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Relativistic Corrections to the Theory of the Invariant Relative Orbits of Satellite Flying Formations

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Abstract: Working with the mean orbit elements, the secular drift of the longitude of the ascending node and the sum of the argument of perigee and mean anomaly are set equal between two neighboring orbits. By having both orbits drift at equal angular rates on the average, they will not separate over time due to the influence of the perturbative effects of the asphericity of the Earth and the relativistic correction to the equations of motion, as is considered in this work. The problem is stated. The expressions for the time rate of change of the longitude of the ascending node and the sum of the argument of perigee and mean anomaly in terms of the Delaunay canonical elements are obtained. The expressions for the second order conditions that guarantee that the drift rates of two neighboring orbits are equal on the average are derived.

Keywords:invariant relative orbits; spacecraft formations; relativistic corrections

1 Introduction

The formation flying concept is the use of several small satellites which work together in a group to accomplish the objective of one larger, usually more expensive, satellite. This actually increases the likelihood of mission success in the event of a malfunction. Although the benefits are numerous, formation flying missions have many difficulties that single platform missions do not. For large formations multiple launches may be required which leads to difficulty in formation establishment and initialization. To ensure a cohesive formation, a common semimajor axis must be maintained for all spacecraft in the formation. Inter-satellite communications are necessary to ensure that the desired relative navigation and pointing are achieved once the formation is operational. There is an obvious increase in the navigational and attitude control requirements to ensure proper relative motion. However, recent advances in spacecraft autonomy in both navigation and attitude control permit the implementation of multiple spacecraft missions that were not feasible even in the early nineties. Navigational complications arise due to orbital mechanics and over long time scales due to perturbations. A static formation, in which there is no relative motion between spacecrafts, is only possible for satellites in circular, coplanar orbits. For non-coplanar, non circular orbits there is relative motion in both the along-track and cross-track directions Hughes, [1].

The problem of designing a satellite formations applies all of the criteria for designing a single satellite orbit. In fact orbit design has no absolute rules. Effective orbit design requires clearly identifying the reasons for orbit selection, reviewing these reasons regularly as mission requirements change or mission definition improves, and continuing to remain open to alternatives. Thus, it is needed to consider whether each satellite is launchable, survivable, and properly in view of ground stations or relay satellites. It is also required to consider the number of satellites, their relative positions, and how these positions change with time, both in the course of an orbit and over the lifetime of the formation. We will reference all the orbits of the spacecraft, namely the deputy orbits to one orbit in the formation namely the chief. The different methods to setup invariant relative orbits for spacecraft flying formations are:- 1- Inclination difference method, 2- Node difference method, 3- Combination of the above two methods.

The literature is wealth with works dealing with designing certain invariant relative orbits for spacecraft flying formations, and it seems worth to sketch some of the most relevant works.

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Easton and Brescia [2] of the united states Naval Research Laboratory analyzed coverage by satellites in two mutually perpendicular orbit planes and concluded that we would need at least six satellites to provide complete Earth coverage. Alfriend, et al. [3] presented two new concepts for minimizing fuel consumption for formation flying satellites. First they derived the changes in the orbital parameters for negating the out-of-plane drift and in track drift due to the J_2 perturbation. Second, they introduced a strategy for equalizing the fuel consumption over all the satellites in the constellation. Schaub and Alfriend [4] presented an analytic method to establish J_2 invariant relative orbits using the mean orbit elements. They obtained two first order conditions between the differences in momenta elements (semi-major axis a, eccentricity e, inclination I) that guarantee that the drift rates of two neighboring orbits are equal on the average. Secular drift of the longitude of the ascending node and the sum of the argument of perigee and mean anomaly were set equal between two neighboring orbits. The resulting orbits require less control and maintenance fuel. Alfriend et al., [5] addressed relative motion orbits for formation flying satellites considering J_2 as the perturbing force only. They categorized the various types of orbits according the number of constraints on the initial conditions of the orbital elements, the various kinds of the orbits in each category are identified and estimates given for fuel requirement to maintain a bounded orbit.

Since the differential gravity perturbations are functions of (a, e, I), the small changes in these elements result in different drift rates for each satellite and the negation of these drifts result in different fuel requirements for each satellite. Since some satellites running out of fuel before others will degrade the system performance it would be advantageous to have the satellites have equal fuel consumption. Using the Hamiltonian framework, the authors aim to tackle the following points:

- Constructing an analytical method to design invariant relative orbits for the flying formation.
- The emphasis will be on secular oblateness perturbations due to the zonal harmonics J_2, J_3, J_4 and the relativistic corrections to the equations of motion.
- The invariance, regarding the perturbations considered, in the longitude of the ascending node and in mean argument of latitude are obtained up to the second order, assuming J_2 is of order 1 and $\frac{1}{c_l^2}$ is of order 2.

2 The transformed Hamiltonian

The Hamiltonian in the PN framework can be written in the form, Heimberger et. al. [6]

$$\mathcal{H} = \bar{v}.\bar{p} - \tfrac{1}{2}v^2 - U_\oplus - \tfrac{1}{c_l^2}\left[\tfrac{1}{8}v^4 - \tfrac{1}{2}U_\oplus^2 + \tfrac{3}{2}U_\oplus v^2\right] \quad (1)$$

where; $\bar{v}$ is the satellite's velocity vector, $U_\oplus$ is the force function due to the Earth's gravitational potential and c_l is the velocity of light. The canonical momentum vector, $\bar{p}$ is given by

$$\bar{p} = \left[1 + \tfrac{1}{2}\tfrac{v^2}{c_l^2} + 3\tfrac{U_\oplus}{c_l^2}\right]\bar{v}$$

So that equation (1) can be written in the form,

$$\mathcal{H} = \tfrac{1}{2}p^2 - U_\oplus + \tfrac{1}{c_l^2}\left[-\tfrac{1}{8}p^4 + \tfrac{1}{2}U_\oplus^2 - \tfrac{3}{2}U_\oplus p^2\right] \quad (2)$$

The disturbing force due to the asphericity of the Earth, $U_\oplus$, (truncating the series at $n = 4$) may be written as:

$$\begin{aligned} U_\oplus &= \tfrac{\mu}{r}\Big\{1 - \tfrac{1}{4}J_2\left(\tfrac{R}{r}\right)^2\left[\left(3S^2-2\right) - 3S^2\cos F_{2,2}\right] - \tfrac{1}{8}J_3\left(\tfrac{R}{r}\right)^3 \\ &\times \left[\left(15S^3 - 12S\right)\sin F_{11} - 5S^3\sin F_{3,3}\right] - \tfrac{1}{64}J_4\left(\tfrac{R}{r}\right)^4 \\ &\times \left[\left(24 - 120S^2 + 105S^4\right) + \left(120S^2 - 140S^4\right)\cos F_{2,2} + 35S^4\cos F_{4,4}\right]\Big\} \end{aligned}$$

where R is the Earth equatorial radius, r is the radial coordinate of the satellite, $J_2,\ J_3,\ J_4$ are the zonal harmonic coefficients, and adopting $C = \cos I$, $S = \sin I$, and $F_{i,j} = if + jg$. Using the Delaunay canonical-variables ($l,\ g,\ h,\ L,\ G, H$)defined by

$$\begin{aligned} l &= M & g &= \omega & h &= \Omega \\ L &= \sqrt{\mu a} & G &= \sqrt{\mu a\left(1-e^2\right)} & H &= \sqrt{\mu a\left(1-e^2\right)}\cos I \end{aligned}$$

