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Numerical Solution of Two Point Boundary Value Problems Using Galerkin-Finite Element Method

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Abstract: The two point boundary value problems with Neumann and mixed Robbin's boundary conditions have great importance in chemical engineering, deflection of beams etc. It is not easy task to solve numerically such type of problems. In this paper, Galerkin-finite element method is proposed for the numerical solution of the boundary value problem having mixed Robbin's and Dirichlet's conditions. Two test problems are taken to show the accuracy of the result. The numerical solutions are compared with the analytic solution available in the literature and found very similar to the analytic solution.

Keywords: Finite Element Method; Dirichlet's boundary conditions; Mixed Robbin's boundary conditions; Crank-Nicolson Scheme; Stability

1 Introduction

Two-point boundary value problems arise in a number of different applications which include, for example, deflection of beams and initial boundary value problem for partial differential equations. The problem is of considerable practical importance, for example, in determine the efficiency of solvent utilization or filtrate recovery in the washing of filter cakes [1]. Many researchers have developed numerical technique to study the numerical solution of two point boundary value problems. Shelly et al. [2] has proposed orthogonal collocation on finite elements for the solution of two point boundary value problems. Villadsen and Stewart [3] proposed solution of boundary value problem by orthogonal collocation method. Jang [4] proposed the solution of two-point boundary value problem by the extended Adomian decomposition method. The Galerkin-finite element method is well known numerical technique for the numerical solution of differential equations. Dogan [5] proposed the Galerkin-finite element approach for the numerical solutions of Burgers' equation. Sengupta et al. [6] carried out Gakerkin finite element methods for wave problems. Kaneko et al. [7] discussed the Discontinuous Galerkin-finite element method for parabolic problems. EI-Gebeily et al. [8] studied the finite element- Galerkin method for singular self-adjoint differential equations. Sharma et al. [9] proposed Galerkin-finite Element Methods for numerical solution of advection- diffusion equation. Reddy, Noye and Hutten explained the detail of the finite element method in their books [10-12]. Onah [13] proved the asymptotic convergence of the solution of a parabolic equation by using two methods namely, the Galerkin method expressed in terms of linear splines and the Finite Element Collocation method expressed by cubic spline basis functions. Galerkin finite element method for the approximation of a nonlinear integro-differential equation associated with the penetration of a magnetic field into a substance is studied by Jangveladze et al. [14] In the present work, we use Galerkin-finite element method for the numerical solution of two point boundary value problem of the form

$$b_1\frac{\partial^2 c}{\partial x^2}+b_2\frac{\partial c}{\partial x}=b_3\frac{\partial c}{\partial t} \tag{1}$$

with boundary conditions

$$a_1 c+a_2\frac{\partial c}{\partial x}=k_1,\ \ at\ \ x=0,$$

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$$a_3 c + a_4 \frac{\partial c}{\partial x} = k_2, \quad at \;\; x = 1$$

In this paper, we have proposed a simple finite element method to find the approximate solutions of two point boundary value problems. The stability of the proposed method is discussed and shown that the method is conditionally stable.

2 Semidiscrete Finite Element Models

The semi discrete formulation involves approximation of the spatial variation of the dependent variable. The first step involves the construction of the weak form of the given problem over a typical element. In second step, we develop the finite element model by seeking approximation of the solution.

2.1 Weak Formulation of the Problem

The weak formulation of the given problem (1) over a typical linear element (x_a, x_b) is given by

$$\int_{x_a}^{x_b} w \left(b_1 \frac{\partial^2 c}{\partial x} + b_2 \frac{\partial c}{\partial x} - b_3 \frac{\partial c}{\partial t} \right) dx = 0 \tag{2}$$

$$\int_{x_a}^{x_b} \left(b_1 \frac{\partial w}{\partial x} \frac{\partial c}{\partial x} - b_2 w \frac{\partial c}{\partial x} + b_3 w \frac{\partial c}{\partial t} \right) dx = 0 \tag{3}$$

