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Solution of Non-linear Sixth-order Two Point Boundary-value Problems Using Parametric Septic Splines

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Abstract: In this paper a family of numerical methods is developed for solution of non-linear sixth-order two-point boundary-value problems using parametric septic spline. It is shown that the new methods give approximations, which are better than existing spline methods. End equations of the spline are derived and truncation error is given. Convergence analysis is discussed through standard procedures. Three numerical examples are given to illustrate the practical applicability and efficiency of our methods.

Keywords: Parametric septic splines; Non-linear boundary-value problems; Spline function approximation

1 Introduction

We consider the numerical solution for the non-linear sixth-order two-point boundary-value problems of the form:

$$D^{(6)}y(x) = f(x, y),\ x \in [a, b],\ D \equiv \frac{d}{dx}, \tag{1.1}$$

subject to the boundary conditions:

$$\left.\begin{aligned} y(a) = A_0,\ y(b) = B_0, \\ D^{(2)}y(a) = A_2,\ D^{(2)}y(b) = B_2, \\ D^{(4)}y(a) = A_4,\ D^{(4)}y(b) = B_4, \end{aligned}\right\} \tag{1.2}$$

where $A_i, B_i,\ (i = 0, 2, 4)$ are finite real arbitrary constants, while y(x) and f(x,y) are continuous functions defined in the interval $x \in [a, b]$.

It is well known that a wide class of boundary value problems arising in various branches of pure and applied sciences including astrophysics, structural engineering, optimization and economics. The literature of the numerical solution of sixth-order boundary-value problems are sparse. These type of problems are generally arise in astrophysics; the narrow convecting layers bounded by stable layers which are believed to surround A-type stars may be modelled by sixth-order boundary-value problems [13].

Chandrasekhar [4] determined that when an infinite horizontal layer of fluid is heated from below and is under the action of rotation, instability sets in. When this instability is an ordinary convection, the ordinary differential equation is a sixth order.

Details of the theorems which listed the conditions for the existence and uniqueness of solutions of sixth-order boundary-value problems are given in Agarwal [1], but no numerical methods are contained therein.

Ramadan et al. [9] and Siraj-ul-Islam et al. [12] have solved the non-linear sixth-order boundary-value problems of the form (1.1)-(1.2) by using non-polynomial septic spline technique. Siddiqi and Twizell [10] presented the solution of sixth-order boundary-value problems using sextic spline. Boutayeb and Twizell [3] developed a family of finite-difference methods of order two, four, six and eight for the solution of special non-linear sixth-order boundary-value problems. Noor

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et al. [8] given a reliable approach for solving linear and non-linear sixth-order boundary-value problems by homotopy perturbation method. Wazwaz [16] used decomposition and modified domain decomposition methods to investigate the solution of sixth-order boundary-value problems. M.El.Gamel et al. [5] used Sinc-Galerkin method, Lamini et al. [7] used spline collocation method and J.H.He [6] has applied variational approach to solve the sixth-order boundary-value problems.

Siddiqi and Akram [11] and Akram and Siddiqi [2] presented a second order method using polynomial and non-polynomial septic spline for the solution of linear sixth-order boundary-value problems with boundary conditions at first and second order derivatives. Twizell [14] developed a second order method for solving special and general sixth-order problems and in later work Twizell and Boutayeb [15] developed finite difference methods of order two, four, six and eight for solving such problems.

The paper is organised into six sections. In section 2, we derive a parametric septic spline method for solving non-linear sixth-order two-point boundary-value problems. In section 3, the end conditions required to complete the definition of spline are derived. In section 4, method of different orders are categorised. In section 5, convergence of the method is established. In section 6, three numerical examples are presented to illustrate the applicability and accuracy of the method developed in the paper.

2 Derivation of the method

To develop the spline approximation to the problem (1.1) with the boundary conditions (1.2), we first introduce a finite set of grid points $x_j = a + jh;\ j = 0, 1, \cdots, N$ by dividing the interval [a,b] into N equal parts where

$$x_0 = a,\ x_N = b \text{ and } h = \frac{(b-a)}{N}. \tag{2.1}$$

A function $S_\triangle(x,\tau)$ of class $C^6[a,b]$ which interpolates y(x) at the mesh points x_j depends on a parameter τ, reduces to septic spline in [a,b] as $\tau \to 0$ is termed as Parametric Septic Spline function.

