \leq & k_n^{N-1} M(\gamma_n^{(N)} + \cdots + \gamma_n^{(2)}) + M\gamma_n^{(1)} \\ & \leq & k_n^N M(\gamma_n^{(N)} + \cdots + \gamma_n^{(1)}) = (1 + a_n) w_n M. \end{array}$$

Set $S_{n,m} = W_{n+m-1} W_{n+m-2} \cdots W_{n+1} W_n : C \to C$, then $x_{n+m} = S_{n,m} x_n$ and for any $x, y \in C$,

$$\|S_{n,m} x - S_{n,m} y\| \leq (1 + a_{n+m-1}) \cdots (1 + a_n) \|x - y\|.$$

Put

$$b_{n,m} = (1 + a_{n+m-1}) \cdots (1 + a_n),$$

then by Lemma 2.1, we get $\lim_{m\to+\infty} b_{n,m} = \prod_{i=n}^{+\infty} (1 + a_i) =: b_n$ and

$$\lim_{n\to+\infty} b_n = \lim_{n\to+\infty} \lim_{m\to+\infty} b_{n,m} = \lim_{n\to+\infty} \prod_{i=n}^{+\infty} (1 + a_i) = 1.$$

We also need the following lemma, which plays a crucial role in dealing with the case of the iteration with errors.

Lemma 3.3

$$\lim_{n\to+\infty} \limsup_{m\to+\infty} \|S_{n,m} f - f\| = 0, \qquad \forall f \in \cap_{i=1}^N F(T_i).$$

Proof. Since

$$\begin{array}{rl} & \|S_{n,2} f - f\| = \|W_{n+1} W_n f - f\| \\ \leq & \|W_{n+1} W_n f - W_{n+1} f\| + \|W_{n+1} f - f\| \\ \leq & (1 + a_{n+1}) \|W_n f - f\| + (1 + a_{n+1}) w_{n+1} M \\ \leq & (1 + a_{n+1})(1 + a_n) w_n M + (1 + a_{n+1}) w_{n+1} M \\ \leq & (1 + a_{n+1})(1 + a_n)(w_n + w_{n+1}) M, \\ \cdots & \cdots \\ & \|S_{n,m-1} f - f\| \\ \leq & (1 + a_{n+m-2}) \cdots (1 + a_n)(w_n + \cdots + w_{n+m-2}) M, \end{array}$$

we get

$$\begin{array}{rl} & \|S_{n,m} f - f\| = \|W_{n+m-1} S_{n,m-1} f - f\| \\ \leq & \|W_{n+m-1} S_{n,m-1} f - W_{n+m-1} f\| + \|W_{n+m-1} f - f\| \\ \leq & (1 + a_{n+m-1}) \|S_{n,m-1} f - f\| + (1 + a_{n+m-1}) w_{n+m-1} M \\ \leq & (1 + a_{n+m-1}) \cdots (1 + a_n)(w_n + \cdots + w_{n+m-1}) M, \end{array}$$

Then fixing n and taking the limsup for m, we obtain

$$\limsup_{m\to+\infty} \|S_{n,m} f - f\| \leq M \cdot b_n \cdot \sum_{i=n}^{+\infty} w_i.$$

Thus, by Lemma 2.1,

$$\lim_{n\to+\infty}\limsup_{m\to+\infty}\|S_{n,m}f-f\|\leq\lim_{n\to+\infty}M\cdot b_n\cdot\sum_{i=n}^{+\infty}w_i=0.$$

This completes the proof. ■

Remark 3.2 *If $\gamma_n^{(i)}\equiv 0$ for all $i=1,2,...,N$ and all $n\geq 1$, then it is easy to see*

$$S_{n,m}f\equiv f.$$

Lemma 3.4 *Let $f,g\in\cap_{i=1}^{N}F(T_i)$ and $\lambda\in[0,1]$, then*

$$h(\lambda)=\lim_{n\to+\infty}\|\lambda x_n+(1-\lambda)f-g\|$$

exists.

