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Numerical Solution of Singular IVPs of Emden-Fowler Type Using Legendre Scaling Functions

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Abstract: In this article a numerical technique is presented for the solution of Emden-Fowler equation. This method uses the Legendre scaling functions. The method consists of expanding the required approximate solution as the elements of Legendre scaling functions. Using the operational matrix of integration, the problem will be reduced to a set of algebraic equations. Some numerical examples are included to demonstrate the validity and applicability of the technique. The method is easy to implement and produces very accurate results.

Keywords: Emden-Fowler equation; singular initial value problem; collocation method; Legendre scaling function; operational matrix of integration

1 Introduction

Many problems in the literature of the diffusion of heat perpendicular to the surfaces of parallel planes are modeled by the heat equation

$$1/x^r\left(x^r y_x\right)_x + af(x,t)g(y) + h(x,t) = y_t,\ 0 < x \le L,\ 0 < t < T,\ r > 0,$$

or equivalently

$$y_{xx} + \frac{r}{x}y_x + af(x,t)g(y) + h(x,t) = y_t,\ 0 < x \le L,\ 0 < t < T,\ r > 0, \tag{1.1}$$

where $f(x,t)g(y)+h(x,t)$ is the nonlinear heat source, $y(x,t)$ is the temperature, and t is the dimensionless time variable. For the steady-state case, and for $k = 2, h(x,t) = 0$, Eq. (1.1), is the Emden-Fowler equation [1-4] given by

$$y'' + \frac{2}{x}y' + af(x)g(y) = 0, \quad y(0) = \alpha, \quad y'(0) = 0. \tag{1.2}$$

In this paper we solve the IVPs of Emden-Fowler type in general case

$$y'' + \frac{\lambda}{x}y' + af(x)g(y) = k(x), \qquad 0 < x, \tag{1.3}$$

$$y(0) = \alpha, \qquad y'(0) = 0, \tag{1.4}$$

where α and λ are constant and $f(x)$, $g(x)$ and $k(x)$ are given functions. When $f(x)$=1 , $k(x)$=0 and a=1, Eq. (1.3) reduces to the Lane-Emden equation, which with specified $g(y)$, was used to model several phenomena in mathematical physics and astrophysics such as the theory of stellar structure, the thermal behavior of a spherical cloud of gas, and theory of thermionic currents [1-3].

The solution of the Emden-Fowler equation (1.3), as well as other various linear and nonlinear singular IVPs in quantum mechanics and astrophysics, is numerically challenging because of the singularity behavior at the origin. The approximate analytical solutions to the Lane-Emden equations were presented by Shawagfeh [5] and Wazwaz [6-9] using ADM [10], and Hashim and et al. [11-13] using homotopy perturbation method, Shang, Wu, and Shao [14] using He's variational iteration method [30-33], and Marzban, Tabrizidooz, and Razzaghi using hybrid functions [15].

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It is worth pointing out that in this work we propose an alternative approach. We shall attempt to omit singularity of this equation and apply collocation method to solve Emden-Fowler equation numerically, in general case. In the current investigation, we reduce the problem to a set of algebraic equations by expanding the unknown function as Legendre scaling functions, with unknown coefficients. The operational matrix of integration is given. The idea of using operational matrices was used in many literatures [16-23] This matrix together with the Legendre scaling functions are then utilized to evaluate the unknown coefficients.

This article is organized as follows: In Section 2, we describe the formulation of the Legendre scaling functions on [0, 1], and derive the operational matrix of integration required for our subsequent development. In Section 3, the proposed method is used to approximate the solution of the problem. As a result, a set of algebraic equations are formed and a solution of the considered problem is introduced. In Section 4, we report our computational results and demonstrate the accuracy of the proposed numerical scheme by presenting several numerical examples. Section 5, ends this article with a brief conclusion. Note that we have computed the numerical results by Maple programming.

