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Homotopy Perturbation Method based on Orthogonal Polynomials

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Abstract: In this paper, the homotopy perturbation method (HPM) on the basis of orthogonal polynomials is applied for solving nonlinear ordinary differential equations. Comparison between homotopy perturbation method based on Legendre, Chebyshev and Taylor expansions is performed by solving some nonlinear differential equations, using them.

Keywords: Homotopy Perturbation Method; Legendre polynomials; Chebyshev polynomials; nonlinear differential equations

1 Introduction

Scientific issues and physical phenomenons often can be modeled in the form of differential equations. The majority of these problems, that we face, are too difficult to find precise analytical solution. In recent years, the methods like the Adomian decomposition, the variational iteration and the $\frac{G'}{G}$ method [1]-[5] have drawn the attention of scientists and engineers. The homotopy perturbation method is one of these, which has been used to solved linear and nonlinear problems, for instance, differential equations both ordinary and partial, linear and nonlinear integro-differential equations. In this paper, we use mentioned method on the basis of orthogonal polynomials to solve various problems. Considering the obtained solutions by HPM, we notice that the more we take distance from $x_0 = 0$, the more we lose the precise for obtained solution. In order to overcome this problem, in this paper, orthogonal polynomials like Legendre and Chebyshev polynomials are used. For exploring efficiency and validity of HPM, results obtained from explained method in this paper and HPM on the basis of Taylor expansion, are compared. In HPM, an approximate solution is considered as a series of functions in which each term can be easily obtained [5]. In this paper using the Legendre and Chebyshev polynomials in HPM, the solution of problem is improved especially in beginning and end of [a, b].

Legendre orthogonal polynomials in [-1,1] are the eigenfunctions of sturm Lioville problem and the solutions of Legendre differential equation $\frac{d}{dx}[(1-x^2)\frac{d}{dx}P_n(x)]+\lambda P_n(x) = 0$, in which $\lambda = n(n+1)$. Also Chebyshev orthogonal polynomials in $[-1, 1]$ are the eigenfunctions of singular sturm Lioville problem $\frac{d}{dx}(\sqrt{1-x^2}\,\frac{d}{dx}\,T_k(x)) + \frac{k^2}{\sqrt{(1-x^2)}}\,T_k(x) = 0$.

2 Homotopy Perturbation method

He's homotopy perturbation method (HPM) [1, 3, 4] is used for solving nonlinear differential equation

$$L(u) + R(u) + N(u) = g(x), \qquad x \in \Omega, \tag{1}$$

subject to the boundary condition

$$B(u, \frac{\partial u}{\partial x}) = 0, \qquad x \in \Gamma, \tag{2}$$

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where L is a linear operator of highest order, R a linear operator of the remaining linear terms, N a nonlinear operator, B a boundary operator, $g(x)$ is a known analytical function and Γ is the boundary of the domain Ω. In homotopy perturbation method, we construct a homotopy $\nu(r,p) : \Omega \times [0,1] \to R$, which satisfies in

$$H(\nu,p) = (1-p)[L(\nu) - L(u_0)] + p\,[L(\nu) + R(\nu) + N(\nu) - g(x)] = 0, \tag{3}$$

or

$$H(\nu,p) = L(\nu) - L(u_0) + p\,L(u_0) + p\,[R(\nu) + N(\nu) - g(x)] = 0, \tag{4}$$

where $p \in [0,1]$ is an embedding parameter, and u_0 is an initial approximation of (1). Obviously, we have

$$H(\nu,0) = L(\nu) - L(u_0) = 0, \tag{5}$$

$$H(\nu,1) = L(\nu) + R(\nu) - g(x) = 0.$$

The changing of p from 0 to 1 is just that of $H(\nu,p)$ from $L(\nu) - L(u_0)$ to $L(\nu) + R(\nu) + N(\nu) - g(x)$. In topology, this is called deformation and $L(\nu) - L(u_0)$ and $L(\nu) + R(\nu) + N(\nu) - g(x)$ are called homotopic. According to the homotopy perturbation method, the parameter p is used as a small parameter, and the solution of (3) can be expressed as a series in p in case of

$$\nu + p\,\nu_1 + p^2\,\nu_2 + p^3\,\nu_3 + \ldots, \tag{6}$$

when $p \to 1$, (6) becomes the approximate solution of (1), i.e.