Proof. It follows from Lemma 2.5 that

$$\begin{aligned}
&\|\lambda x_{n+m}+(1-\lambda)f-g\|=\|\lambda S_{n,m}x_n+(1-\lambda)f-g\|\\
\leq\ &\|[\lambda S_{n,m}x_n+(1-\lambda)S_{n,m}f]-S_{n,m}[\lambda x_n+(1-\lambda)f]\|+(1-\lambda)\|S_{n,m}f-f\|\\
&+\|S_{n,m}[\lambda x_n+(1-\lambda)f]-S_{n,m}g\|+\|S_{n,m}g-g\|\\
\leq\ &b_{n,m}\gamma^{-1}(\|x_n-f\|-\frac{1}{b_{n,m}}\|S_{n,m}x_n-S_{n,m}f\|)+\|S_{n,m}f-f\|\\
&+b_{n,m}\|\lambda x_n+(1-\lambda)f-g\|+\|S_{n,m}g-g\|\\
\leq\ &b_{n,m}\gamma^{-1}(\|x_n-f\|-\|x_{n+m}-f\|+\frac{b_{n,m}-1}{b_{n,m}}\|x_{n+m}-f\|+\frac{1}{b_{n,m}}\|S_{n,m}f-f\|)\\
&+b_{n,m}\|\lambda x_n+(1-\lambda)f-g\|+\|S_{n,m}f-f\|+\|S_{n,m}g-g\|.
\end{aligned}$$

For any fiexed n, we can take the limsup for m and obtain

$$\begin{aligned}
&\limsup_{n\to+\infty}\|\lambda x_n+(1-\lambda)f-g\|\\
\leq\ &b_n\gamma^{-1}(\|x_n-f\|-r+\frac{b_n-1}{b_n}r+\frac{1}{b_n}\limsup_{m\to+\infty}\|S_{n,m}f-f\|)\\
&+b_n\|\lambda x_n+(1-\lambda)f-g\|+\limsup_{m\to+\infty}\|S_{n,m}f-f\|+\limsup_{m\to+\infty}\|S_{n,m}g-g\|.
\end{aligned}$$

Hence, by Lemma 2.1 and Lemma 3.3, we have

$$\begin{aligned}
&\limsup_{n\to+\infty}\|\lambda x_n+(1-\lambda)f-g\|\\
\leq\ &\gamma^{-1}(r-r+\lim_{n\to+\infty}\limsup_{m\to+\infty}\|S_{n,m}f-f\|)+\liminf_{n\to+\infty}\|\lambda x_n+(1-\lambda)f-g\|\\
&+\lim_{n\to+\infty}\limsup_{m\to+\infty}\|S_{n,m}f-f\|+\lim_{n\to+\infty}\limsup_{m\to+\infty}\|S_{n,m}g-g\|\\
=\ &\liminf_{n\to+\infty}\|\lambda x_n+(1-\lambda)f-g\|,
\end{aligned}$$

which implies that

$$h(\lambda)=\lim_{n\to+\infty}\|\lambda x_n+(1-\lambda)f-g\|$$

exists. This completes the proof. ■

Now we can prove the weak convergence theorem of the iterative sequence $\{x_n\}$. It is remarked that a short proof can be found in [10]. For completeness and easy reference, we shall give the full proof.

Theorem 3.1 *Let C be a nonempty bounded closed convex subset of uniformly convex Banach space X and $P:X\to C$ be a nonexpansive retraction from X onto C. Let $T_1,T_2,...,T_N:C\to X$ be finite family of nonself asymptotically* *nonexpansive mappings with* $\cap_{i=1}^{N} F(T_i) \neq \emptyset$ *and the sequences* $\{k_n\} \subset [1, +\infty)$ *satisfying* $\sum_{n=1}^{+\infty}(k_n - 1) < +\infty$. *Let* $\{x_n\}$ *be defined by*

$$\begin{array}{rcl} & & x_1 \in C \\ x_{n+1} & = & P[\alpha_n^{(1)} x_n + \beta_n^{(1)} T_1 (PT_1)^{n-1} y_n^{(N-2)} + \gamma_n^{(1)} u_n^{(1)}]; \\ y_n^{(N-2)} & = & P[\alpha_n^{(2)} x_n + \beta_n^{(2)} T_2 (PT_2)^{n-1} y_n^{(N-3)} + \gamma_n^{(2)} u_n^{(2)}]; \\ \cdots & \cdots & \cdots \\ y_n^{(1)} & = & P[\alpha_n^{(N-1)} x_n + \beta_n^{(N-1)} T_{N-1} (PT_{N-1})^{n-1} y_n^{(0)} + \gamma_n^{(N-1)} u_n^{(N-1)}]; \\ y_n^{(0)} & = & P[\alpha_n^{(N)} x_n + \beta_n^{(N)} T_N (PT_N)^{n-1} x_n + \gamma_n^{(N)} u_n^{(N)}]. \end{array}$$