Considering J_2 as a small parameter of the problem, the orders of magnitude, up to the second order, of the involved parameters are defined as follows: $J_2 \equiv \mathcal{O}(1);\ J_3,\ J_4,\ 1/\ c_l^2 \equiv \mathcal{O}(2)$. The Hamiltonian, up to the second order, can now be expressed as a power series in J_2 as follows

$$\mathcal{H} \;=\; \mathcal{H}_0 + \sum_{n=1}^{2} \frac{J_2^n}{n!}\mathcal{H}_N + \sum_{n=2}^{5} \frac{J_2^n}{n!}\mathcal{H}_{PN} \tag{3}$$

where $\mathcal{H}_0$ is the integrable part of the Hamiltonian, $\mathcal{H}_N$ is the Newtonian contribution, and $\mathcal{H}_{PN}$ is the contribution due to the relativistic corrections. The author used a perturbation technique based on Lie series and Lie transform, Kamel [7], to eliminate the short as well as the long periodic terms, respectively from the Hamiltonian (3). The transformed Hamiltonian, for different orders 0,1,2,3, is obtained by Abd El-Salam [8].

$$\mathcal{H}^{**} \;=\; \mathcal{H}_0^{**} + \mathcal{H}_1^{**} + \mathcal{H}_2^{**} + \mathcal{H}_{PN}^{**} \tag{4}$$

with

$$\begin{aligned}
\mathcal{H}_0^{**} &= -\frac{\mu^2}{2L^2} \\
\mathcal{H}_1^{**} &= \frac{1}{4}A_{1,2}\eta_{3,3}\left(1-3C^2\right) \\
\mathcal{H}_2^{**} &= \frac{3}{64}\frac{A_{1,2}^2}{\mu^2}\left\{\left[5-10C^2-35C^4\right]\eta_{3,7}-\left[4-24C^2+36C^4\right]\eta_{4,6}\right. \\
&- \left.\left[5-18C^2+5C^4\right]\eta_{5,5}\right\} + \frac{3}{128}A_{2,4}\left\{\left[15-150C^2+175C^4\right]\eta_{3,7}\right. \\
&- \left.\left[9-90C^2+105C^4\right]\eta_{5,5}\right\} - 3B_2\left\{\eta_{3,1}-\frac{5}{8}\eta_{4,0}\right\} \\
\mathcal{H}_{PN}^{**} &= B_{3,2}\left(3S^2-2\right)\left\{\frac{3}{4}\eta_{3,5}-\frac{5}{8}\eta_{5,3}\right\} + \frac{A_{1,2}}{\mu^2}B_2\left(3S^2-2\right) \\
&\times \left\{\frac{45}{4}\eta_{3,5}+\frac{27}{4}\eta_{4,4}-9\eta_{5,3}\right\}
\end{aligned}$$

where $A_{i,j},\ B_{i,j}$ the dimensionless parameters are defined by

$$\begin{aligned}
A_{1,2} &= \mu^4R^2 & A_{2,4} &= \frac{2\mu^6R^4J_4}{J_2^2} & \Psi &= \left(\frac{a}{r}\right) \\
B_2 &= \frac{2\mu^4c_l^{-2}}{J_2^2} & B_{3,2} &= 3!\frac{c_l^{-2}\mu^6R^2J_2}{J_2^3} & \eta_{i,j} &= L^{-i}G^{-j}
\end{aligned}$$

3 Statement of the problem

To design an invariant relative orbits for spacecraft flying formations we follow:-

1. Compute the secular drift of the longitude of the ascending node $\left(\dot{h} = \dot{\Omega}\right)$, and the sum of the argument of perigee and mean anomaly $\left(\dot{l}+\dot{g}\right)$.
2. These secular drift rates are set equal between two neighboring orbits.
3. Having both orbits drift at equal angular rates on the average, they will not separate over time due to the influence of the perturbative effects of the asphericity of the Earth and the relativistic correction to the equations of motion.

Using the canonical equations of motion

$$\dot{u} \;=\; \frac{\partial\mathcal{H}^{**}}{\partial U} \qquad \dot{U} \;=\; -\frac{\partial\mathcal{H}^{**}}{\partial u} \tag{5}$$

where $(u = l, g, h;\ \ U = L, G, H)$. Substituting the transformed Hamiltonian (4) into equation (5) yields

$$\dot{u} \;=\; \frac{\partial\mathcal{H}_0^{**}}{\partial U} + J_2\frac{\partial\mathcal{H}_1^{**}}{\partial U} + \frac{J_2^2}{2}\frac{\partial\mathcal{H}_2^{**}}{\partial U} + \frac{J_2^3}{3!}\frac{\partial\mathcal{H}_{PN}^{**}}{\partial U} \tag{6}$$

Since the argument of mean latitude (θ) is the sum of the mean anomaly and the argument of perigee (i.e. $\theta = l + g$). Evaluating the required derivatives yields the sum of the argument of perigee and mean anomaly, $\dot{\theta}$ and the secular drift rates of the longitude of the ascending node, $\dot{h}$

$$\dot{\theta} = \sum_{n=0}^{3} \frac{J_2^n}{n!}\mathcal{D}_n^{\theta}, \qquad \dot{h} = \sum_{n=1}^{3} \frac{J_2^n}{n!}\mathcal{D}_n^{h} \tag{7}$$

with

$$\begin{aligned}
\mathcal{D}_0^{\theta} &= \mu^2 \bar{K}_0 & \mathcal{D}_2^{\theta} &= \frac{3}{64}\frac{A_{1,2}^2}{\mu^2}\sum_{n=3}^{6}\bar{K}_n + \frac{3}{128}A_{2,4}\sum_{n=7}^{10}\bar{K}_n + 3B_2\bar{K}_{11} \\
\mathcal{D}_1^{\theta} &= \frac{A_{1,2}}{4}\sum_{n=1}^{2}\bar{K}_n & \mathcal{D}_3^{\theta} &= B_{3,2}\sum_{n=12}^{13}\bar{K}_n + \frac{A_{1,2}}{\mu^2}B_2\sum_{n=14}^{15}\bar{K}_n \\
\mathcal{D}_1^{h} &= \frac{A_{1,2}}{4}Z_1 & \mathcal{D}_2^{h} &= \frac{3}{64}\frac{A_{1,2}^2}{\mu^2}\sum_{n=2}^{4}Z_n + \frac{3}{128}A_{2,4}\sum_{n=5}^{6}Z_n, \\
& & \mathcal{D}_3^{h} &= B_{3,2}Z_7 + \frac{A_{1,2}}{\mu^2}B_2 Z_8
\end{aligned}$$

where the nonvanishing coefficients $Z_i,\ i = 1,...6$ and $\bar{K}_j,\quad j = 1,...,10$ are given by Abd El-Salam, et. al. [9] in the coefficients of equations (10), (11) there. We will write only the newly computed coefficients which account for the relativistic contributions