We consider a uniform mesh $\triangle = [x_j,\ j = 0(1)N]$, with mesh size h. The spline function $S_\triangle(x,\tau) = S_\triangle(x)$ satisfies the differential equation in the subinterval $[x_{j-1}, x_j]$;

$$\begin{aligned} S_\triangle^{(6)}(x) + \tau^2 S_\triangle^{(4)}(x) &= (Q_j + \tau^2 F_j)\frac{x - x_{j-1}}{h} + (Q_{j-1} + \tau^2 F_{j-1})\frac{x_j - x}{h} \\ &= V_j z + V_{j-1}\bar{z}, \end{aligned} \tag{2.2}$$

where

$$V_j = Q_j + \tau^2 F_j, \quad S_\triangle^{(4)}(x_j) = F_j, \quad S_\triangle^{(6)}(x_j) = Q_j$$

and $\tau > 0$ then it is termed as septic spline in compression.
Solving the differential equation (2.2), we get

$$\begin{aligned} S_\triangle(x) = {} & a_j + b_j x + c_j x^2 + d_j x^3 + e_j cos\tau x + f_j sin\tau x \\ & + \frac{(Q_j + \tau^2 F_j)}{\tau^2}\left\{\frac{(x - x_{j-1})^5}{5!h} - \frac{(x - x_{j-1})^3}{6h\tau^2} + \frac{(x - x_{j-1})}{h\tau^4}\right\} \\ & + \frac{(Q_{j-1} + \tau^2 F_{j-1})}{\tau^2}\left\{\frac{(x_j - x)^5}{5!h} - \frac{(x_j - x)^3}{6h\tau^2} + \frac{(x_j - x)}{h\tau^4}\right\}. \end{aligned} \tag{2.3}$$

To determine the six constants of integration of (2.3) in terms of $y_j,\ y_{j-1},\ M_j,\ M_{j-1},\ F_j,\ F_{j-1}$ we define the interpolatory conditions at x_j and x_{j-1}

$$\left.\begin{aligned} S_\triangle(x_j) = y_j,\ S_\triangle(x_{j-1}) = y_{j-1}, \\ S_\triangle^{(2)}(x_j) = M_j,\ S_\triangle^{(2)}(x_{j-1}) = M_{j-1}, \\ S_\triangle^{(4)}(x_j) = F_j,\ S_\triangle^{(4)}(x_{j-1}) = F_{j-1}. \end{aligned}\right\} j = 1(1)N-1 \tag{2.4}$$

From algebraic manipulation we get the following expressions for coefficients:

$$\begin{aligned}
a_j &= \frac{(y_{j-1}x_j - y_j x_{j-1})}{h} + \frac{x_j x_{j-1}}{2h}[M_{j-1}x_j - M_j x_{j-1}] + \frac{x_j x_{j-1}(x_j + x_{j+1})}{6h}[M_j - M_{j+1}] \\
&\quad + \left(\frac{x_j x_{j-1} h}{2h\tau^2} - \frac{1}{h\tau^4}\right)[F_{j-1}x_j - F_j x_{j-1}] + \frac{x_j x_{j-1}(x_j + x_{j+1})}{6h\tau^2}[F_j - F_{j-1}] \\
&\quad + \left(\frac{x_j x_{j-1} h}{12\tau^2} - \frac{x_j x_{j-1}(x_j + x_{j+1})h}{36\tau^2} + \frac{h^3}{\tau^2 5!} - \frac{h}{6\tau^4}\right) \\
&\quad [(Q_j + \tau^2 F_j)x_{j-1} - (Q_{j-1} + \tau^2 F_{j-1})x_j],
\end{aligned}$$

$$\begin{aligned}
b_j &= \frac{(y_j - y_{j-1})}{h} - \frac{(x_j + x_{j+1})}{2h}[M_{j-1}x_j - M_j x_{j-1}] - \frac{(x_j^2 + x_{j-1}^2 + x_j x_{j-1})}{6h}[M_j - M_{j-1}] \\
&\quad - \frac{(x_j + x_{j+1})}{2h\tau^2}[F_{j-1}x_j - F_j x_{j-1}] - \left(\frac{(x_j^2 + x_{j-1}^2 + x_j x_{j-1})}{6h\tau^2} + \frac{1}{h\tau^4}\right)[F_j - F_{j-1}] \\
&\quad - \left(\frac{(x_j + x_{j-1})h}{12\tau^2} - \frac{(x_j^2 + x_{j-1}^2 + x_j x_{j-1})h}{36\tau^2} + \frac{h^3}{\tau^2 5!} - \frac{h}{6\tau^4}\right) \\
&\quad [(Q_j + \tau^2 F_j)x_{j-1} - (Q_{j-1} + \tau^2 F_{j-1})x_j],
\end{aligned}$$

$$c_j = \frac{M_{j-1}x_j - M_j x_{j-1}}{2h} + \frac{F_{j-1}x_j - F_j x_{j-1}}{2h\tau^2} + \frac{h[(Q_j + \tau^2 F_j)x_{j-1} - (Q_{j-1} + \tau^2 F_{j-1})x_j]}{12\tau^2},$$

$$\begin{aligned}
d_j &= \frac{M_j - M_{j-1}}{6h} + \frac{F_j - F_{j-1}}{6h\tau^2} - \frac{h[(Q_j + \tau^2 F_j) - (Q_{j-1} + \tau^2 F_{j-1})]}{36\tau^2}, \\
e_j &= \frac{1}{\tau^6 sin\omega}\left(Q_j sin\tau x_{j-1} - Q_{j-1} sin\tau x_j\right), \\
f_j &= \frac{-1}{\tau^6 sin\omega}\left(Q_j cos\tau x_{j-1} - Q_{j-1} cos\tau x_j\right). \qquad (2.5)
\end{aligned}$$