2.2 Finite Element Formulation of the Problem

The finite-element model may be obtained from equation (3) by substituting finite element approximations in the decoupled form

$$c(x,t) = \sum_{j=1}^{N} c_j^e(t_n) \psi_j^e(x); n = 1, 2 : \tag{4}$$

Substituting $w = \psi_i(x)$ and (4) in equation (3) to obtain the i^{th} equation of the system, we have

$$\int_{x_a}^{x_b} \left[b_1 \frac{\partial \psi_i}{\partial x} \left(\sum_{j=1}^{N} c_j \frac{d\psi_i}{dx} \right) c_j - b_2 \psi_i \left(\sum_{j=1}^{N} c_j \frac{d\psi_j}{dx} \right) + b_3 \psi_i \left(\sum_{j=1}^{N} \frac{dc_j}{dt} \psi_j \right) \right] dx = 0 \tag{5}$$

$$\sum_{j=1}^{N} \left[b_1 \left(\int_{x_a}^{x_b} \frac{d\psi_i}{dx} \frac{d\psi_j}{dx} dx \right) c_j - b_2 \left(\int_{x_a}^{x_b} \psi_j \frac{d\psi_j}{dx} dx \right) c_j + b_3 \left(\int_{x_a}^{x_b} \psi_i \psi_j dx \right) \frac{dc_j}{dt} \right] = 0 \tag{6}$$

The system (6) can be written in the matrix form

$$[K^1]\{c\} - [K^2]\{c\} + [M]\{\dot{c}\} = 0 \tag{7}$$

where

$$K_{ij}^1 = \int_{x_a}^{x_b} b_1 \frac{d\psi_i}{dx} \frac{d\psi_j}{dx} dx$$

$$K_{ij}^2 = \int_{x_a}^{x_b} b_2 \psi_i \frac{d\psi_j}{dx} dx$$

$$M_{ij} = \int_{x_a}^{x_b} b_3 \psi_i \psi_j dx \tag{8}$$

The system (7) can be written as

$$[K]\{c\} + [M]\{\dot{c}\} = 0 \tag{9}$$

where

$$[K] = [K^1] - [K^2]$$

We use the linear piecewise approximation in the space variable and the Galerkin method to obtain the semi discrete approximation to equation (1)

$$c^e(x,t) = \psi_{i-1}(x)\, c_{i-1}(t) + \psi_i(x)\, c_i(t) \tag{10}$$

where

$$\psi_{i-1}(x) = \frac{x - x_{i-1}}{x_i - x_{i-1}}, \quad \psi_i(x) = \frac{x_i - x}{x_i - x_{i-1}} \tag{11}$$

We have used the linear piecewise approximation (10) and (11) to find out the integral in the equation (8). Then, the system (9) become

$$[M]\{\dot{c}\} + [K]\{c\} = 0 \tag{12}$$

where

$$[K] = \frac{b_1}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \frac{b_2}{2}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}, [M] = \frac{b_3 h}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{13}$$

$$\{\dot{c}\} = \begin{bmatrix} \dot{c_{i-1}} \\ \dot{c_i} \end{bmatrix}, \{c\} = \begin{bmatrix} c_{i-1} \\ c_i \end{bmatrix} \tag{14}$$

3 Fully Discretized Finite Element Equations

We have the system of ordinary differential equations as follows

$$[M]\left\{\dot{C}\right\} + [K]\{C\} = 0 \tag{15}$$

subject to the initial condition

$$\{C\}_0 = \phi(x) = \{C_0\}$$

where $\{C\}_0$ denotes the vector of nodal values of C at time $t = 0$ whereas$\{C_0\}$ denotes the column of nodal values c_{j0}.