$$\begin{aligned}
Z_7 &= 6C\left(-\tfrac{3}{4}\eta_{3,6} + \tfrac{5}{8}\eta_{5,4}\right), \\
Z_8 &= 6C\left(-\tfrac{45}{4}\eta_{3,6} - \tfrac{27}{4}\eta_{4,5} + 9\eta_{5,4}\right) \\
\bar{K}_{11} &= \left(3\eta_{4,1} - \tfrac{5}{2}\eta_{5,0} + \eta_{3,2}\right), \\
\bar{K}_{12} &= \left(\tfrac{25}{8}\eta_{6,3} + \tfrac{15}{8}\eta_{5,4} - \tfrac{9}{4}\eta_{4,5} - \tfrac{15}{4}\eta_{3,6}\right), \\
\bar{K}_{13} &= 3C^2\left(\tfrac{21}{4}\eta_{3,6} - \tfrac{25}{8}\eta_{5,4} + \tfrac{9}{4}\eta_{4,5} - \tfrac{25}{8}\eta_{6,3}\right), \\
\bar{K}_{14} &= \left(45\eta_{6,3} - \tfrac{243}{4}\eta_{4,5} - \tfrac{225}{4}\eta_{3,6}\right), \\
\bar{K}_{15} &= 3C^2\left(-45\eta_{6,3} - 18\eta_{5,4} + \tfrac{297}{4}\eta_{4,5} + \tfrac{315}{4}\eta_{3,6}\right)
\end{aligned}$$

4 The second order constraints

In order to prevent two neighboring orbits from drifting apart, the average secular growth needs to be equal. To keep the satellites from drifting apart over time, it would be desirable to match all three rates $\left(\dot{l}, \dot{g}, \dot{h}\right)$. Therefore

$$\dot{\theta}_i = \dot{l} + \dot{g} = \dot{\theta}_j, \qquad \dot{h}_i = \dot{h}_j, \qquad \forall i \neq j \tag{8}$$

where θ is the mean argument of latitude. Denoting the reference mean orbital elements with the subscript "0' the Taylor expansion for the drift rates $\left\{\dot{\Xi}_i, \Xi \equiv \theta, h\right\}$ of a neighboring orbit about the reference orbit element, retaining the terms up the second order derivatives, can be written as

$$\begin{aligned}
\dot{\Xi}_i &= \left(\dot{\Xi}_i\right)_0 + \delta L\left(\dot{\Xi}_{i,L}\right)_0 + \delta G\left(\dot{\Xi}_{i,G}\right)_0 + \delta H\left(\dot{\Xi}_{i,H}\right)_0 + \frac{1}{2!}\left\{(\delta L)^2\left(\dot{\Xi}_{i,LL}\right)_0\right. \\
&+ (\delta G)^2\left(\dot{\Xi}_{i,GG}\right)_0 + (\delta H)^2\left(\dot{\Xi}_{i,HH}\right)_0 + (\delta L)(\delta G)\left(\dot{\Xi}_{i,LG}\right)_0 \\
&+ \left.(\delta G)(\delta H)\left(\dot{\Xi}_{i,GH}\right)_0 + (\delta H)(\delta L)\left(\dot{\Xi}_{i,HL}\right)_0\right\}
\end{aligned} \tag{9}$$

where $\left\{\left(\dot{\Xi}_{i,x}\right)_0 = \frac{\partial \dot{\Xi}_i}{\partial x}\Big|_{x=x_0},\quad x = L, G, H\right\}$, are the required partial derivatives, evaluated at the mean orbital element L_o, G_o, H_o and $\delta x = x_i - x_0$

The required derivatives are evaluated in terms of the elements of the reference orbit as;

$$\left(\dot{\theta}_{i,L}\right)_0 = \sum_{n=0}^{3}\frac{J_2^n}{n!}\mathcal{D}^\theta_{n,L}, \qquad \left(\dot{h}_{i,L}\right)_0 = \sum_{n=0}^{3}\frac{J_2^n}{n!}\mathcal{D}^h_{n,L},$$

$$\left(\dot{h}_{i,G}\right)_0 = \sum_{n=0}^{3}\frac{J_2^n}{n!}\mathcal{D}^h_{n,G}, \qquad \left(\dot{\theta}_{i,H}\right)_0 = \sum_{n=0}^{3}\frac{J_2^n}{n!}\mathcal{D}^\theta_{n,H},$$

$$\left(\dot{\theta}_{i,LL}\right)_0 = \sum_{n=0}^{3}\frac{J_2^n}{n!}\mathcal{D}^\theta_{n,LL}, \qquad \left(\dot{h}_{i,LL}\right)_0 = \sum_{n=0}^{3}\frac{J_2^n}{n!}\mathcal{D}^h_{n,LL},$$

$$\left(\dot{h}_{i,GG}\right)_0 = \sum_{n=0}^{3}\frac{J_2^n}{n!}\mathcal{D}^h_{n,GG}, \qquad \left(\dot{\theta}_{i,HH}\right)_0 = \sum_{n=0}^{3}\frac{J_2^n}{n!}\mathcal{D}^\theta_{n,HH},$$

$$\left(\dot{\theta}_{i,LG}\right)_0 = \sum_{n=0}^{3}\frac{J_2^n}{n!}\mathcal{D}^\theta_{n,LG}, \qquad \left(\dot{h}_{i,LG}\right)_0 = \sum_{n=0}^{3}\frac{J_2^n}{n!}\mathcal{D}^h_{n,LG},$$

$$\left(\dot{h}_{i,GH}\right)_0 = \sum_{n=0}^{3}\frac{J_2^n}{n!}\mathcal{D}^h_{n,GH}, \qquad \left(\dot{\theta}_{i,HL}\right)_0 = \sum_{n=0}^{3}\frac{J_2^n}{n!}\mathcal{D}^\theta_{n,HL},$$

$$\left(\dot{\theta}_{i,G}\right)_0 = \sum_{n=0}^{3}\frac{J_2^n}{n!}\mathcal{D}^\theta_{n,G}, \qquad \left(\dot{h}_{i,H}\right)_0 = \sum_{n=0}^{3}\frac{J_2^n}{n!}\mathcal{D}^h_{n,H},$$

$$\left(\dot{\theta}_{i,GG}\right)_0 = \sum_{n=0}^{3}\frac{J_2^n}{n!}\mathcal{D}^\theta_{n,GG}, \qquad \left(\dot{h}_{i,HL}\right)_0 = \sum_{n=0}^{3}\frac{J_2^n}{n!}\mathcal{D}^h_{n,HL},$$

$$\left(\dot{\theta}_{i,GH}\right)_0 = \sum_{n=0}^{3}\frac{J_2^n}{n!}\mathcal{D}^h_{n,HH}, \qquad \left(\dot{h}_{i,HH}\right)_0 = \sum_{n=0}^{3}\frac{J_2^n}{n!}\mathcal{D}^\theta_{n,GH}.$$

where the nonvanishing coefficients $\mathcal{D}^\theta$ and $\mathcal{D}^h$ are given by Abd El-Salam, et. al. [9] in the appendix. We will write only the modified coefficients which include the relativistic contributions to the theory of invariant relative orbits