$$u = \lim_{p\to 1} \nu \;=\; \nu_0 + \nu_1 + \nu_2 + \nu_3 + \ldots. \tag{7}$$

3 HPM Based on Legendre and Chebyshev Polynomials

To solve differential equations by the homotopy perturbation method, for an arbitrary integer number m, $g(x)$ can be expressed in the Taylor series, Chebyshev and Legendre series, that is pointed by $g_{T,m}(x)$, $g_{C,m}(x)$, and $g_{L,m}(x)$, respectively, where

$$g(x) \;\approx\; g_{T,m}(x) = \sum_{n=0}^{m} \frac{g^n(0)}{n!}\,x^n, \tag{8}$$

$$g(x) \;\approx\; g_{C,m}(x) = \sum_{n=0}^{m} a_n T_n(x), \tag{9}$$

$$g(x) \;\approx\; g_{L,m}(x) = \sum_{n=0}^{m} c_n\,P_n(x),. \tag{10}$$

In (9), Chebyshev polynomials $T_n(x)$, satisfy the following recursive relation

$$T_0(x) = 1,\ T_1(x) = x,\ T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x),\ n \geq 2$$

and

$$a_0 = \frac{1}{\pi}\int_{-1}^{1} \frac{g(x)T_0(x)}{\sqrt{1-x^2}}dx,\quad a_i = \frac{2}{\pi}\int_{-1}^{1} \frac{g(x)T_i(x)}{\sqrt{1-x^2}}dx,\quad i = 1,2,\cdots,m.$$

If $g(x)$ is defined in $[a,b]$, then (9) is written

$$g(x) \approx g_{C,m}(x) = \sum_{n=0}^{m} a_n T_n(\frac{2}{b-a} - \frac{b+a}{b-a}),$$

where

$$a_0 = \frac{1}{\pi}\int_{-1}^{1} \frac{g(\frac{b-a}{2})T_0(x)}{\sqrt{1-x^2}}dx,\quad a_i = \frac{2}{\pi}\int_{-1}^{1} \frac{g(\frac{b-a}{2})T_i(x)}{\sqrt{1-x^2}}dx,\quad i = 1,2,\cdots m.$$

In (10), the Legendre polynomials $P_n(x)$ satisfy the following recursive relation

$$P_0(x)=1,\ P_1(x)=x,\ P_n(x)=\frac{2n-1}{n}xP_{n-1}(x)-\frac{n-1}{n}P_{n-2}(x),\quad n\geq 2,$$

and

$$c_i=\frac{2i+1}{2}\int_{-1}^{1}g(x)P_i(x)\,dx.$$

The nonlinear term $N(v)$ is written by means of He's polynomials by

$$N(\nu_0,\nu_1,\cdots,\nu_n)=\frac{1}{n!}\frac{\partial^n}{\partial p^n}N(\sum_{k=0}^{n}p^k\nu_k)_{p=0},\qquad n=0,1,2,\cdots, \tag{11}$$

where satisfy the following relation

$$N(\nu)=N(\nu_0)+N(\nu_0,\nu_1)p+N(\nu_0,\nu_1,\nu_2)p^2+\cdots+N(\nu_0,\nu_1,\nu_2,\cdots,\nu_n)p^n+\cdots. \tag{12}$$

Substituting (6), (10) and (12) in (4), and equating coefficients of equal powers of p, gives

$$p^0\ :\ L(\nu_0)-L(u_0)=0, \tag{13}$$
$$p^1\ :\ L(\nu_1)+L(u_0)+R(\nu_0)+N(\nu_0)-g_{L,m}(x)=0, \tag{14}$$
$$p^2\ :\ L(\nu_2)+R(\nu_1)+N(\nu_0,\nu_1)=0, \tag{15}$$
$$p^3\ :\ L(\nu_3)+R(\nu_2)+N(\nu_0,\nu_1,\nu_2)=0, \tag{16}$$
$$\vdots$$
$$p^{n+1}\ :\ L(\nu_{n+1})+R(\nu_n)+N(\nu_0,\nu_1,\nu_2,\cdots,\nu_n)=0. \tag{17}$$

By solving the above equations with suitable initial conditions, the approximative solution of equation (1) is obtained according to $U_N=\sum_{k=0}^{N}\nu_k, (N\leq m-1)$.
Also in HPM based on Chebyshev, $g_{C,m}(x)$ from (9) is used instead of $g_{L,m}(x)$ in (14).