As applied to a vector of time derivatives of the nodal values the weighted average of approximation on the equation (15), we have

$$[M]\left(\frac{\{C\}^{n+1} - \{C\}^n}{\Delta t}\right) + \theta[K]\{C\}^{n+1} + (1-\theta)[K]\{C\}^n = 0 \tag{16}$$

The equation (16) can be written in simple form as

$$([M] + \Delta t\theta[K])\{C\}^{n+1} = [M]\{C\}^n - \Delta t(1-\theta)[K]\{C\}^n \tag{17}$$

The algebraic system (17) is solved by Gauss elimination method by taking Crank-Nicolson Scheme i.e $\theta = \frac{1}{2}$ in equation (17).

3.1 Stability Analysis of the Scheme

After assembling and using Crank-Nicolson Scheme, we have

$$\begin{bmatrix} \frac{b_1}{h} + \frac{b_2(3+2\Delta th)}{6} & \frac{-b_1}{h} + \frac{b_2(-6+h\Delta t)}{12} & 0 \\ \frac{-b_1}{h} + \frac{b_2(6+h\Delta t)}{12} & \frac{2b_1}{h} + \frac{b_2 h\Delta t}{3} & \frac{-b_1}{h} + \frac{b_2(-6+h\Delta t)}{12} \\ 0 & \frac{-b_1}{h} + \frac{b_2(6+h\Delta t)}{12} & \frac{b_1}{h} + \frac{b_2(-3+2\Delta th)}{6} \end{bmatrix}\begin{bmatrix} c_{i-1}^{n+1} \\ c_{i-1}^{n+1} \\ c_{i-1}^{n+1} \end{bmatrix} =$$

$$\begin{bmatrix} \frac{b_1}{h} + \frac{b_2(3+2\Delta th)}{6} & \frac{-b_1}{h} + \frac{b_2(-6+h\Delta t)}{12} & 0 \\ \frac{-b_1}{h} + \frac{b_2(6+h\Delta t)}{12} & \frac{2b_1}{h} + \frac{b_2 h\Delta t}{3} & \frac{-b_1}{h} + \frac{b_2(-6+h\Delta t)}{12} \\ 0 & \frac{-b_1}{h} + \frac{b_2(6+h\Delta t)}{12} & \frac{b_1}{h} + \frac{b_2(-3+2\Delta th)}{6} \end{bmatrix}\begin{bmatrix} c_{i-1}^{n} \\ c_{i-1}^{n} \\ c_{i-1}^{n} \end{bmatrix} \tag{18}$$

where h and Δt are step sizes along x-axis and time direction respectively. The finite element difference-differential equation at the i^{th} node is given by

$$\alpha_1 c_{i-1}^{n+1} + \alpha_2 c_i^{n+1} + \alpha_3 c_{i+1}^{n+1} = \beta_1 c_{i-1}^n + \beta_2 c_i^n + \beta_3 c_{i+1}^n \tag{19}$$

where

$$\alpha_1 = \frac{-b_1}{h} + \frac{b_2(6 + h\Delta t)}{12}, \alpha_2 = \frac{2b_1}{h} + \frac{b_2 h\Delta t}{3}, \alpha_3 = \frac{-b_1}{h} + \frac{b_2(-6 + h\Delta t)}{12}$$

and

$$\beta_1 = \frac{-b_1}{h} + \frac{b_2(6 - h\Delta t)}{12}, \beta_2 = \frac{2b_1}{h} - \frac{b_2 h\Delta t}{3}, \beta_3 = \frac{-b_1}{h} + \frac{b_2(-6 - h\Delta t)}{12}$$

Substituting $c_i^n = A\xi^n exp(j\beta h)$ where β the mode number, h is the element size and $j = \sqrt{-1}$, we obtain

$$\xi(\alpha_1 + \alpha_2 exp(j\beta h) + \alpha_3 exp(2j\beta h)) = \beta_1 + \beta_2 exp(j\beta h) + \beta_3 exp(2j\beta h) \tag{20}$$

Dividing the equation (20) by $exp(2j\beta h)$ both and simplification we have

$$\xi = \frac{(\beta_1 exp(-2j\beta h) + \beta_2 exp(-j\beta h) + beta_3)}{(\alpha_1 exp(-2j\alpha h) + \alpha_2 exp(-j\alpha h) + alpha_3)} \tag{21}$$

The scheme will be stable if $|\xi| \leq 1$, hence the scheme is conditionally stable.