$$\begin{aligned}
\mathcal{D}^\theta_{2,L} &= \tfrac{3}{64}\tfrac{A_{1,2}^2}{\mu^2}\left[\eta_{5,7}\left(-36+648C^2-1860C^4\right)+\eta_{6,6}\left(-205+1110C^2\right.\right.\\
&- \left.930C^4\right)+\eta_{7,5}\left(-150+540C^2-150C^4\right)+\eta_{4,8}\left(105-270C^2\right.\\
&- \left.\left.1155C^4\right)\right]+\tfrac{3}{128}A_{2,4}\left[\eta_{5,7}\left(180-1800C^2+2100C^4\right)\right.\\
&- \eta_{7,5}\left(270-2700C^2+3150C^4\right)+\eta_{4,8}\left(315-4050C^2+5775C^4\right)\\
&- \left.\eta_{6,6}\left(225-3150C^2+4725C^4\right)\right]-3B_2\left(12\eta_{5,1}-\tfrac{25}{2}\eta_{6,0}+3\eta_{4,2}\right)
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}^\theta_{3,L} &= B_{3,2}\left\{\left(-\tfrac{75}{4}\eta_{7,3}-\tfrac{75}{8}\eta_{6,4}+9\eta_{5,5}+\tfrac{45}{4}\eta_{4,6}\right)\right.\\
&+ \left.3C^2\left(-\tfrac{63}{4}\eta_{4,6}+\tfrac{125}{8}\eta_{6,4}-9\eta_{5,5}+\tfrac{75}{4}\eta_{7,3}\right)\right\}\\
&+ \tfrac{A_{1,2}}{\mu^2}B_2\left\{\left(-270\eta_{7,3}+243\eta_{5,5}+\tfrac{675}{4}\eta_{4,6}\right)\right.\\
&+ \left.3C^2\left(270\eta_{7,3}+90\eta_{6,4}-297\eta_{5,5}-\tfrac{945}{4}\eta_{4,6}\right)\right\},
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}^\theta_{2,G} &= \tfrac{3}{64}\tfrac{A_{1,2}^2}{\mu^2}\left[\eta_{4,8}\left(-63+1458C^2-5115C^4\right)+\eta_{5,7}\left(-246+1776C^2\right.\right.\\
&- \left.1860C^4\right)+\eta_{6,6}\left(-125+630C^2-225C^4\right)+\eta_{3,9}\left(280-900C^2\right.\\
&- \left.\left.4620C^4\right)\right]+\tfrac{3}{128}A_{2,4}\left[\eta_{4,8}\left(315-4050C^2+5775C^4\right)\right.\\
&+ \eta_{6,6}\left(-225+1125C^2-4725C^4\right)+\eta_{3,9}\left(840-13500C^2\right.\\
&+ \left.\left.23100C^4\right)+\eta_{5,7}\left(-270+5040C^2-9450C^4\right)\right]\\
&+ 3B_2\left(-3\eta_{4,2}-2\eta_{3,3}\right),
\end{aligned}$$

$$
\begin{aligned}
\mathcal{D}^{\theta}_{3,G} &= B_{3,2}\left\{\left(-\tfrac{75}{8}\eta_{6,4}-\tfrac{15}{2}\eta_{5,5}+\tfrac{45}{4}\eta_{4,6}+\tfrac{45}{2}\eta_{3,7}\right)-3B_2\left(3\eta_{4,2}+2\eta_{3,3}\right)\right.\\
&+ \left.3C^2\left(-42\eta_{3,7}+\tfrac{75}{4}\eta_{5,5}-\tfrac{63}{4}\eta_{4,6}+\tfrac{125}{4}\eta_{6,4}\right)\right\}\\
&+ \tfrac{A_{1,2}}{\mu^2}B_2\left\{\left(-135\eta_{6,4}+\tfrac{1215}{4}\eta_{4,6}+\tfrac{675}{2}\eta_{3,7}\right)\right.\\
&+ \left.3C^2\left(225\eta_{6,4}+108\eta_{5,5}-\tfrac{2079}{4}\eta_{4,6}-\tfrac{2520}{4}\eta_{3,7}\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{D}^{\theta}_{3,H} &= 3B_{3,2}C\left(\tfrac{21}{2}\eta_{3,7}-\tfrac{25}{4}\eta_{5,5}+\tfrac{9}{2}\eta_{4,6}-\tfrac{25}{4}\eta_{6,4}\right)\\
&+ 3\tfrac{A_{1,2}}{\mu^2}B_2C\left(-90\eta_{6,4}-36\eta_{5,5}+\tfrac{297}{2}\eta_{4,6}+\tfrac{315}{2}\eta_{3,7}\right),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{D}^{\theta}_{2,LL} &= \tfrac{3}{64}\tfrac{A^2_{1,2}}{\mu^2}\left[\eta_{6,7}\left(180-3240C^2+9300C^4\right)+\eta_{7,6}\left(1230-6660C^2\right.\right.\\
&+ \left.5580C^4\right)+\eta_{8,5}\left(1050-3780C^2+1050C^4\right)+\eta_{5,8}\left(-420\right.\\
&+ \left.\left.1080C^2+4620C^4\right)\right]+\tfrac{3}{128}A_{2,4}\left[\eta_{6,7}\left(-900+9000C^2\right.\right.\\
&- \left.10500C^4\right)+\eta_{8,5}\left(1890-18900C^2+22050C^4\right)\eta_{5,8}+\left(-1260\right.\\
&- \left.\left.16200C^2-2300C^4\right)+\eta_{7,6}\left(1350-18900C^2+28350C^4\right)\right]\\
&+ 3B_2\left(60\eta_{6,1}-75\eta_{7,0}+12\eta_{5,2}\right),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{D}^{\theta}_{3,LL} &= B_{3,2}\left\{\left(\tfrac{525}{4}\eta_{8,3}+\tfrac{225}{4}\eta_{7,4}-45\eta_{6,5}-45\eta_{5,6}\right)\right.\\
&+ \left.3C^2\left(63\eta_{5,6}-\tfrac{375}{4}\eta_{7,4}+45\eta_{6,5}-\tfrac{525}{4}\eta_{8,3}\right)\right\}\\
&+ \tfrac{A_{1,2}}{\mu^2}B_2\left\{\left(2160\eta_{8,3}-1215\eta_{6,5}-675\eta_{5,6}\right)\right.\\
&+ \left.3C^2\left(-18900\eta_{8,3}-540\eta_{7,4}+1485\eta_{6,5}+945\eta_{5,6}\right)\right\},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{D}^{\theta}_{3,GG} &= B_{3,2}\left\{\left(\tfrac{225}{4}\eta_{7,4}+\tfrac{75}{2}\eta_{6,5}-45\eta_{5,6}-\tfrac{135}{2}\eta_{4,7}\right)\right.\\
&+ \left.3C^2\left(378\eta_{3,8}-\tfrac{525}{4}\eta_{5,6}+\tfrac{252}{2}\eta_{4,7}-\tfrac{750}{4}\eta_{6,5}\right)\right\}\\
&+ \tfrac{A_{1,2}}{\mu^2}B_2\left\{\left(540\eta_{6,5}-\tfrac{3645}{2}\eta_{4,7}-\tfrac{4725}{2}\eta_{3,8}\right)\right.\\
&+ \left.3C^2\left(-1350\eta_{6,5}-756\eta_{5,6}+4158\eta_{4,7}+5670\eta_{3,8}\right)\right\},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{D}^{\theta}_{3,HH} &= 3B_{3,2}\left(\tfrac{21}{2}\eta_{3,8}-\tfrac{25}{4}\eta_{5,6}+\tfrac{9}{2}\eta_{4,7}-\tfrac{25}{4}\eta_{6,5}\right)\\
&+ 3\tfrac{A_{1,2}}{\mu^2}B_2\left(-90\eta_{6,5}-36\eta_{5,6}+\tfrac{297}{2}\eta_{4,7}+\tfrac{315}{2}\eta_{3,8}\right),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{D}^{\theta}_{2,LG} &= \tfrac{3}{64}\tfrac{A^2_{1,2}}{\mu^2}\left[\eta_{5,8}\left(252-5832C^2+20460C^4\right)+\eta_{6,7}\left(1230-8880C^2\right.\right.\\
&+ \left.9300C^4\right)+\eta_{7,6}\left(150-3780C^2+1350C^4\right)+\eta_{4,9}\left(-84+2700C^2\right.\\
&+ \left.\left.13860C^4\right)\right]+\tfrac{3}{128}A_{2,4}\left[\eta_{5,8}\left(-1260+16200C^2-23100C^4\right)\right.\\
&+ \eta_{7,6}\left(1350-6750C^2+28350C^4\right)-\eta_{4,9}\left(2520-40500C^2\right.\\
&+ \left.\left.+69300C^4\right)+\eta_{6,7}\left(1350-25200C^2+47250C^4\right)\right]\\
&+ 3B_2\left(12\eta_{5,2}+6\eta_{4,3}\right),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{D}^{\theta}_{3,LG} &= B_{3,2}\left\{\left(\tfrac{225}{4}\eta_{7,4}+\tfrac{75}{2}\eta_{6,5}-45\eta_{5,6}-\tfrac{135}{2}\eta_{4,7}\right)\right.\\
&+ \left.3C^2\left(\tfrac{189}{2}\eta_{4,7}-\tfrac{375}{4}\eta_{6,5}+63\eta_{5,6}-\tfrac{375}{4}\eta_{7,4}\right)\right\}\\
&+ \tfrac{A_{1,2}}{\mu^2}B_2\left\{\left(810\eta_{7,4}-1215\eta_{5,6}-\tfrac{2025}{2}\eta_{4,7}\right)\right.\\
&+ \left.3C^2\left(-1350\eta_{7,4}-540\eta_{6,5}+2079\eta_{5,6}+1890\eta_{4,7}\right)\right\},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{D}^{\theta}_{3,GH} &= B_{3,2}\left\{3C\left(-84\eta_{3,8}+\tfrac{75}{2}\eta_{5,6}-\tfrac{63}{2}\eta_{4,7}+\tfrac{125}{2}\eta_{6,5}\right)\right\}\\
&+ \tfrac{A_{1,2}}{\mu^2}B_2\left\{3C\left(450\eta_{6,5}+216\eta_{5,6}-\tfrac{2079}{2}\eta_{4,7}-\tfrac{2520}{2}\eta_{3,8}\right)\right\}
\end{aligned}
$$