4 Numerical results

In this section, three initial nonlinear differential equations of second order are solved by homotopy perturbation method. In order to compare the precision of HPM on the basis of Taylor, Chebyshev and Legendre, their figure are drown.

Example 1 *Consider the following equation with the exact solution $u(x)=e^{x^2}$ [1].*

$$u''+xu'+x^2u^3=(2+6x^2)e^{x^2}+x^2e^{3x^2},\ \ 0\leq x\leq 1, \tag{18}$$
$$u(0)=1,\ \ u'(0)=0. \tag{19}$$

The operator form of (18) is

$$L(u)+R(u)+N(u)=g(x),\ \ 0\leq x\leq 1 \tag{20}$$

where

$$L=\frac{d^2}{dx^2},\ R=x\frac{d}{dx},\ N(u)=x^2u^3$$

and

$$g(x)=(2+6x^2)e^{x^2}+x^2e^{3x^2}.$$

Homotopy (4) is

$$\nu''-L(u_0)+\mathrm{p}L(u_0)+\mathrm{p}[x\nu'+x^2\nu^3-g(x)]=0, \tag{21}$$

which expansions of $g(x)$ in three cases Taylor, Chebyshev and Legendre, respectively are obtained by

$$g_{T,6}(x) \approx 2+9x^2+10x^4+\frac{47}{6}x^6, \tag{22}$$

$$g_{C,6}(x) \approx 2.031686-2.897768x+51.49054x^2-227.0229x^3+560.3514x^4 -623.3735x^5+281.1969x^6, \tag{23}$$

$$g_{L,6}(x) \approx 2.071452-3.74041x+55.3264x^2-229.4513x^3+546.1548x^4 -597.4087x^5+268.7491x^6. \tag{24}$$

In (21), u_0 is the initial approximation of $u(x)$ that is obtained by use of Hermit interpolation on conditions $u(0)=1$, $u'(0)=0$, $u''(0)=2$. So $u_0(x)=1+x^2$, thus $L(u_0)=2$.
According to (11), He's polynomials are

$$\begin{aligned}
&N(\nu_0)=x^2\nu_0^3,\\
&N(\nu_0,\nu_1)=x^2(3\nu_0^2\nu_1),\\
&N(\nu_0,\nu_1,\nu_2)=x^2(3\nu_0^2\nu_2\ +\ 3\nu_0\nu_1^2),\\
&N(\nu_0,\nu_1,\nu_2,\nu_3)=x^2(3\nu_0^2\nu_3\ +\ 6\nu_0\nu_1\nu_2\ +\ \nu_1^3),\\
&N(\nu_0,\nu_1,\nu_2,\nu_3,\nu_4)=x^2(3\nu_1^2\nu_2\ +\ 3\nu_0\nu_2^2\ +6\nu_0\nu_1\nu_3\ +\ 3\nu_0^2\nu_4),\\
&N(\nu_0,\nu_1,\nu_2,\nu_3,\nu_4,\nu_5)=x^2(3\nu_1\nu_2^2\ +\ \nu_1^2\nu_3\ +\ 6\nu_0\nu_2\nu_3\ +\ 6\nu_0\nu_1\nu_4\ +\ 3\nu_0^2\nu_5).\\
&\vdots
\end{aligned} \tag{25}$$

Substituting above relations in (21), gives

$$N(\nu)=x^2\nu^3 = x^2\nu_0^3 + x^2(3\nu_0^2\nu_1)\mathrm{p} + x^2(3\nu_0^2\nu_2+3\nu_0\nu_1^2)\mathrm{p}^2 + x^2(3\nu_0^2\nu_3\ +\ 6\nu_0\nu_1\nu_2\ +\ \nu_1^3)\mathrm{p}^3\ +\ \cdots, \tag{26}$$

and substituting (26) and (6) in homotopy (21), gives

$$\mathrm{p}^0 : \begin{cases} \nu_0''-2=0,\\ \nu_0(0)=1,\ \nu_0'(0)=0 \end{cases} \tag{27}$$