where $\{\alpha_n^{(i)}\}, \{\beta_n^{(i)}\}, \{\gamma_n^{(i)}\}$ *are sequences in* $[0, 1]$ *with* $0 < p \leq \alpha_n^{(i)}, \beta_n^{(i)} \leq q < 1$, $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$, $\sum_{n=1}^{+\infty} \gamma_n^{(i)} < +\infty$ *and* $\{u_n^{(i)}\}$ *are bounded sequences in* X, $i = 1, 2, \cdots, N$. *Assume that one of the following conditions holds:*

(1) X *satisfies the Opial's condition;*

(2) X *has a Fréchet differentiable norm;*

(3) X^* *has the Kadec-Klee property.*

Then $\{x_n\}$ *converges weakly to a common fixed point of* $\{T_i\}_{i=1}^{N}$.

Proof. Denote the set of weak limit points of $\{x_n\}$ by $\omega_\omega(\{x_n\})$, i.e. $\omega_\omega(\{x_n\}) = \{p \in X$: there exists subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup p\}$. To prove the weak convergence of $\{x_n\}$, we only need to prove that $\omega_\omega(\{x_n\})$ is singleton. Since X is reflexive and C is bounded, we obtain $\omega_\omega(\{x_n\}) \neq \emptyset$. Assuming that $f, g \in \omega_\omega(\{x_n\})$, in the following, we shall show $f = g$. By the Lemma 2.6 and Lemma 3.2, we know $f, g \in \cap_{i=1}^{N} F(T_i)$. Also, there exist two subsequence $\{x_{n_i}\}$ and $\{x_{n_j}\}$ in $\{x_n\}$ such that $x_{n_i} \rightharpoonup f$ and $x_{n_j} \rightharpoonup g$.

(1) Assume that (1) is given and $f \neq g$, then by the Opial's condition and Lemma 3.1, we get

$$\begin{array}{rl} & \lim_{n\to+\infty} \|x_n - f\| = \lim_{i\to+\infty} \|x_{n_i} - f\| \\ < & \lim_{i\to+\infty} \|x_{n_i} - g\| = \lim_{n\to+\infty} \|x_n - g\| = \lim_{j\to+\infty} \|x_{n_j} - g\| \\ < & \lim_{i\to+\infty} \|x_{n_j} - f\| = \lim_{n\to+\infty} \|x_n - f\|. \end{array}$$

This contraction implies $f = g$.

(2) By the Lemma 3.4, $h(\lambda) = \lim_{n\to+\infty} \|\lambda x_n + (1-\lambda)f - g\|$ exists. Then for any $\varepsilon > 0$, there exists n_0 such that for all $n \geq n_0$,

$$\|\lambda x_n + (1-\lambda)f - g\| \leq h(\lambda) + \varepsilon.$$

Hence

$$\langle \lambda x_n + (1-\lambda)f - g, J(f-g)\rangle \leq \|f - g\|(h(\lambda) + \varepsilon)$$

for all $n \geq n_0$. Since $f \in \bar{co}\{x_{n_i}, i \geq n_0\}$, we get

$$\langle \lambda f + (1-\lambda)f - g, J(f-g)\rangle \leq \|f - g\|(h(\lambda) + \varepsilon)$$

and hence $\|f - g\|^2 \leq \|f - g\|(h(\lambda) + \varepsilon)$. Therefore, we have

$$\|f - g\| \leq h(\lambda). \tag{3. 8}$$

Since X has a Fréchet differentiable norm, we get

$$\begin{array}{rl} & \lim_{\lambda\to 0} (2\lambda)^{-1}(\|\lambda x_n + (1-\lambda)f - g\| - \|f - g\|)\|f - g\| \\ = & \langle x_n - f, J(f-g)\rangle \end{array}$$

uniformly in $n \in N$. This implies that

$$\liminf_{n\to+\infty} \langle x_n - f, J(f-g)\rangle \geq 0.$$

Now for arbitrary $\varepsilon > 0$, we can choose n_0' such that

$$\langle x_n - f, J(f-g)\rangle \geq -\varepsilon$$

for all $n \geq n_0'$. It follows from $g \in \bar{co}\{x_{n_j}, j \geq n_0'\}$ that

$$\langle g-f, J(f-g)\rangle \geq -\varepsilon.$$

Thus for arbitrary $\varepsilon > 0$, $\|f-g\|^2 \leq \varepsilon$, which implies $f = g$.