$$\begin{aligned}
\mathcal{D}^{\theta}_{3,HL} &= 3B_{3,2}C\left(-\tfrac{63}{2}\eta_{4,7}+\tfrac{125}{4}\eta_{6,5}-18\eta_{5,6}+\tfrac{125}{2}\eta_{7,4}\right)\\
&+ 3\tfrac{A_{1,2}}{\mu^2}B_2C\left(540\eta_{7,4}+180\eta_{6,5}-558\eta_{5,6}-\tfrac{945}{2}\eta_{3,7}\right),\\
\mathcal{D}^{h}_{3,L} &= 6B_{3,2}C\left(\tfrac{9}{4}\eta_{4,6}-\tfrac{25}{8}\eta_{6,4}\right)\\
&+ 6\tfrac{A_{1,2}}{\mu^2}B_2C\left(\tfrac{135}{4}\eta_{4,6}+27\eta_{5,5}-45\eta_{6,4}\right),\\
\mathcal{D}^{h}_{3,G} &= 6B_{3,2}C\left(\tfrac{21}{4}\eta_{3,7}-\tfrac{25}{8}\eta_{5,5}\right)\\
&+ 6\tfrac{A_{1,2}}{\mu^2}B_2C\left(\tfrac{315}{4}\eta_{3,7}+\tfrac{162}{4}\eta_{4,6}-54\eta_{5,5}\right),\\
\mathcal{D}^{h}_{3,H} &= -6B_{3,2}\left(\tfrac{3}{4}\eta_{3,7}-\tfrac{5}{8}\eta_{5,5}\right)\\
&- 6\tfrac{A_{1,2}}{\mu^2}B_2\left(\tfrac{45}{4}\eta_{3,7}+\tfrac{27}{4}\eta_{4,6}-9\eta_{5,5}\right),\\
\mathcal{D}^{h}_{3,LL} &= -6B_{3,2}C\left(9\eta_{5,6}-\tfrac{75}{4}\eta_{7,4}\right)\\
&- 6\tfrac{A_{1,2}}{\mu^2}B_2C\left(135\eta_{5,6}+135\eta_{6,5}-270\eta_{7,4}\right)\\
\mathcal{D}^{h}_{3,GG} &= -6B_{3,2}C\left(\tfrac{21}{4}\eta_{3,8}-\tfrac{25}{8}\eta_{5,6}\right)\\
&- 6\tfrac{A_{1,2}}{\mu^2}B_2C\left(\tfrac{315}{4}\eta_{3,8}+\tfrac{162}{4}\eta_{4,7}-54\eta_{5,6}\right),\\
\mathcal{D}^{h}_{3,LG} &= -6B_{3,2C}\left(\tfrac{63}{4}\eta_{4,7}-\tfrac{225}{8}\eta_{6,5}\right),\\
&- 6\tfrac{A_{1,2}}{\mu^2}B_2C\left(\tfrac{945}{4}\eta_{4,7}+162\eta_{5,6}-225\eta_{6,5}\right),\\
\mathcal{D}^{h}_{3,GH} &= 6B_{3,2}C\left(\tfrac{21}{4}\eta_{3,8}-\tfrac{25}{8}\eta_{5,6}\right)\\
&+ 6\tfrac{A_{1,2}}{\mu^2}B_2C\left(\tfrac{315}{4}\eta_{3,8}+\tfrac{162}{4}\eta_{4,7}-54\eta_{5,6}\right),\\
\mathcal{D}^{h}_{3,HL} &= 6B_{3,2}\left(\tfrac{9}{4}\eta_{4,7}-\tfrac{25}{8}\eta_{6,5}\right)\\
&+ 6\tfrac{A_{1,2}}{\mu^2}B_2\left(\tfrac{135}{4}\eta_{4,7}+27\eta_{5,6}-45\eta_{6,5}\right),
\end{aligned}$$