$$\mathrm{p}^1 : \begin{cases} \nu_1''+2+x\nu_0'+x^2\nu_0^3-g(x)=0,\\ \nu_1(0)=0,\ \nu_1'(0)=0, \end{cases} \tag{28}$$

$$\mathrm{p}^2 : \begin{cases} \nu_2''+x\nu_1'+3x^2\nu_0^2\nu_1=0,\\ \nu_2(0)=0,\ \nu_2'(0)=0, \end{cases} \tag{29}$$

$$\mathrm{p}^3 : \begin{cases} \nu_3''+x\nu_2'+3x^2(\nu_0^2\nu_2+\nu_0\nu_1^2)=0,\\ \nu_3(0)=0,\ \nu_3'(0)=0, \end{cases} \tag{30}$$

$$\mathrm{p}^4 : \begin{cases} \nu_4''+x\nu_3'+x^2(3\nu_0^2\nu_3+6\nu_0\nu_1\nu_2+\nu_1^3)=0,\\ \nu_4(0)=0,\ \nu_4'(0)=0, \end{cases} \tag{31}$$

$$\mathrm{p}^5 : \begin{cases} \nu_5''+x\nu_4'+x^2(3\nu_1^2\nu_2+3\nu_0\nu_2^2+6\nu_0\nu_1\nu_3+3\nu_0^2\nu_4)=0,\\ \nu_5(0)=0,\ \nu_5'(0)=0, \end{cases} \tag{32}$$

$$\mathrm{p}^6 : \begin{cases} \nu_6''+x\nu_5'+x^2(3\nu_1\nu_2^2+\nu_1^2\nu_3+6\nu_0\nu_2\nu_3+6\nu_0\nu_1\nu_4+3\nu_0^2\nu_5)=0,\\ \nu_6(0)=0,\ \nu_6'(0)=0. \end{cases} \tag{33}$$

$$\vdots$$

Substituting (22) in $g(x)$ at (28), and solving differential equation (27)-(33) with Matlab, the approximate solution based *on Taylor expansion is given by*

$$U_{T,6}(x)=\sum_{i=0}^{6}\nu_i = 0.916437\times10^{-12}x^{50} - 0.208106\times10^{-10}x^{48} + 0.973285\times10^{-11}x^{46} + 0.107307\times10^{-8}x^{44} + 0.594508\times10^{-8}x^{42} + 0.175404\times10^{-8}x^{40} - 0.165370\times10^{-6}x^{38} - 0.115087\times10^{-5}x^{36} - 0.459765\times10^{-5}x^{34} - 0.125830\times10^{-4}x^{32} - 0.231139\times10^{-4}x^{30} - 0.217812\times10^{-4}x^{28} + 0.283609\times10^{-4}x^{26} + 0.185292\times10^{-4}x^{24} + \cdots + 0.416667\times10^{-4}x^{8} + 0.166667x^{6} + 0.50000x^{4} + x^{2} + 1.$$

Similarly, placing (23) in $g(x)$ at (28), the approximate solution based on solution based on Chebyshev polynomials is

$$U_{C,6}(x)=\sum_{i=0}^{6}\nu_i = 0.916433\times10^{-12}x^{50} - 0.150239\times10^{-8}x^{48} + 0.528641\times10^{-8}x^{47} + 0.799210\times10^{-6}x^{46} - 0.562944\times10^{-5}x^{45} - 0.130278\times10^{-3}x^{44} + \cdots + 1.82018x^{9} + 2.81702x^{8} - 13.4651x^{7} + 18.0384x^{6} - 11.2787x^{5} + 4.03821x^{4} - 0.482963x^{3} + 1.01585x^{2} + 1.$$

Finally, placing (24) in $g(x)$ at (28), the approximate solution based on Legendre polynomials is as form

$$U_{L,6}(x)=\sum_{i=0}^{6}\nu_i = 0.916433\times10^{-12}x^{50} - 0.145157\times10^{-8}x^{48} + 0.512248\times10^{-8}x^{47} + \cdots + 2.63112x^{8} - 12.8249x^{7} + 17.5209x^{6} - 11.3790x^{5} + 4.35455x^{4} - 0.623397x^{3} + 1.03572x^{2} + 1.$$

For comparison of results in three different expansions, the figure of absolute errors $ET=|U_{T,6}(x)-e^{x^2}|$, $EC=|U_{C,6}(x)-e^{x^2}|$, $EL=|U_{L,6}(x)-e^{x^2}|$ are presented in figure 1.