Substituting the values of coefficients from equation (2.5) in (2.3), we obtain

$$\begin{aligned}
S_\Delta(x) &= z y_j + \bar{z} y_{j-1} + \frac{h^2}{3!}\{q_3(z)M_j + q_3(\bar{z})M_{j-1}\} + \frac{h^4}{5!}\{q_5(z)F_j + q_5(\bar{z})F_{j-1}\} \\
&\quad + \left(\frac{h}{\omega}\right)^6 \left\{\frac{\omega^4}{5!}q_5(z) - \frac{\omega^2}{3!}q_3(z) + q_1(z)\right\} Q_j \\
&\quad + \left(\frac{h}{\omega}\right)^6 \left\{\frac{\omega^4}{5!}q_5(\bar{z}) - \frac{\omega^2}{3!}q_3(\bar{z}) + q_1(\bar{z})\right\} Q_{j-1}, \qquad (2.6)
\end{aligned}$$

where

$z = \frac{x - x_{j-1}}{h}$, $\bar{z} = \frac{x_j - x}{h}$ or $\bar{z} = 1 - z$,

$S_\Delta(x_j) = y_j$, $\omega = \tau h$,

$q_5(z) = z^5 - (10/3)z^3 + (7/3)z$,

$q_3(z) = z^3 - z$, $q_1(z) = z - \frac{\sin(\omega z)}{\sin\omega}$.

The function $S_{\triangle}(x)$ in interval $[x_j, x_{j+1}]$ is obtained with (j+1) replacing j in (2.6) so that

$$\begin{aligned} S_{\Delta}(x) &= \bar{z}y_j + zy_{j+1} + \frac{h^2}{3!}\{q_3(\bar{z})M_j + q_3(z)M_{j+1}\} + \frac{h^4}{5!}\{q_5(\bar{z})F_j + q_5(z)F_{j+1}\} \\ &+\left(\frac{h}{\omega}\right)^6\left\{\frac{\omega^4}{5!}q_5(\bar{z}) - \frac{\omega^2}{3!}q_3(\bar{z}) + q_1(\bar{z})\right\}Q_j \\ &+\left(\frac{h}{\omega}\right)^6\left\{\frac{\omega^4}{5!}q_5(z) - \frac{\omega^2}{3!}q_3(z) + q_1(z)\right\}Q_{j+1}, \end{aligned} \tag{2.7}$$

where

$z = \frac{x-x_j}{h}, \quad \bar{z} = \frac{x_{j+1}-x}{h}$ or $\bar{z} = 1 - z$.

The continuity of first, third and fifth derivatives at $x = x_j$ requires

$$S_{\triangle}^{(1)}(x_j^-) = S_{\triangle}^{(1)}(x_j^+), \;\; S_{\triangle}^{(3)}(x_j^-) = S_{\triangle}^{(3)}(x_j^+), \;\; S_{\triangle}^{(5)}(x_j^-) = S_{\triangle}^{(5)}(x_j^+). \tag{2.8}$$

Applying these derivative continuities, the following consistency relations are derived:

$$\begin{aligned} M_{j+1} + 4M_j + M_{j-1} &= \frac{6}{h^2}(y_{j+1} - 2y_j + y_{j-1}) + \frac{h^2}{60}(7F_{j+1} + 16F_j + 7F_{j-1}) \\ &\quad +6h^4(\alpha_2 Q_{j+1} + 2\beta_2 Q_j + \alpha_2 Q_{j-1}); \;\; j = 1(1)N-1, \end{aligned} \tag{2.9}$$

$$\begin{aligned} M_{j+1} - 2M_j + M_{j-1} &= \frac{h^2}{6}(F_{j+1} + 4F_j + F_{j-1}) \\ &\quad +h^4(\alpha_1 Q_{j+1} + 2\beta_1 Q_j + \alpha_1 Q_{j-1}); \;\; j = 1(1)N-1, \end{aligned} \tag{2.10}$$

$$F_{j+1} - 2F_j + F_{j-1} = h^2(\alpha Q_{j+1} + 2\beta Q_j + \alpha Q_{j-1}); \;\; j = 1(1)N-1, \tag{2.11}$$

where

$\alpha_2 = \frac{7}{360\omega^2} + \frac{1}{\omega^2}\left\{\frac{1}{6\omega^2} - \frac{1}{\omega^4}(\omega cosec\omega - 1)\right\}, \quad \beta_2 = \frac{8}{360\omega^2} + \frac{1}{\omega^2}\left\{\frac{1}{3\omega^2} - \frac{1}{\omega^4}(1 - \omega cot\omega)\right\},$