Let $\eta=\frac{G}{L}$, therefore we can write

$$\delta G = \eta\,(\delta L)+L\,(\delta\eta) \tag{10}$$

$$\delta H = \eta C\,(\delta L)+LC\,(\delta\eta)-L\eta S\,(\delta I) \tag{11}$$

It worth to note here that δL, δG, and δH are independent. Therefore equation (9) can be rewritten in terms of δL, $\delta\eta$, and δI in the form

$$\begin{aligned}
\delta\dot{\Xi} &= \left[\left(\dot{\Xi}_{i,L}\right)_0+\eta\left(\dot{\Xi}_{i,G}\right)_0+\eta C\left(\dot{\Xi}_{i,H}\right)_0\right]\delta L+\left[L\left(\dot{\Xi}_{i,G}\right)_0+LC\left(\dot{\Xi}_{i,H}\right)_0\right]\delta\eta\\
&+ \left[L\eta S\left(\dot{\Xi}_{i,H}\right)_0\right]\delta I+\left[\tfrac{1}{2}\left(\dot{\Xi}_{i,LL}\right)_0+\tfrac{1}{2}\eta^2\left(\dot{\Xi}_{i,GG}\right)_0+\tfrac{1}{2}\eta^2C^2\left(\dot{\Xi}_{i,HH}\right)_0\right.\\
&+ \left.\eta\left(\dot{\Xi}_{i,LG}\right)_0+\eta^2C\left(\dot{\Xi}_{i,GH}\right)_0+\eta C\left(\dot{\Xi}_{i,HL}\right)_0\right](\delta L)^2+\left[\tfrac{1}{2}L^2\left(\dot{\Xi}_{i,GG}\right)_0\right.\\
&+ \left.\tfrac{1}{2}L^2C^2\left(\dot{\Xi}_{i,HH}\right)_0+\tfrac{1}{2}L^2C\left(\dot{\Xi}_{i,GH}\right)_0\right](\delta\eta)^2+\left[\tfrac{1}{2}L^2\eta^2S^2\left(\dot{\Xi}_{i,HH}\right)_0\right](\delta I)^2\\
&+ \left[L\eta\left(\dot{\Xi}_{i,GG}\right)_0+L\eta C^2\left(\dot{\Xi}_{i,HH}\right)_0+L\left(\dot{\Xi}_{i,LG}\right)_0+2L\eta C\left(\dot{\Xi}_{i,GH}\right)_0\right.\\
&+ \left.LC\left(\dot{\Xi}_{i,HL}\right)_0\right](\delta L)(\delta\eta)-\left[L^2\eta C\left(\dot{\Xi}_{i,HH}\right)_0+L^2\eta C\left(\dot{\Xi}_{i,GH}\right)_0\right](\delta\eta)(\delta I)\\
&- \left[LC\eta\left(\dot{\Xi}_{i,HH}\right)_0+L\eta^2S\left(\dot{\Xi}_{i,GH}\right)_0+L\eta S\left(\dot{\Xi}_{i,LG}\right)_0\right](\delta I)(\delta L)
\end{aligned} \tag{12}$$

where $\delta\dot{\Xi}\equiv\delta\dot{\theta}=\dot{\theta}_i-\dot{\theta}_0$, $\delta\dot{h}=\dot{h}_i-\dot{h}_0$ are the difference between the drift rate the argument of mean latitude of the reference orbit and that one of the neighboring orbit, and the difference between the drift rate of the ascending node of the reference orbit and that one of the neighboring orbit respectively. The conditions satisfying the invariance property for the relative orbits are $\delta\dot{\Xi}=0$. Substituting the included derivatives, previously computed, into equations (12) yields

$$\begin{aligned}\delta\dot{\Xi} &= \mathcal{A}_L^{\Xi}\delta L+\mathcal{A}_\eta^{\Xi}\delta\eta+\mathcal{A}_I^{\Xi}\delta I+\mathcal{A}_{LL}^{\Xi}(\delta L)^2+\mathcal{A}_{\eta\eta}^{\Xi}(\delta\eta)^2+\mathcal{A}_{II}^{\Xi}(\delta I)^2\\ &+ \mathcal{A}_{L\eta}^{\Xi}(\delta L)(\delta\eta)+\mathcal{A}_{\eta I}^{\Xi}(\delta\eta)(\delta I)+\mathcal{A}_{IL}^{\Xi}(\delta I)(\delta L)=0 \end{aligned} \tag{13}$$

where

$$\begin{aligned}
\mathcal{A}_L^\theta &= \sum_{n=1}^{3} J_2^n\mathcal{I}_n^\theta, & \mathcal{A}_\eta^\theta &= \sum_{n=1}^{3} J_2^n\mathcal{K}_n^\theta, & \mathcal{A}_I^\theta &= \sum_{n=1}^{3} J_2^n\mathcal{L}_n^\theta,\\
\mathcal{A}_{II}^\theta &= \sum_{n=1}^{3} J_2^n\mathcal{P}_n^\theta, & \mathcal{A}_{L\eta}^\theta &= \sum_{n=1}^{3} J_2^n\mathcal{Q}_n^\theta, & \mathcal{A}_{IL}^\theta &= \sum_{n=1}^{3} J_2^n\mathcal{S}_n^\theta,\\
\mathcal{A}_I^h &= \sum_{n=1}^{3} J_2^n\mathcal{L}_n^h, & \mathcal{A}_\eta^h &= \sum_{n=1}^{3} J_2^n\mathcal{K}_n^h, & \mathcal{A}_{LL}^\theta &= \sum_{n=1}^{3} J_2^n\mathcal{M}_n^\theta,\\
\mathcal{A}_{LL}^h &= \sum_{n=1}^{3} J_2^n\mathcal{M}_n^h, & \mathcal{A}_{\eta\eta}^h &= \sum_{n=1}^{3} J_2^n\mathcal{N}_n^h, & \mathcal{A}_{II}^h &= \sum_{n=1}^{3} J_2^n\mathcal{P}_n^h,\\
\mathcal{A}_{\eta\eta}^\theta &= \sum_{n=1}^{3} J_2^n\mathcal{N}_n^\theta, & \mathcal{A}_L^h &= \sum_{n=1}^{3} J_2^n\mathcal{I}_n^h, & \mathcal{A}_{\eta I}^\theta &= \sum_{n=1}^{3} J_2^n\mathcal{R}_n^\theta,\\
\mathcal{A}_{L\eta}^h &= \sum_{n=1}^{3} J_2^n\mathcal{Q}_n^h, & \mathcal{A}_{\eta I}^h &= \sum_{n=1}^{3} J_2^n\mathcal{R}_n^h, & \mathcal{A}_{IL}^h &= \sum_{n=1}^{3} J_2^n\mathcal{S}_n^h
\end{aligned}$$

where the included coefficients $\mathcal{S},\mathcal{I},\mathcal{K},\mathcal{L},\mathcal{P},\mathcal{M},\mathcal{N},\ \mathcal{R},\mathcal{Q}$ are obtained previously by the Abd El-Salam et. al. [9] equation (22) there

4.1 Special case: first order constraints

Neglecting the terms higher than the first order, and also neglecting the squared and the mixed differences of δL, $\delta\eta$, and δI in equations (13), we get

$$\mathcal{I}_0^\theta\delta L+J_2(\mathcal{I}_1^\theta\delta L+\mathcal{K}_1^\theta\delta\eta+\mathcal{L}_1^\theta\delta I) = 0 \tag{14}$$

$$J_2(\mathcal{I}_1^h\delta L+\mathcal{K}_1^h\delta\eta+\mathcal{L}_1^h\delta I) = 0 \tag{15}$$