Example 2 *Consider differential equation*

$$u''+uu'=x\sin(2x^2)-4x^2\sin(x^2)+2\cos(x^2),\quad 0\le x\le 1, \tag{34}$$

with initial values $u(0)=0$, $u'(0)=0$.
The analytical solution of this equation is $u(x)=\sin(x^2)$ [5].
The operator form of (34) is

$$L(u)+N(u)=g(x)$$

that

$$L(u)=\frac{d^2u}{dx^2}=u'',\ N(u)=uu',\ g(x)=x\sin(2x^2)-4x^2\sin(x^2)+2\cos(x^2).$$

According (4), the homotopy form of above differential equation is

$$\nu''-2+\mathrm{p}(2+\nu\nu'-g(x))=0. \tag{35}$$

In above equation, $g(x)$ is replaced with expansions of $g_T(x), g_C(x), g_L(x)$. He's polynomials are in the forms of

$$\begin{aligned}
&N(\nu_0)=\nu_0\nu_0',\\
&N(\nu_0,\nu_1)=\nu_1\nu_0'+\nu_1'\nu_0,\\
&N(\nu_0,\nu_1,\nu_2)=\nu_2\nu_0'+\nu_1\nu_1'+\nu_0\nu_2',\\
&N(\nu_0,\nu_1,\nu_2,\nu_3)=\nu_3\nu_0'+\nu_2\nu_1'+\nu_1\nu_2'+\nu_0\nu_3',\\
&N(\nu_0,\nu_1,\nu_2,\nu_3,\nu_4)=\nu_4\nu_0'+\nu_3\nu_1'+\nu_2\nu_2'+\nu_1\nu_3'+\nu_0\nu_4',\\
&\vdots\\
&N(\nu_0,\cdots,\nu_6,\nu_7,\nu_8)=\nu_8\nu_0'+\nu_7\nu_1'+\nu_6\nu_2'+\nu_5\nu_3'+\nu_4\nu_4'+\nu_3\nu_5'+\nu_2\nu_6'+\nu_1\nu_7'+\nu_0\nu_8',\\
&N(\nu_0,\cdots,\nu_9)=\nu_9\nu_0'+\nu_8\nu_1'+\nu_7\nu_2'+\nu_6\nu_3'+\nu_5\nu_4'+\nu_4\nu_5'+\nu_3\nu_6'+\nu_2\nu_7'+\nu_1\nu_8'+\nu_0\nu_9'.
\end{aligned}$$