$\alpha = \frac{1}{\omega^2}\left(\omega cosec\omega - 1\right), \qquad \beta = \frac{1}{\omega^2}\left(1 - \omega\cot\omega\right),$

$\alpha_1 = \frac{1}{\omega^2}\left(\frac{1}{6} - \alpha\right), \qquad \beta_1 = \frac{1}{\omega^2}\left(\frac{1}{3} - \beta\right).$

Using equations (2.9)-(2.11), we obtain the following consistency relation in terms of sixth derivative of spline Q_j and $y_j, \; j = 0(1)N$

$$\begin{aligned} &y_{j+3} - 6y_{j+2} + 15y_{j+1} - 20y_j + 15y_{j-1} - 6y_{j-2} + y_{j-3} \\ &\quad = h^6(p\,Q_{j+3} + q\,Q_{j+2} + r\,Q_{j+1} + s\,Q_j + r\,Q_{j-1} + q\,Q_{j-2} + p\,Q_{j-3}); \quad j = 3, 4, \cdots, N-3. \end{aligned} \tag{2.12}$$

where

$$\left.\begin{aligned} p &= \left(\frac{\omega - sin\omega}{\omega^6 sin\omega} - \frac{1}{6\omega^3 sin\omega} + \frac{1}{120\omega sin\omega}\right), \\ q &= \left(\frac{6}{\omega^6} - \frac{2(2+cos\omega)}{\omega^5 sin\omega} - \frac{1-cos\omega}{3\omega^3 sin\omega} + \frac{13-cos\omega}{60\omega sin\omega}\right), \\ r &= \left(\frac{-15}{\omega^6} - \frac{7+8cos\omega}{\omega^5 sin\omega} + \frac{5+4cos\omega}{6\omega^3 sin\omega} + \frac{67-52cos\omega}{120\omega sin\omega}\right), \\ s &= \left(\frac{20}{\omega^6} - \frac{4(2+3cos\omega)}{\omega^5 sin\omega} - \frac{2(1+3cos\omega)}{3\omega^3 sin\omega} + \frac{13-33cos\omega}{30\omega sin\omega}\right). \end{aligned}\right\} \quad (2.13)$$

As $\omega \to 0$ then $(\alpha, \beta, \alpha_1, \beta_1, \alpha_2, \beta_2) \to (\frac{1}{6}, \frac{1}{3}, -\frac{7}{360}, -\frac{8}{360}, -\frac{31}{15120}, -\frac{2}{945})$ and $(p, q, r, s) \to (\frac{1}{5040}, \frac{120}{5040}, \frac{1191}{5040}, \frac{2416}{5040})$. then, the spline defined by (2.12) reduces to a ordinary septic spline and the spline relations reduce to corresponding ordinary septic spline relations.
The relation (2.12) gives (N-5) linear algebraic equations in (N-1) unknowns $(y_j,\ j = 1(1)N-1)$, therefore we need four more equations, two at each end of the range of integration to have complete solution of $y_j\ 's$ appearing in equation (2.12), are derived in the following section.

3 Development of End Conditions

For the discretization of the boundary condition, we define

$$\begin{aligned}
&(i) && \sum_{k=0}^{4} a_k y_k + c_1 h^2 y_0^{(2)} + c_2 h^4 y_0^{(4)} + h^6 \sum_{k=0}^{5} v_k y_k^{(6)} + t_1 = 0, && j = 1. \\
&(ii) && \sum_{k=0}^{5} b_k y_k + c_3 h^2 y_0^{(2)} + c_4 h^4 y_0^{(4)} + h^6 \sum_{k=1}^{6} w_k y_k^{(6)} + t_2 = 0, && j = 2. \\
&(iii) && \sum_{k=N-5}^{N} b_k y_k + c_3 h^2 y_N^{(2)} + c_4 h^4 y_N^{(4)} + h^6 \sum_{k=N-6}^{N-1} w_k y_k^{(6)} + t_{N-2} = 0, && j = N-2. \\
&(iv) && \sum_{k=N-4}^{N} a_k y_k + c_1 h^2 y_N^{(2)} + c_2 h^4 y_N^{(4)} + h^6 \sum_{k=N-5}^{N} v_k y_k^{(6)} + t_{N-1} = 0, && j = N-1,
\end{aligned} \quad (3.1)$$

where $c_1,\ c_2,\ c_3,\ c_4,\ a_k\ , b_k,\ v_k$ and w_k are arbitrary parameters to be determined. To obtain the local truncation error $t_j;\ j = 3, 4 \cdots, N-3$, associated with the scheme (2.12), we first rewrite it in the form:

$$\begin{aligned} &y_{j+3} - 6y_{j+2} + 15y_{j+1} - 20y_j + 15y_{j-1} - 6y_{j-2} + y_{j-3} \\ &\quad = h^6(p\, y_{j+3}^{(6)} + q\, y_{j+2}^{(6)} + r\, y_{j+1}^{(6)} + s\, y_j^{(6)} + r\, y_{j-1}^{(6)} + q\, y_{j-2}^{(6)} + p\, y_{j-3}^{(6)}) + t_j; \quad j = 3, 4, \cdots, N-3. \end{aligned} \quad (3.2)$$