Considering δL of order $\mathcal{O}(J_2)$ as suggested by Schaub and Alfriend [4], terms factored by $J_2\delta L$ can be neglected, we get

$$\mathcal{I}_0^\theta\delta L + J_2(\mathcal{K}_1^\theta\delta\eta+\mathcal{L}_1^\theta\delta I) = 0 \tag{16}$$

$$J_2(\mathcal{K}_1^h\delta\eta+\mathcal{L}_1^h\delta I) = 0 \tag{17}$$

rearranging the terms, equation (17) can be written as

$$\delta\eta = -\frac{\mathcal{L}_1^h}{\mathcal{K}_1^h}\delta I \tag{18}$$

using equation (18) for δI into equation (16), we obtain

$$\mathcal{I}_0^\theta\delta L + J_2(\mathcal{K}_1^\theta\delta\eta-\mathcal{L}_1^\theta\frac{\mathcal{K}_1^h}{\mathcal{L}_1^h}\delta\eta) = 0 \tag{19}$$

rearranging the terms, equation (19) can be written in the form

$$\delta L = -\frac{J_2}{\mathcal{I}_0^\theta}(\mathcal{K}_1^\theta-\mathcal{L}_1^\theta\frac{\mathcal{K}_1^h}{\mathcal{L}_1^h})\delta\eta \tag{20}$$

using the coefficients $\mathcal{L}_1^h,\ \mathcal{K}_1^h,\ \mathcal{I}_0^\theta,\ \mathcal{K}_1^\theta,\ \mathcal{L}_1^\theta$, defined previously, with $\eta_{i,j}=\frac{1}{L^{i+j}\eta^j}$, assuming the distances are measured in Earth radius and the time is normalized by the mean motion of a satellite at one Earth radius, $R=1$ and $\mu=1$, equations (18) and (20) give

$$\delta\eta = -\frac{\eta}{4}(\tan I)\,\delta I \tag{21}$$

$$\delta L = \frac{J_2}{4L^4\eta^5}\left[(3\eta+4)\left(5C^2+1\right)\right]L\delta\eta \tag{22}$$

which are the same first order constraints obtained by Schaub and Alfriend [4].

4.2 Solution of the second order constraints

Equations (13) are two simultaneous nonlinear algebraic equations in three unknowns, namely δL, $\delta\eta$, δI. When one of these three unknowns is assumed known (say δI), these two equations can be solved as follows. Rewriting equations (13) as

$$\begin{aligned}
&\mathcal{A}^{\theta}_{LL}(\delta L)^2 + \mathcal{A}^{\theta}_{L\eta}(\delta L)(\delta\eta) + \mathcal{A}^{\theta}_{\eta\eta}(\delta\eta)^2 + [\mathcal{A}^{\theta}_{L} + \mathcal{A}^{\theta}_{IL}(\delta I)](\delta L) \\
&+[\mathcal{A}^{\theta}_{\eta} + \mathcal{A}^{\theta}_{\eta I}(\delta I)](\delta\eta) + [\mathcal{A}^{\theta}_{I}\delta I + \mathcal{A}^{\theta}_{II}(\delta I)^2] = 0 \qquad (23)\\
&\mathcal{A}^{h}_{LL}(\delta L)^2 + \mathcal{A}^{h}_{L\eta}(\delta L)(\delta\eta) + \mathcal{A}^{h}_{\eta\eta}(\delta\eta)^2 + [\mathcal{A}^{h}_{L} + \mathcal{A}^{h}_{IL}(\delta I)](\delta L) \\
&+[\mathcal{A}^{h}_{\eta} + \mathcal{A}^{h}_{\eta I}(\delta I)](\delta\eta) + [\mathcal{A}^{h}_{I}\delta I + \mathcal{A}^{h}_{II}(\delta I)^2] = 0 \qquad (24)
\end{aligned}$$

Multiplying equation (23) by $\mathcal{A}^{h}_{LL}$ and equation (24) by $\mathcal{A}^{\theta}_{LL}$ and then subtracting yields

$$c_1(\delta L)(\delta\eta) + c_2(\delta\eta)^2 + c_3(\delta L) + c_4(\delta\eta) + c_5 = 0 \qquad (25)$$

where

$$\begin{aligned}
c_1 &= (\mathcal{A}^{\theta}_{L\eta})(\mathcal{A}^{h}_{LL}) - (\mathcal{A}^{\theta}_{LL})(\mathcal{A}^{h}_{L\eta})\\
c_2 &= (\mathcal{A}^{\theta}_{\eta\eta})(\mathcal{A}^{h}_{LL}) - (\mathcal{A}^{\theta}_{LL})(\mathcal{A}^{h}_{\eta\eta})\\
c_3 &= [\mathcal{A}^{\theta}_{L} + \mathcal{A}^{\theta}_{IL}(\delta I)](\mathcal{A}^{h}_{LL}) - (\mathcal{A}^{\theta}_{LL})[\mathcal{A}^{h}_{L} + \mathcal{A}^{h}_{IL}(\delta I)]\\
c_4 &= [\mathcal{A}^{\theta}_{\eta} + \mathcal{A}^{\theta}_{\eta I}(\delta I)](\mathcal{A}^{h}_{LL}) - (\mathcal{A}^{\theta}_{LL})[\mathcal{A}^{h}_{\eta} + \mathcal{A}^{h}_{\eta I}(\delta I)]\\
c_5 &= [\mathcal{A}^{\theta}_{I}\delta I + \mathcal{A}^{\theta}_{II}(\delta I)^2](\mathcal{A}^{h}_{LL}) - (\mathcal{A}^{\theta}_{LL})[\mathcal{A}^{h}_{I}\delta I + \mathcal{A}^{h}_{II}(\delta I)^2]
\end{aligned}$$

From equation (25), we can get

$$(\delta L) = -\frac{c_2(\delta\eta)^2 + c_4(\delta\eta) + c_5}{c_1(\delta\eta) + c_3} \qquad (26)$$

Substituting equation (26) into equation (23) yields an algebraic equation of fourth degree in $\delta\eta$ only in the form

$$d_1(\delta\eta)^4 + d_2(\delta\eta)^3 + d_3(\delta\eta)^2 + d_4\delta\eta + d_5 = 0 \qquad (27)$$