4 Numerical Experiment And Discussion

In this section, we have studied two test examples to check the accuracy of the proposed numerical scheme. Problem 1: Consider a diffusion reaction problem with mixed boundary conditions as:

$$\frac{\partial C}{\partial t} = \frac{1}{P}\frac{\partial^2 C}{\partial x^2} - \frac{\partial C}{\partial x} \tag{22}$$

$$C - \frac{1}{P}\frac{\partial C}{\partial x} = 0, \;\; at \;\; x = 0, \;\; for \;\; all \;\; t \geq 0,$$

$$\frac{\partial C}{\partial x} = 0, \;\; at \;\; x = 1, \;\; for \;\; all \;\; t \geq 0,$$

$$C = 1, \;\; at \;\; t = 0, \;\; for \;\; all \;\; x$$

The problem is solved by the proposed method for different values of P and at different times up to $t = 2$. Figures 1-4 show the behavior of approximated solutions at different values of P. Figure 5 shows the approximated solutions at $t = 1$ for different values of P. The Figures show the approximated solutions are becoming smaller as we increase the value of P (i.e. the role of diffusion term decreases) up to $x = 0.9$. The Figure 6 compares the analytic and approximated solutions given in [4] at $P = 6$.

Problem 2: In this problem, we have considered the problem (22) with the following initial and Dirichlet's boundary conditions as follows

$$C = 1, \;\; at \;\; t = 0, \;\; for \;\; all \;\; x$$

$$C = 0 \;\; at \;\; x = 0, \;\; and \;\; \frac{\partial C}{\partial x} = 0, \;\; at \;\; x = 1, \;\; for \;\; all \;\; t \geq 0,$$

The results of the problem are shown in the Figures 7-12. Figures 7-10 show the behavior of approximated solutions at different values of P. Figure 11 shows the approximated solutions at $t = 1$ for different values of P. The Figures show the approximated solutions are becoming smaller as we increase the value of P (i.e. the role of diffusion term decrease in the right of the domain after $x = 0.6$. The Figure 12 compares the analytic and approximated solutions given in [4] at $P = 8$.

5 Conclusion

In this article, Galerkin-finite element method is proposed to find the approximate solutions of two point boundary value problems (BVP). In the solution procedure, the first step is to make weak formulation and then develop finite element formulation. Lastly, weighted average is used for fully discretization. The stability of the proposed method is discussed and shown that the method is conditionally stable. As test problem, two different solutions of two point BVP are chosen. Also, a comparison of numerical and analytical solutions is made and found that the proposed scheme has good accuracy.

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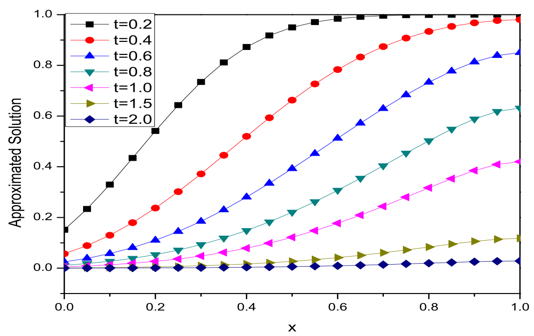


Fig.1. The physical behavior of approximated solution of Example 1 for $P = 5.0$.