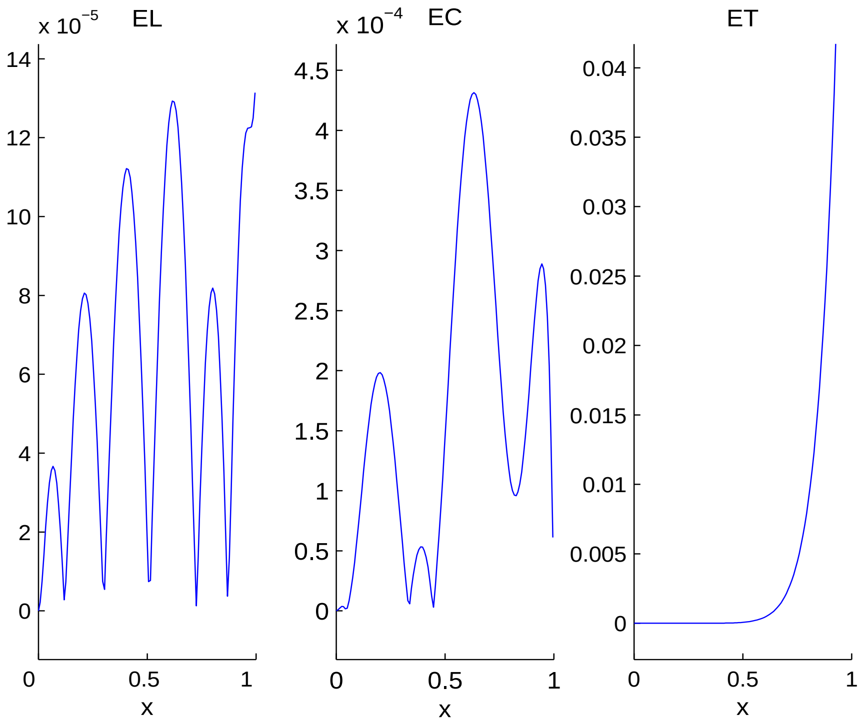

Figure 1: Absolute errors of HPM by Lengendre, Chebyshev and Taylor polynomials respectively of example 1.

Substituting above relations in (12), and equating coefficients of equal powers of p *gives*

$$
\begin{aligned}
\mathrm{p}^0 &: \left\{ \begin{array}{l} \nu_0'' - 2 = 0, \\ \nu_0'(0) = 0,\ \nu_0(0) = 0. \end{array} \right. \\
\mathrm{p}^1 &: \left\{ \begin{array}{l} \nu_1'' + 2 + \nu_0\nu_0' - g(x) = 0, \\ \nu_1'(0) = 0,\ \nu_1(0) = 0. \end{array} \right. \\
\mathrm{p}^2 &: \left\{ \begin{array}{l} \nu_2'' + \nu_1\nu_0' + \nu_1'\nu_0 = 0, \\ \nu_2'(0) = 0,\ \nu_2(0) = 0. \end{array} \right. \\
\mathrm{p}^3 &: \left\{ \begin{array}{l} \nu_3'' + \nu_2\nu_0' + \nu_1'\nu_1 + \nu_0\nu_2' = 0, \\ \nu_3'(0) = 0,\ \nu_3(0) = 0. \end{array} \right. \\
\mathrm{p}^4 &: \left\{ \begin{array}{l} \nu_4'' + \nu_3\nu_0' + \nu_1'\nu_2 + \nu_1\nu_2' + \nu_0\nu_3' = 0, \\ \nu_4'(0) = 0,\ \nu_4(0) = 0. \end{array} \right. \\
&\ \vdots \\
\mathrm{p}^{10} &: \left\{ \begin{array}{l} \nu_{10}'' + \nu_9\nu_0' + \nu_1'\nu_8 + \nu_7\nu_2' + \nu_6\nu_3' + \nu_5\nu_4' + \nu_4\nu_5' + \nu_3\nu_6' + \nu_2\nu_7' + \nu_1\nu_8' + \nu_0\nu_9' = 0, \\ \nu_{10}'(0) = 0,\ \nu_{10}(0) = 0. \end{array} \right.
\end{aligned}
\tag{36}
$$

If $g(x)$ is replaced by $g_{T,10}(x)$, then by solving above equations, we obtain

$$
\begin{aligned}
g_{T,10}(x) &= \sum_{i=0}^{10} C_i T_i(2x-1) = 2 + 2x^3 - 5x^4 - \frac{4}{3}x^7 + \frac{3}{4}x^8 + O(x^{11}), \\
U_T(x) = \sum_{i=0}^{10} \nu_i &= -0.7715918926 \times 10^{-15} x^{55} \ +\ 0.1910694455 \times 10^{-14} x^{54} + \cdots \\
&\quad -0.1709401709 \times 10^{-2} x^{13} +\ 0.008333333333 x^{10} \ -\ 0.1666666667 x^6 + x^2.
\end{aligned}
$$

And also if $g(x)$ is replaced with $g_{C,10}(x)$, then

$$\begin{aligned} g_{C,10}(x) =& \sum_{i=0}^{10} c_i T_i(2x-1) = 2.000079 - 0.004575138x + 0.07259847x^2 + 1.491726x^3 \\ & -3.269352x^4 - 2.286241x^5 - 2.548907x^6 + 11.98909x^7 - 18.43044x^8 \\ & +12.57527x^9 - 2.965324x^{10}. \end{aligned}$$

$$\begin{aligned} U_{C,10}(x) = \sum_{i=0}^{10} \nu_i &= 0.8266260308 \times 10^{-15}x^{67} - 0.2270536396 \times 10^{-13}x^{66} \\ & + 0.2936802291 \times 10^{-12}x^{65} + \cdots - 0.02542160014x^5 \\ & + 0.6049872501 \times 10^{-2}x^4 - 0.7625230001 \times 10^{-3}x^3 \\ & + 1.39500 \times 10^{-4}x^2. \end{aligned}$$