(3) From Lemma 2.3 and (3.8), we have

$$\begin{aligned} & \|\lambda x_n + (1-\lambda)f - g\|^2 \\ \leq \ & \|f-g\|^2 + 2\lambda\langle x_n - f, J(\lambda x_n + (1-\lambda)f - g)\rangle. \end{aligned}$$

Then we can get for all $\lambda \in [0,1]$,

$$\liminf_{n\to+\infty}\langle x_n - f, J(\lambda x_n + (1-\lambda)f-g)\rangle \geq 0.$$

Hence for all $\lambda \in [0,1]$,

$$\liminf_{j\to+\infty}\langle x_{n_j} - f, J(\lambda x_{n_j} + (1-\lambda)f-g)\rangle \geq 0.$$

Thus for arbitrary $k \in N$, there exists $j_k \geq k$, $\{j_k\}\uparrow$, such that

$$\langle x_{n_{j_k}} - f, J(\frac{1}{k}x_{n_{j_k}} + (1-\frac{1}{k})f - g)\rangle \geq -\frac{1}{k}. \quad (3.\ 9)$$

Obviously, $x_{n_{j_k}} \rightharpoonup g$. Put

$$j_k = J(\frac{1}{k}x_{n_{j_k}} + (1-\frac{1}{k})f-g),$$

since X^* is reflexive, the set of all weak limit points of $\{j_k\}$ is nonempty. Hence we may assume that, without loss of generality, j_k is weakly convergent to some point $j \in X^*$. Therefore $\|j\| \leq \liminf_{k\to+\infty}\|j_k\| = \|f-g\|$. Since

$$\langle f-g, j_k\rangle = \|\frac{1}{k}x_{n_{j_k}} + (1-\frac{1}{k})f - g\|^2 - \frac{1}{k}\langle x_{n_{j_k}} - f, j_k\rangle,$$

passing the limit for k, we have $\langle f-g, j\rangle = \|f-g\|^2$. Hence $\|j\| \geq \|f-g\|$, we get

$$\langle f-g, j\rangle = \|f-g\|^2 = \|j\|^2,$$

which means $j = J(f-g)$. Thus we can conclude that $j_k \rightharpoonup j$ and $\|j_k\| \to \|f-g\| = \|j\|$. Since X^* has Kadec-Klee property, we have $j_k \to j$. Taking the limit for k in (3.9), we get $\langle g-f, j\rangle \geq 0$, i.e., $\|f-g\|^2 \leq 0$, which implies $f = g$. This completes the proof. ■

Remark 3.3 *Theorems 3.1 generalizes and improves many recent important results. For instance,* (1) *If* $\gamma_n^{(i)} \equiv 0$, *then we can obtain Theorem 3.6 in* [5]*;* (2) *If* $\gamma_n^{(i)} \equiv 0$, $T_i : C \to C$, *then we can get Theorem 3.4 in* [4]*;* (3) *If* $N \leq 3$ *and* $T_i : C \to C$, *then we can get Theorem 3.10 in [6], Theorem 2.1 in* [8]*, Theorem 1 in* [16]*, Theorem 2.9 in* [22]*, Theorem 3.3 in* [26]*, Theorem 3.1-3.2 in* [27]*, Theorem 3.5 in* [30] *and many others.*

If $\{T_i\}_{i=1}^N$ is a family of nonexpansive mappings, we can have the following theorem, which is an extension of Theorem 3.9 and Theorem 4.2 in [5], Theorem 4.1 in [10], Theorem 1 in [16], Theorem 3.5 and Theorem 4.1 in [24], Theorem 3.2 in [26] and others. The proof is immediate corollaries of the lemmas and Theorem 3.1.