Using the Taylor's series expansion, the terms $y_{j+3}^{(6)}$, $y_{j+2}^{(6)}$, etc. are expanded around the point x_j and the expression for $t_j,\ j = 3, 4, \cdots, N-3$ is obtained:

$$t_j = C_6 h^6 y_j^{(6)} + C_8 h^8 y_j^{(8)} + C_{10} h^{10} y_j^{(10)} + C_{12} h^{12} y_j^{(12)} + C_{14} h^{14} y_j^{(14)} + O(h^{16}), \quad (3.3)$$

where

$$\left.\begin{array}{rcl} C_6 &=& (1-2p-2q-2r-s), \\ C_8 &=& \left(\frac{(1-36p-16q-4r)}{4}\right), \\ C_{10} &=& \left(\frac{105840}{10!} - \frac{(81p+16q+r)}{12}\right), \\ C_{12} &=& \left(\frac{1013760}{12!} - \frac{(729p+64q+r)}{360}\right), \\ C_{14} &=& \left(\frac{9369360}{14!} - \frac{(6561p+256q+r)}{20160}\right). \end{array}\right\} \quad (3.4)$$

Thus for different choices of parameters p, q, r, s in scheme (2.12), method of different orders are obtained.

4 Numerical method of different orders

(I) ***Second-order method***

In order to obtain the second-order method we find that

$(a_0, a_1, a_2, a_3, a_4, v_0, v_1, v_2, v_3, v_4, v_5) = (-5, 14, -14, 6, -1, -\frac{29}{180}, 1, 0, 0, 0, 0)$,

$(b_0, b_1, b_2, b_3, b_4, b_5, w_1, w_2, w_3, w_4, w_5, w_6) = (4, -14, 20, -15, 6, -1, -\frac{1}{180}, \frac{361}{360}, 0, 0, 0, 0)$,

$(b_{N-5}, b_{N-4}, b_{N-3}, b_{N-2}, b_{N-1}, b_N, w_{N-6}, w_{N-5}, w_{N-4}, w_{N-3}, w_{N-2}, w_{N-1})$
$= (-1, 6, -15, 20, -14, 4, 0, 0, 0, 0, \frac{361}{360}, -\frac{1}{180})$,

$(a_{N-4}, a_{N-3}, a_{N-2}, a_{N-1}, a_N, v_{N-5}, v_{N-4}, v_{N-3}, v_{N-2}, v_{N-1}, v_N)$
$= (-1, 6, -14, 14, -5, 0, 0, 0, 0, 1, -\frac{29}{180})$,
$(c_1, c_2, c_3, c_4) = (2, -\frac{5}{6}, -1, -\frac{1}{12})$

and the local truncation error is

$$t_j = \begin{cases} \frac{-9580}{8!} h^8 y_j^{(8)} + O(h^9), & j = 1, N-1, \\ \frac{-9966}{8!} h^8 y_j^{(8)} + O(h^9), & j = 2, N-2. \end{cases} \quad (4.1)$$

For $(p, q, r, s) = (\frac{1}{5040}, \frac{120}{5040}, \frac{1191}{5040}, \frac{2416}{5040})$ the truncation error is given by

$$t_j = -\frac{1}{12} h^8 y_j^{(8)} + O(h^{10}), \quad j = 3(1)N-3. \quad (4.2)$$

(II) ***Fourth-order method***

In order to obtain the fourth-order method we find that

$(a_0, a_1, a_2, a_3, a_4, v_0, v_1, v_2, v_3, v_4, v_5) = (-5, 14, -14, 6, -1, \frac{323}{5040}, \frac{1133}{2016}, \frac{101}{504}, \frac{25}{2016}, 0, 0)$,

$(b_0, b_1, b_2, b_3, b_4, b_5, w_1, w_2, w_3, w_4, w_5, w_6) = (4, -14, 20, -15, 6, -1, \frac{811}{3397}, \frac{803}{1553}, \frac{229}{960}, \frac{29}{10080}, 0, 0)$,

$(b_{N-5}, b_{N-4}, b_{N-3}, b_{N-2}, b_{N-1}, b_N, w_{N-6}, w_{N-5}, w_{N-4}, w_{N-3}, w_{N-2}, w_{N-1})$
$= (-1, 6, -15, 20, -14, 4, 0, 0, \frac{29}{10080}, \frac{229}{960}, \frac{803}{1553}, \frac{811}{3397})$,

$$(a_{N-4}, a_{N-3}, a_{N-2}, a_{N-1}, a_N, v_{N-5}, v_{N-4}, v_{N-3}, v_{N-2}, v_{N-1}, v_N) = (-1, 6, -14, 14, -5, 0, 0, \tfrac{25}{2016}, \tfrac{101}{504}, \tfrac{1133}{2016}, \tfrac{323}{5040}),$$

$$(c_1, c_2, c_3, c_4) = (2, -\tfrac{5}{6}, -1, -\tfrac{1}{12})$$

and the local truncation error is

$$t_j = \begin{cases} \tfrac{13046}{10!} h^{10} y_j^{(10)} + O(h^{11}), & j = 1, N-1, \\ \tfrac{27275830}{10!} h^{10} y_j^{(10)} + O(h^{11}), & j = 2, N-2. \end{cases} \tag{4.3}$$