And substituting $g_{L,10}(x)$ in $g(x)$, gives

$$\begin{aligned} g_{L,10}(x) &= \sum_{i=0}^{10} a_i P_i(2x-1) = 1.999997 + 0.0002588031x - 0.00820343x^2 \\ & +2.111178x^3 - 5.800379x^4 + 3.398617x^5 - 8.952536x^6 + 13.47777x^7 \\ & - 14.20152x^8 + 8.289320x^9 - 1.690496x^{10}. \\ U_{L,10}(x) &= \sum_{i=0}^{10} \nu_i = 0.4977589206 \times 10^{-16}x^{67} - 0.1580876830 \times 10^{-14}x^{66} + \cdots \\ & + 0.4313385003 \times 10^{-4}x^3 + 0.9999985000x^2. \end{aligned}$$

In order to compare the obtained results, figures of absolute errors $ET = |U_T(x) - sin(x^2)|$, $EC = |U_C(x) - sin(x^2)|$ and $EL = |U_L(x) - sin(x^2)|$ are drown in figure 2.

Example 3 *Consider the differential equation [5]*

$$\begin{aligned} & u'' + 3u - 2u^3 = cos(x)sin(2x), \quad x \in [0,1] \\ & u(0) = 0, \; u'(0) = 1, \end{aligned}$$

with the analytical solution $u(x) = sin(x)$.
An operator form of the above equation is

$$L(u) + R(u) + N(u) = g(x)$$

that

$$L(u) = u'', \; R(u) = 3u, \; N(u) = -2u^3, \; g(x) = cos(x)sin(2x),$$

and it's homotopy form is

$$\nu'' + \mathrm{p}(3\nu - 2\nu^3 - g(x)) = 0. \tag{37}$$

Similar to explanations of previous two examples, approximate solutions are obtained in three cases A, B, C.
A: Taylor expansion

$$\begin{aligned} g_T(x) &= 2x - \frac{7}{3}x^3 + \frac{61}{60}x^5 - \frac{547}{2520}x^7. \\ U_T(x) &= \sum_{i=0}^{7} \nu_i = -0.1564801829 \times 10^{-17}x^{49} + 0.7476702325 \times 10^{-16}x^{47} + \cdots \\ & +0.008333333333x^5 - 0.1666666667x^3 + x. \end{aligned}$$

B: Chebyshev expansion

$$\begin{aligned} g_C(x) &= -0.2716389000\times 10^{-5} + 2.000313114x - 0.006372349090x^2 - 2.283576753x^3 \\ &\quad -0.1899709556x^4 + 1.404547623x^5 - 0.4207188586x^6 - 0.01292492203x^7. \\ U_C(x) &= \sum_{i=0}^{7} = -0.1171291730\times 10^{-23}x^{49} + \cdots - 0.5306895422\times 10^{-3}x^4 \\ &\quad -0.1666144810x^3 - 0.1358194500\times 10^{-5}x^2 + x. \end{aligned}$$

C: Legendre expansion

$$\begin{aligned} g_L(x) &= -0.5864032943\times 10^{-5} + 2.000419978x - 0.007291800371x^2 - 2.280515932x^3 \\ &\quad -0.1940267703x^4 + 1.405257663x^5 - 0.4180999470x^6 - 0.01444776816x^7. \\ U_L(x) &= -0.2044227557\times 10^{-23}x^{49} + \cdots - 0.2932016471\times 10^{-5}x^2 + x. \end{aligned}$$

Figures of absolute errors of example 3 are shown in figure 3.