Theorem 3.2 *Let* C *be a nonempty closed convex subset of uniformly convex Banach space* X *and* $P : X \to C$ *be a nonexpansive retraction from* X *onto* C*. Let* $T_1, T_2, ..., T_N : C \to X$ *be finite family of nonself nonexpansive mappings with* $\cap_{i=1}^N F(T_i) \neq \emptyset$*. Let* $\{x_n\}$ *be defined by*

$$\begin{aligned} & & x_1 \in C; \\ x_{n+1} &=& P[\alpha_n^{(1)}x_n + \beta_n^{(1)}T_1y_n^{(N-2)} + \gamma_n^{(1)}u_n^{(1)}]; \\ y_n^{(N-2)} &=& P[\alpha_n^{(2)}x_n + \beta_n^{(2)}T_2y_n^{(N-3)} + \gamma_n^{(2)}u_n^{(2)}]; \\ \cdots & \cdots & \cdots \\ y_n^{(1)} &=& P[\alpha_n^{(N-1)}x_n + \beta_n^{(N-1)}T_{N-1}y_n^{(0)} + \gamma_n^{(N-1)}u_n^{(N-1)}]; \\ y_n^{(0)} &=& P[\alpha_n^{(N)}x_n + \beta_n^{(N)}T_Nx_n + \gamma_n^{(N)}u_n^{(N)}], \end{aligned}$$

where $\{\alpha_n^{(i)}\}, \{\beta_n^{(i)}\}, \{\gamma_n^{(i)}\}$ *are sequences in* $[0,1]$ *with* $0 < p \le \alpha_n^{(i)}, \beta_n^{(i)} \le q < 1$, $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$, $\sum_{n=1}^{+\infty} \gamma_n^{(i)} < +\infty$ *and* $\{u_n^{(i)}\}$ *are sequences in* C, $i = 1, 2, \cdots, N$. *Assume that one of the following conditions holds:*

(1) X *satisfies the Opial's condition;*

(2) X *has a Fréchet differentiable norm;*

(3) X^* *has the Kadec-Klee property.*

Then $\{x_n\}$ *converges weakly to a common fixed point of* $\{T_i\}_{i=1}^N$.

To prove our strong convergence theorem, we need the following notions: We shall say that a finite family of nonself mappings $T_1, T_2, ..., T_N : C \to X$ with $F = \cap_{i=1}^N F(T_i) \neq \emptyset$ satisfies Condition $(\widetilde{C})$ if there exists a nondecreasing function $f : [0, +\infty) \to [0, +\infty)$ with $f(0) = 0$, $f(r) > 0$ for $r \in (0, +\infty)$, such that at least one of the $\{T_i\}_{i=1}^N$ satisfies condition $(\widetilde{I})$, i.e.,

$$\|T_i x - x\| \ge f(d(x, F)), \qquad \forall x \in C,$$

for at least one T_i, $1 \le i \le N$, where $d(x, F) = \inf\{\|x - p\| : p \in F\}$.

A mapping $T : C \to X$ is said to be demicompact if, for any bounded sequence $\{x_n\}$ in C such that $x_n - Tx_n$ converges, there exists a subsequence say $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to some p in C. T is said to be completely continuous if it is continuous and for every bounded sequence $\{x_n\}$, there exists a subsequence say $\{x_{n_j}\}$ of $\{x_n\}$ such that the sequence Tx_{n_j} converges to some element of the range of T.

Remark 3.4 *It is well known that every continuous and demicompact mapping must satisfy condition* $(\widetilde{I})$. *Since every completely continuous* $T : C \to C$ *is continuous and demicompact so that it satisfies condition* $(\widetilde{I})$. *Therefore, the condition* $(\widetilde{I})$ *is weaker than the demicompactness and complete continuousness* (*see* [5]).

Theorem 3.3 *Let* C *be a nonempty bounded closed convex subset of uniformly convex Banach space* X *and* $P : X \to C$ *be a nonexpansive retraction from* X *onto* C. *Let* $T_1, T_2, ..., T_N : C \to X$ *be finite family of nonself asymptotically nonexpansive mappings and* $\{x_n\}$ *be as in Theorem 3.1. If the family* $\{T_1, T_2, ..., T_N\}$ *satisfies condition* $(\widetilde{C})$, *i.e., at least one of* $\{T_1, T_2, ..., T_N\}$ *satisfies* $(\widetilde{I})$, *then* $\{x_n\}$ *converges strongly to a common fixed point of* $\{T_i\}_{i=1}^N$.