For $(p, q, r, s) = (0, 0, \frac{1}{4}, \frac{1}{2})$ the truncation error is given by

$$t_j = \frac{1}{120} h^{10} y_j^{(10)} + O(h^{12}), \qquad j = 3(1)N-3. \tag{4.4}$$

(III) ***Sixth-order method***

In order to obtain the sixth-order method we find that

$$(a_0, a_1, a_2, a_3, a_4, v_0, v_1, v_2, v_3, v_4, v_5) = (-5, 14, -14, 6, -1, \tfrac{123}{2144}, \tfrac{3063}{5174}, \tfrac{805}{5454}, \tfrac{88}{1517}, -\tfrac{324}{16867}, \tfrac{31}{9927}),$$

$$(b_0, b_1, b_2, b_3, b_4, b_5, w_1, w_2, w_3, w_4, w_5, w_6) = (4, -14, 20, -15, 6, -1, \tfrac{301}{1249}, \tfrac{729}{1432}, \tfrac{697}{2811}, \tfrac{6}{270257}, -\tfrac{41}{22143}, \tfrac{11}{10720}),$$

$$(b_{N-5}, b_{N-4}, b_{N-3}, b_{N-2}, b_{N-1}, b_N, w_{N-6}, w_{N-5}, w_{N-4}, w_{N-3}, w_{N-2}, w_{N-1}) = (-1, 6, -15, 20, -14, 4, \tfrac{11}{10720}, -\tfrac{41}{22143}, \tfrac{6}{270257}, \tfrac{697}{2281}, \tfrac{729}{1432}, \tfrac{301}{1249}),$$

$$(a_{N-4}, a_{N-3}, a_{N-2}, a_{N-1}, a_N, v_{N-5}, v_{N-4}, v_{N-3}, v_{N-2}, v_{N-1}, v_N) = (-1, 6, -14, 14, -5, \tfrac{31}{9927}, -\tfrac{324}{16867}, \tfrac{88}{1517}, \tfrac{805}{5454}, \tfrac{3063}{5174}, \tfrac{123}{2144}),$$
$$(c_1, c_2, c_3, c_4) = (2, -\tfrac{5}{6}, -1, -\tfrac{1}{12})$$

and the local truncation error is

$$t_j = \begin{cases} \tfrac{1283115}{12!} h^{12} y_j^{(12)} + O(h^{13}), & j = 1, N-1, \\ \tfrac{2515468}{12!} h^{12} y_j^{(12)} + O(h^{13}), & j = 2, N-2. \end{cases} \tag{4.5}$$

For $(p, q, r, s) = (0, \frac{1}{120}, \frac{26}{120}, \frac{66}{120})$ the truncation error is given by

$$t_j = \frac{1}{30240} h^{12} y_j^{(12)} + O(h^{14}), \qquad j = 3(1)N-3. \tag{4.6}$$

(IV) ***Eighth-order method***

For $(p, q, r, s) = (\frac{1}{30240}, \frac{246}{30240}, \frac{6567}{30240}, \frac{16612}{30240})$

the local truncation error is given by

$$t_j = \frac{367}{4838400} h^{14} y_j^{(14)} + O(h^{16}), \qquad j = 3(1)N-3. \tag{4.7}$$

Boundary equations of eighth-order method are not constructed, however these equations can be obtained by using the same procedure adapted in previous cases.

Remark: When $(p,q,r,s) = (\frac{1}{5040}, \frac{120}{5040}, \frac{1191}{5040}, \frac{2416}{5040})$, our method reduces to Siddiqi and Akram [11].

5 Convergence

Let $y = y(x_j),\ Y = (y_j),\ V = (v_j),\ T = (t_j),\ E = (e_j) = y - Y,\ j = 1, 2, \cdots, N-1$ be an exact column vectors, where y, Y, T, E are exact, approximate, local truncation error, discretization error respectively.
For solving non-linear sixth-order boundary value problem $D^{(6)}y(x) = f(x,y),$ the family of numerical methods is described by the set of equations given by (2.12) and (3.1), then the solution vector $Y = [y_1, y_2, \cdots, y_{N-1}]^T$, T denoting transpose, is obtained by solving a non-linear algebric system of order (N-1) has the form:

$$M_0 Y - h^6 B f(x,Y) = V. \tag{5.1}$$

The seven-band matrix M_0 has the form

$$M_0 = \begin{bmatrix} 14 & -14 & 6 & -1 & & & & & & \\ -14 & 20 & -15 & 6 & -1 & & & & & \\ -6 & 15 & -20 & 15 & -6 & 1 & & & & \\ 1 & -6 & 15 & -20 & 15 & -6 & 1 & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & & 1 & -6 & 15 & -20 & 15 & -6 & 1 \\ & & & & 1 & -6 & 15 & -20 & 15 & -6 \\ & & & & & -1 & 6 & -15 & 20 & -14 \\ & & & & & & -1 & 6 & -14 & 14 \end{bmatrix} \tag{5.2}$$

and the matrix B has the form

$$B = \begin{bmatrix} -v_1 & -v_2 & -v_3 & -v_4 & -v_5 & & & & & \\ -w_1 & -w_2 & -w_3 & -w_4 & -w_5 & -w_6 & & & & \\ q & r & s & r & q & p & & & & \\ p & q & r & s & r & q & p & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & & p & q & r & s & r & q & p \\ & & & & p & q & r & s & r & q \\ & & & & -w_{N-6} & -w_{N-5} & -w_{N-4} & -w_{N-3} & -w_{N-2} & -w_{N-1} \\ & & & & & -v_{N-5} & -v_{N-4} & -v_{N-3} & -v_{N-2} & -v_{N-1} \end{bmatrix}. \tag{5.3}$$

The (N-1) column vector V is defined by

$$v_j = \begin{cases} -a_0 y_0 - c_1 h^2 y_0^{(2)} - c_2 h^4 y_0^{(4)} - h^6(v_0 f_0), & j = 1, \\ -b_0 y_0 - c_3 h^2 y_0^{(2)} - c_4 h^4 y_0^{(4)}, & j = 2, \\ h^6(p f_0) - y_0, & j = 3, \\ 0, & 4 \le j \le (N-4), \\ h^6(p f_N) - y_N, & j = N-3, \\ -b_N y_N - c_3 h^2 y_N^{(2)} - c_4 h^4 y_N^{(4)}, & j = N-2, \\ -a_N y_N - c_1 h^2 y_N^{(2)} - c_2 h^4 y_N^{(4)} - h^6(v_N f_N), & j = N-1. \end{cases} \tag{5.4}$$

The vector $y = [y(x_1), y(x_2), \cdots, y(x_{N-1})]^T$, T denoting transpose satisfies

$$M_0 y - h^6 B f(x, y) = V + T, \tag{5.5}$$

where $T = [t_1, t_2, \cdots, t_{N-1}]^T$ is the local truncation error vector. In [15], a conventional convergence analysis shows that the norm of the vector

$$E = y - Y$$

satisfies:

$$\| E \|_\infty \le \frac{(b-a)^6}{512 - (b-a)^6 B^* F^*} [C_6 h^6 y_j^{(6)} + C_8 h^8 y_j^{(8)} + C_{10} h^{10} y_j^{(10)} + C_{12} h^{12} y_j^{(12)} + \cdots], \tag{5.6}$$

where

$$C_j = max_{a \le x \le b} \left| \frac{d^j y(x)}{dx^j} \right| \ \ for \ \ j = 1, 2, \cdots \ ,$$

$$B^* = \| B \| \ \ and \ \ F^* = max_{a \le x \le b} \left| \frac{df}{dy(x)} \right|.$$

The order of convergence of the numerical method is, thus p, if C_{p+6} is the first non-vanishing constant on the right hand side of (3.3) provided

$$F^* < \frac{512}{(b-a)^6 B^*}.$$

6 Numerical Examples

We now consider three numerical problems for non-linear sixth order two-point BVP and compared the results with the existing methods. All calculations were done by using Symbolic Toolbox MATLAB7.

Example 1. Consider the boundary value problem, which is discussed in [9]:

$$D^{(6)} y(x) + y^2(x) = 12x cos(x) + (31 - x^2) sin(x) + (x^4 - 2x^2 + 1)[sin(x)]^2, \quad x \in [0, 1],$$

subject to the boundary conditions:

$$\left.\begin{aligned} y(0) = y(1) = 0, \\ D^{(2)}y(0) = 0, \quad D^{(2)}y(1) = 2sin(1) + 4cos(1), \\ D^{(4)}y(0) = 0, \quad D^{(4)}y(1) = -12sin(1) - 8cos(1). \end{aligned}\right\} \tag{6.1}$$

The analytical solution of the above problem is

$$y(x) = (x^2 - 1)sin(x). \tag{6.2}$$

Example 2. Consider the boundary value problem, which is discussed in [12]:

$$D^{(6)}y(x) = 20exp[-36y(x)] - 40(1+x)^{-6}, \quad x \in [0,1],$$

subject to the boundary conditions:

$$\left.\begin{aligned} y(0) = 1, \quad y(1) = \tfrac{1}{6}log2, \\ D^{(2)}y(0) = -\tfrac{1}{6}, \quad D^{(2)}y(1) = -\tfrac{1}{24}, \\ D^{(4)}y(0) = -1, \quad D^{(4)}y(1) = -\tfrac{1}{16}. \end{aligned}\right\} \tag{6.3}$$