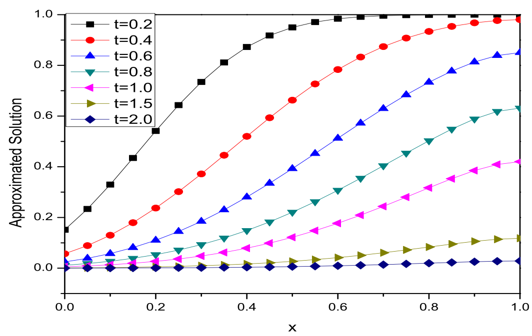


Fig.2. The physical behavior of approximated solution of Example 1 for $P = 10.0$.

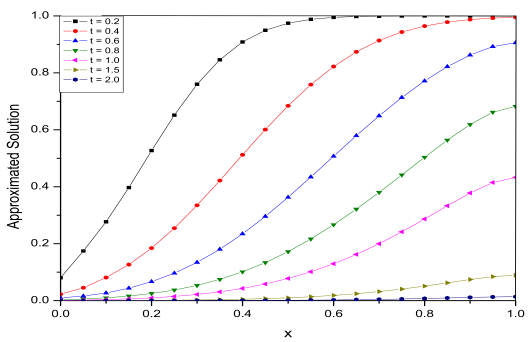


Fig.3. The physical behavior of approximated solution of Example 1 for $P = 15.0$.

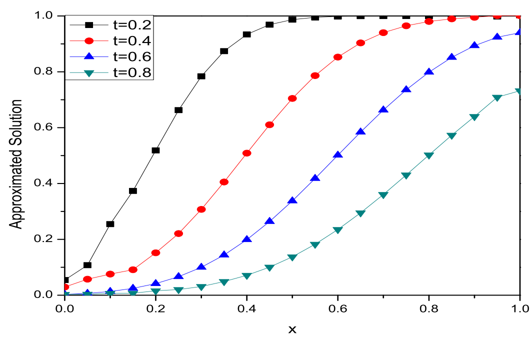
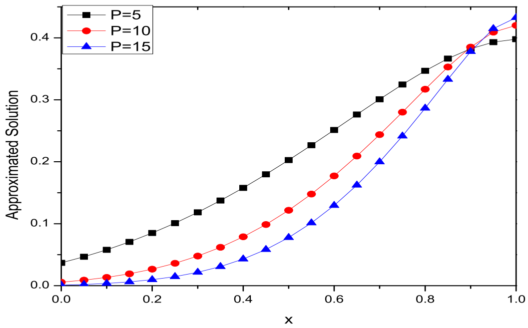


Fig.4. The physical behavior of approximated solution of Example 1 for $P = 20.0$.



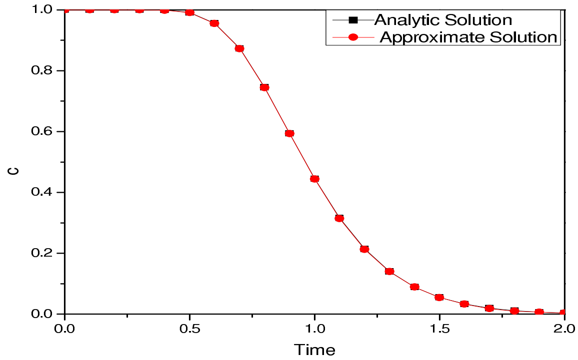
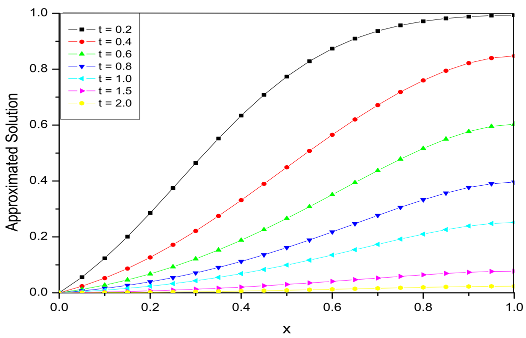


Fig.5. The physical behavior of approximated solution of Example 1 for different P at $T = 1$.