Proof. By inequality (3.4), we get

$$d(x_{n+1}, F) \le (1 + a_n) d(x_n, F) + L^N M w_n,$$

where $\sum_{n=1}^{+\infty} a_n < +\infty$ and $\sum_{n=1}^{+\infty} w_n < +\infty$. Then it follows from Lemma 2.2 that $\lim_{n\to+\infty} d(x_n, F)$ exists. Assume that T_i satisfy condition $(\widetilde{I})$, i.e.,

$$\|T_i x - x\| \ge f(d(x, F)), \qquad \forall x \in C,$$

then by Lemma 3.1, we get $\lim_{n\to+\infty} f(d(x_n, F)) = 0$. Hence $\lim_{n\to+\infty} d(x_n, F) = 0$. Thus for any $\varepsilon > 0$, there exists $n_0 \in N$ such that for all $n \ge n_0$,

$$d(x_n, F) < \frac{\varepsilon}{4\alpha} \qquad \text{and} \qquad \sum_{i=n}^{+\infty} w_i < \frac{\varepsilon}{3\alpha L^N M},$$

where $\alpha = \prod_{n=1}^{+\infty}(1 + a_n)$. Then for any $n \ge n_0$, there exists $f \in F$ such that $\|x_n - f\| < \frac{\varepsilon}{3\alpha}$. Therefore, for any $m \in N$, by (3.2),

$$\begin{aligned}
&\|x_{n+m} - x_n\| \le \|x_{n+m} - f\| + \|x_n - f\| \\
\le\ & (1 + a_{n+m-1})(\|x_{n+m-1} - f\| + L^N M w_{n+m-1}) + \|x_n - f\| \\
\le\ & (1 + a_{n+m-1})\|x_{n+m-1} - f\| + \alpha L^N M w_{n+m-1} + \|x_n - f\| \\
\le\ & (1 + a_{n+m-1})[(1 + a_{n+m-2})\|x_{n+m-1} - f\| + L^N M w_{n+m-1}] \\
& + \alpha L^N M w_{n+m-1} + \|x_n - f\| \\
\le\ & (1 + a_{n+m-1})(1 + a_{n+m-2})\|x_{n+m-2} - f\| \\
& + \alpha L^N M (w_{n+m-2} + w_{n+m-1}) + \|x_n - f\| \\
\cdots\ & \cdots \\
\le\ & (1 + a_{n+m-1})(1 + a_{n+m-2}) \cdots (1 + a_n)\|x_n - f\| \\
& + \alpha L^N M (w_n + \cdots w_{n+m-2} + w_{n+m-1}) + \|x_n - f\| \\
\le\ & \alpha\|x_n - f\| + \alpha L^N M (w_n + \cdots w_{n+m-2} + w_{n+m-1}) + \|x_n - f\| \\
\le\ & \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,
\end{aligned}$$

which implies that $\{x_n\}$ is a Cauchy sequence in C and it must converge to a point of C. Set $\lim_{n\to+\infty} x_n = p$, since $\lim_{n\to+\infty} d(x_n, F) = 0$ and F is closed, we get $p \in F$. This completes the proof. ■

For completeness, we conclude with the following strong convergence theorem for finite family of nonexpansive nonself mappings. The proof is similar to that of Theorem 3.2.

Theorem 3.4 *Let C be a nonempty bounded closed convex subset of uniformly convex Banach space X and $P : X \to C$ be a nonexpansive retraction from X onto C. Let $T_1, T_2, ..., T_N : C \to X$ be finite family of nonself nonexpansive mappings and $\{x_n\}$ be as in Theorem 3.2. If the family $\{T_1, T_2, ..., T_N\}$ satisfies condition $(\widetilde{C})$, then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^N$.*

Remark 3.5 *Theorem 3.3 and Theorem 3.4 generalize and improve many recent important results such as Theorem 3.5 in* [4]*, Theorem 3.4 and Theorem 4.1 in* [5]*, Theorem 3.7 in* [6]*, Theorem 2.4 in* [8]*, Theorem 4.2 in* [10]*, Theorem 2.6 in* [15]*, Theorem 2 in* [16]*, Theorem 2.4 in* [22]*, Theorem 3.6 in* [24]*, Theorem 3.3-3.4 in* [30]*, Theorem 2.1-2.3 in* [31] *and others.*

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