The analytical solution of the above problem is

$$y(x) = \frac{1}{6}log(1+x). \tag{6.4}$$

Example 3. Consider the boundary value problem:

$$D^{(6)}y(x) = e^{-x}y^2(x), \quad x \in [0,1],$$

subject to the boundary conditions:

$$\left.\begin{aligned} y(0) = D^{(2)}y(0) = D^{(4)}y(0) = 1, \\ y(0) = D^{(2)}y(0) = D^{(4)}y(0) = e. \end{aligned}\right\} \tag{6.5}$$

The analytical solution of the above problem is

$$y(x) = e^x. \tag{6.6}$$

The interval $0 \le x \le 1$ was divided into N equal subintervals each of width $h = 2^{-m}$ with m=3, 4, 5.

The observed maximum errors (in absolute values), corresponding to the problem (6.1),(6.3) and (6.5) was computed for each value of N and the results for our second, fourth and sixth-order methods are listed in Tables 1-3. In Table 1, the new method (2.12) is applied to problem (6.1) and the results are compared with Ramadan et al. [9], where as in Table 2, the new method is applied to problem (6.3) and the results are compared with Siraj-ul-Islam et al. [12].

These tables show that performance of our second and fourth order method is better where as comparable in case of sixth order method.

7 Conclusion

In this paper we have presented new methods for solving non-linear sixth-order two-point boundary value problems using parametric septic spline. The present methods enable us to approximate the solution at every point of the range of integration. The comparison of the method is also depicted through Tables 1-3, which shown our methods have an improvement

Table 1: Observed maximum absolute errors, Example(6.1)

Methods $\downarrow$	$\mathbf{h = \frac{1}{8}}$	$\mathbf{h = \frac{1}{16}}$	$\mathbf{h = \frac{1}{32}}$
Second Order Method $(p,q,r,s) = (\frac{1}{5040}, \frac{120}{5040}, \frac{1191}{5040}, \frac{2416}{5040})$	3.75×10^{-5}	8.30×10^{-6}	3.20×10^{-6}
Fourth Order Method $(p,q,r,s) = (0, 0, \frac{1}{4}, \frac{1}{2})$	9.58×10^{-8}	7.49×10^{-9}	3.47×10^{-10}
Sixth Order Method $(p,q,r,s) = (0, \frac{1}{120}, \frac{26}{120}, \frac{66}{120})$	9.69×10^{-9}	2.04×10^{-10}	5.43×10^{-11}
Ramadan et al. [9]	1.65×10^{-8}	2.60×10^{-10}	6.86×10^{-11}

Table 2: Observed maximum absolute errors, Example(6.3)

Methods $\downarrow$	$\mathbf{h = \frac{1}{8}}$	$\mathbf{h = \frac{1}{16}}$	$\mathbf{h = \frac{1}{32}}$
Second Order Method $(p,q,r,s) = (\frac{1}{5040}, \frac{120}{5040}, \frac{1191}{5040}, \frac{2416}{5040})$	2.27×10^{-4}	7.31×10^{-6}	1.87×10^{-6}
Siraj-ul-Islam et al. [12]	2.3×10^{-3}	3.4×10^{-4}	1.54×10^{-4}
Fourth Order Method $(p,q,r,s) = (0, 0, \frac{1}{4}, \frac{1}{2})$	3.06×10^{-6}	4.34×10^{-7}	3.53×10^{-8}
Siraj-ul-Islam et al. [12]	3.0×10^{-3}	1.38×10^{-4}	4.30×10^{-6}
Sixth Order Method $(p,q,r,s) = (0, \frac{1}{120}, \frac{26}{120}, \frac{66}{120})$	1.09×10^{-6}	5.51×10^{-8}	1.36×10^{-9}
Siraj-ul-Islam et al. [12]	4.55×10^{-7}	5.96×10^{-9}	2.68×10^{-11}

over existing methods.

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Table 3: Observed maximum absolute errors, Example(6.5)

Methods $\downarrow$	$\mathbf{h = \frac{1}{8}}$	$\mathbf{h = \frac{1}{16}}$	$\mathbf{h = \frac{1}{32}}$
Second Order Method $(p, q, r, s) = (\frac{1}{5040}, \frac{120}{5040}, \frac{1191}{5040}, \frac{2416}{5040})$	2.19×10^{-4}	3.88×10^{-5}	1.59×10^{-6}
Fourth Order Method $(p, q, r, s) = (0, 0, \frac{1}{4}, \frac{1}{2})$	7.02×10^{-6}	4.35×10^{-6}	7.87×10^{-7}
Sixth Order Method $(p, q, r, s) = (0, \frac{1}{120}, \frac{26}{120}, \frac{66}{120})$	3.79×10^{-10}	2.51×10^{-11}	5.10×10^{-10}

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