Fig.6. Comparison of analytic and numerical solution of Example 1 for $P = 6$.



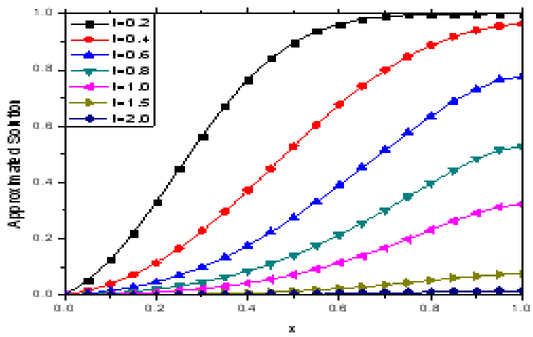
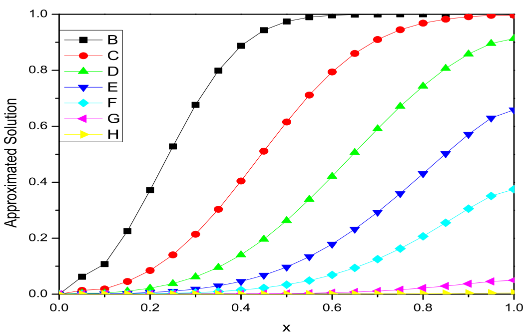


Fig.7. The physical behavior of approximated solution of Example 2 for $P = 5.0$.

Fig.8. The physical behavior of approximated solution of Example 2 for $P = 10.0$.



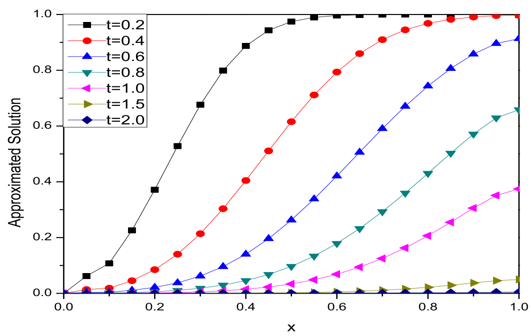


Fig.9. The physical behavior of approximated solution of Example 2 for $P = 15.0$.

Fig.10. The physical behavior of approximated solution of Example 2 for $P = 20.0$.

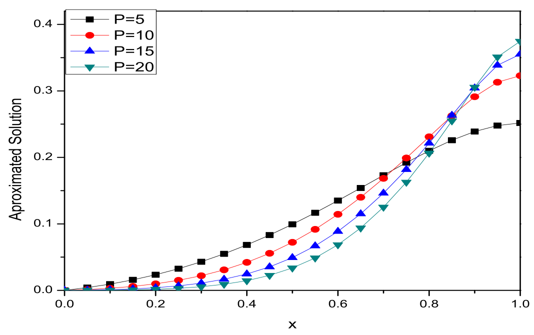


Fig.11. The physical behavior of approximated solution of Example 2 for different P at $T = 1$.

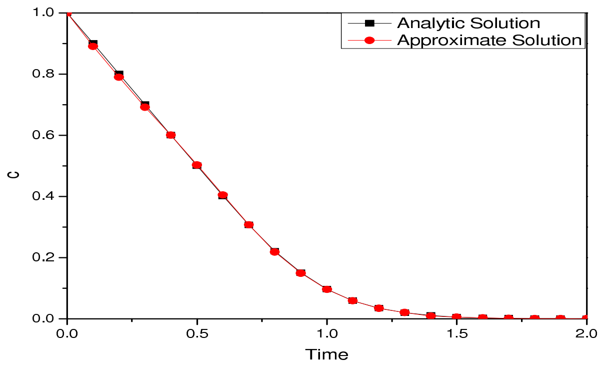


Fig.12. Comparison of analytic and numerical solution of Example 2 for $P = 8$.

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