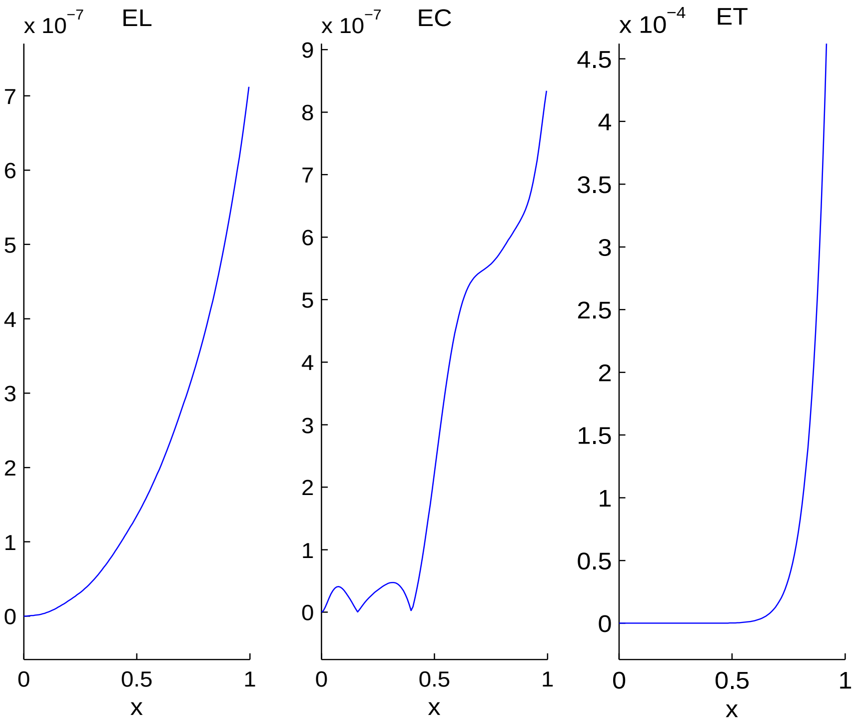

Figure 2: Absolute errors of HPM by Lengendre, Chebyshev and Taylor polynomials respectively of example 2.

5 Conclusion

Considering presented examples and figures 1-3, we conclude that obtained solutions of homotopy perturbation method on the basis of orthogonal polynomials expansion are better than Taylor expansion and obtained solution of the Legendre basis is better than approximate solution of the Chebyshev basis.

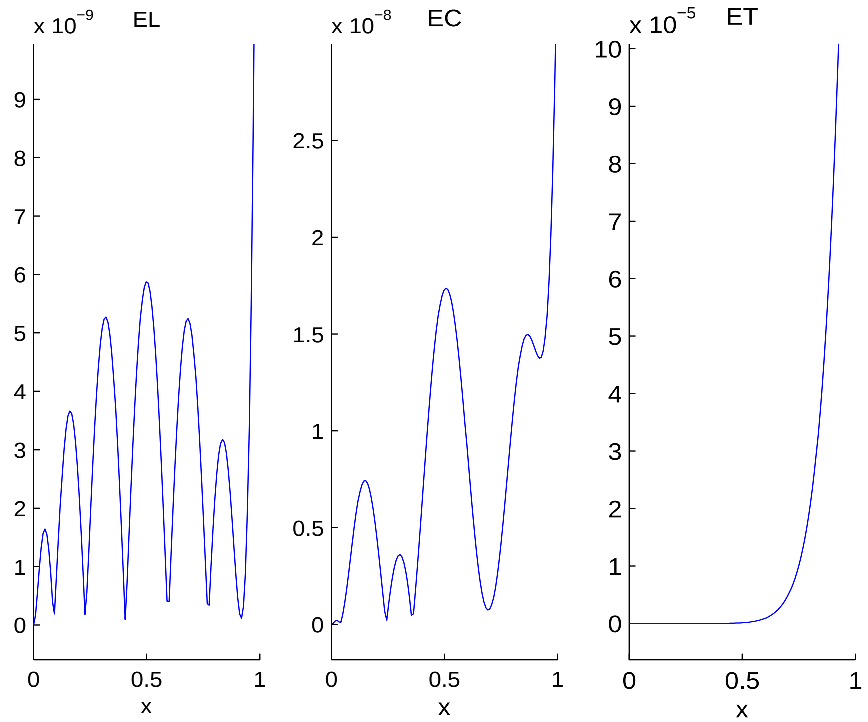

Figure 3: Absolute errors of HPM by Lengendre, Chebyshev and Taylor polynomials respectively of example 3.

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