where

$$\begin{aligned}
d_1 &= c_2^2(\mathcal{A}^{\theta}_{LL}) - c_2c_1(\mathcal{A}^{\theta}_{L\eta}) + c_1^2(\mathcal{A}^{\theta}_{\eta\eta})\\
d_2 &= 2c_2c_4(\mathcal{A}^{\theta}_{LL}) - [c_2c_3 + c_4c_1](\mathcal{A}^{\theta}_{L\eta}) - c_2c_1[\mathcal{A}^{\theta}_{L} + \mathcal{A}^{\theta}_{IL}(\delta I)]\\
&+ c_1^2[\mathcal{A}^{\theta}_{\eta} + \mathcal{A}^{\theta}_{\eta I}(\delta I)] + 2c_1c_3(\mathcal{A}^{\theta}_{\eta\eta})\\
d_3 &= [c_4^2 + 2c_2c_5](\mathcal{A}^{\theta}_{LL}) - [c_4c_3 + c_5c_1](\mathcal{A}^{\theta}_{L\eta}) + c_3^2(\mathcal{A}^{\theta}_{\eta\eta}) - [c_2c_3 + c_4c_1]\\
&\times [\mathcal{A}^{\theta}_{L} + \mathcal{A}^{\theta}_{IL}(\delta I)] + 2c_1c_3[\mathcal{A}^{\theta}_{\eta} + \mathcal{A}^{\theta}_{\eta I}(\delta I)] + c_1^2[\mathcal{A}^{\theta}_{I}\delta I + \mathcal{A}^{\theta}_{II}(\delta I)^2]\\
d_4 &= 2c_4c_5(\mathcal{A}^{\theta}_{LL}) - c_5c_3(\mathcal{A}^{\theta}_{L\eta}) - [c_4c_3 + c_5c_1][\mathcal{A}^{\theta}_{L} + \mathcal{A}^{\theta}_{IL}(\delta I)]\\
&+ c_3^2[\mathcal{A}^{\theta}_{\eta} + \mathcal{A}^{\theta}_{\eta I}(\delta I)] + 2c_1c_3[\mathcal{A}^{\theta}_{I}\delta I + \mathcal{A}^{\theta}_{II}(\delta I)^2]\\
d_5 &= c_5^2(\mathcal{A}^{\theta}_{LL}) - c_5c_3[\mathcal{A}^{\theta}_{L} + \mathcal{A}^{\theta}_{IL}(\delta I)] + c_3^2[\mathcal{A}^{\theta}_{I}\delta I + \mathcal{A}^{\theta}_{II}(\delta I)^2]
\end{aligned}$$

5 Solution of the quartic equation (27)

The roots of the quartic equation (27) can be written as

$$\begin{aligned}
(\delta\eta)_{1,2} &= \mathbb{A} - \mathbb{B} \mp \mathbb{C}\\
(\delta\eta)_{3,4} &= \mathbb{A} + \mathbb{B} \mp \mathbb{D}
\end{aligned}$$

where

$$\begin{aligned} \mathbb{A} &= -\tfrac{d_2}{4d_1}, \\ \mathbb{B} &= \tfrac{1}{2}\sqrt{\mathcal{T}+\mathcal{C}+\mathcal{F}}, \\ \mathbb{C}_{1,2} &= \tfrac{1}{2}\sqrt{2\mathcal{T}-\mathcal{C}-\mathcal{F}\mp\mathcal{E}} \end{aligned}$$

with

$$\begin{aligned} \mathcal{T} &= \tfrac{d_2^2}{4d_1^2}-\tfrac{2d_3}{3d_1}, \\ \mathcal{U} &= \left(d_3^2-3d_2d_4+12d_1d_5\right) \\ \mathcal{C} &= \tfrac{2^{\frac{1}{3}}\mathcal{U}}{3d_1\mathcal{Z}}, \\ \mathcal{V} &= \left(2d_3^3-9d_2d_3d_4+27d_1d_4^2+27d_2^2d_5-72d_1d_3d_5\right) \\ \mathcal{Z} &= \left(\mathcal{V}+\sqrt{-4\mathcal{U}^3+\mathcal{V}^2}\right)^{\frac{1}{3}} \\ \mathcal{F} &= \tfrac{1}{3*2^{\frac{1}{3}}d_1}\mathcal{Z}, \\ \mathcal{G} &= -\left(\tfrac{d_2}{d_1}\right)^3+\tfrac{4d_2d_3}{d_1^2}-\tfrac{8d_4}{d_1} \\ \mathcal{E} &= \tfrac{\mathcal{G}}{8\mathbb{B}} \end{aligned}$$

Substituting the four roots $\delta\eta's$ into equation (26) yields the four constraints $\delta L's$ that guarantee the invariance of the relative motion of certain satellite constellation

$$(\delta L)_{1,2} = -\frac{c_2\left(\mathbb{F}-2\mathbb{A}\mathbb{B}\pm 2\mathbb{B}\mathbb{C}_1\mp 2\mathbb{A}\mathbb{C}_1\right)+c_4\left(\mathbb{A}-\mathbb{B}-\mathbb{C}_1\right)+c_5}{c_1\left(\mathbb{A}-\mathbb{B}-\mathbb{C}_1\right)+c_3} \tag{28}$$

$$(\delta L)_{3,4} = -\frac{c_2\left(\mathbb{G}+2\mathbb{A}\mathbb{B}\mp 2\mathbb{B}\mathbb{C}_2\mp 2\mathbb{A}\mathbb{D}\right)+c_4\left(\mathbb{A}+\mathbb{B}-\mathbb{C}_2\right)+c_5}{c_1\left(\mathbb{A}+\mathbb{B}-\mathbb{C}_2\right)+c_3} \tag{29}$$

where

$$\mathbb{F} = \mathbb{A}^2+\mathbb{B}^2+\mathbb{C}^2 \qquad \mathbb{G} = \mathbb{A}^2+\mathbb{B}^2+\mathbb{C}_2^2$$

6 Computational algorithm

Given Data

1. a : The semi -major axis of the master orbit, e : The eccentricity of the master orbit, I : The inclination of the master orbit, δI : The desired difference in inclination between the master orbit and the neighboring orbit, μ : Earth's gravitational constant, R : The mean equatorial radius, J_2 : The second zonal harmonic, J_4 : The fourth zonal harmonic, ,c_l : The velocity of light.
2. Compute the following expressions $C=\cos I$, $S=\sin I$, $L=\sqrt{\mu a}$, $\eta_{i,j}=\frac{1}{L^{i+j}\eta^j}$, $A_{1,2}=\mu^4R^2$, $A_{2,4}=\frac{2\mu^6R^4J_2}{J_2}$, $\eta=\sqrt{1-e^2}$, $B_2=\frac{2\mu^4c_l^{-2}}{J_2^2}$, $B_{3,2}=3!\frac{c_l^{-2}\mu^6R^2J_2}{J_2^3}$
3. Compute the following coefficients $\mathcal{D}'s$ (using the expressions obtained in the required derivatives of the equation (8)).
4. Compute the following coefficients $\mathcal{I}'s$, $\mathcal{K}'s$, $\mathcal{L}'s$, $\mathcal{M}'s$, $\mathcal{N}'s$, $\mathcal{P}'s$, $\mathcal{Q}'s$, $\mathcal{R}'s$, and $\mathcal{S}'s$ (equations (13)).
5. Compute the following coefficients $\mathcal{A}'s$ (in equations (13)).

6. Solving equation (27) either numerically or analytically, as given by the expressions (28) and (29) yields the four roots for $\delta\eta$.

7. Compute the four constraints $\delta L's$ that guarantee the invariance of the relative motion of certain satellite constellation as given by the expressions (28) and (29)

7 Conclusion

Using the oblate Earth model, truncating its potential series at J_4, plus the relativistic corrections to the equations of motion, eight second order conditions between the differences in momenta elements (semi-major axis a, eccentricity e, inclination I) are obtained. These conditions guarantee that the drift rates of two neighboring orbits are equal on the average. The resulting orbits require less control and maintenance fuel. The two first order conditions obtained by Schaub and Alfriend [4] are derived as a special case of the here presented work. Finally a computational algorithm is presented.

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