2 Legendre scaling functions

The r-th order Legendre scaling functions are the set of $r+1$ functions $\phi^0(x), ..., \phi^r(x)$ where $\phi^i(x)$ is a polynomial of i-th order [24-29] and all ϕ's form orthonormal basis, that is, for $i = 0, 1, ..., r$,

$$\phi^i(x) = \sum_{k=0}^{i} a_{ik} x^k, \qquad i = 0, 1, ..., r, \tag{2.5}$$

where

$$\int_0^1 \phi^i(x)\phi^k(x)dx = \delta_{i,k}, \qquad i, k = 0, 1, ..., r, \tag{2.6}$$

and

$$\delta_{i,k} = \begin{cases} 1, & i = k, \\ 0, & i \neq k. \end{cases}$$

In fact the variable a_{ik} are chosen so that the function ϕ^i, $i = 0, 1, ..., r$, form an orthonormal basis.

For example the cubic Legendre scaling functions ($r = 3$), consist of the four functions as

$$\begin{cases} \phi^0(x) = 1, & 0 \le x \le 1, \\ \phi^1(x) = \sqrt{3}(2x - 1), & 0 \le x \le 1, \\ \phi^2(x) = \sqrt{5}(6x^2 - 6x + 1), & 0 \le x \le 1, \\ \phi^3(x) = \sqrt{7}(20x^3 - 30x^2 + 12x - 1), & 0 \le x \le 1. \end{cases} \tag{2.7}$$

2.1 Function approximation

For a fixed J, a function $f(x)$ defined over $[0, 1]$ can be approximated using Legendre scaling functions of order r as

$$f(x) = \sum_{k=0}^{2^J-1} \sum_{m=0}^{r} c_{J,k} \phi^m_{J,k}(x) = C^T \Phi_J(x), \tag{2.8}$$

where

$$C = \left[c^0_{J,0}, ..., c^r_{J,0} | ... | c^0_{J,2^J-1}, ..., c^r_{J,2^J-1}\right]^T, \tag{2.9}$$

$$\Phi_J(x) = \left[\phi^0_{J,0}(x), ..., \phi^r_{J,0}(x) | ... | \phi^0_{J,2^J-1}(x), ..., \phi^r_{J,2^J-1}(x)\right]^T, \tag{2.10}$$

with

$$c^m_{J,k} = \int_0^1 f(x)\phi^m_{J,k}(x)dx, \qquad k = 0, 1, ..., 2^J - 1, \qquad m = 0, 1, ..., r, \tag{2.11}$$

and

$$\phi^m_{J,k} = \phi^m(2^J x - k), \qquad k = 0, 1, ..., 2^J - 1, \qquad i = 0, 1, ..., r.$$

The dimensions of vectors C and $\Phi_J(x)$ are $n = r2^J$.

2.2 The operational matrix of integration

The integral of vector $\Phi_J(x)$ in (2.10), can be expressed as

$$\int_0^x \Phi_J(t)dt = I_\phi \Phi_J(x), \tag{2.12}$$

where I_ϕ is $n \times n$ operational matrix of integration for Legendre scaling functions. The matrix I_ϕ can be obtained by the following process.

Let

$$\begin{cases} a_0 = 1, \\ a_i = \sqrt{a_{i-1}^2 + 2}, \quad i = 1, 2, \ldots, r-1, \end{cases}$$

$$b_i = 1/(a_{i-1}a_i), \qquad i = 1, 2, \ldots, r-1,$$

$$A_r = \begin{bmatrix} 0 & b_1 & 0 & & & & \\ & 0 & b_2 & 0 & & & \\ & & 0 & b_3 & 0 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & 0 & b_{r-2} & 0 \\ & & & & & 0 & b_{r-1} \\ & & & & & & 0 \end{bmatrix}_{r\times r},$$

$$B_r = 2^{-J} \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}_{r\times r},$$

$$M_r = 2^{-(J+1)} \left(A_r - A_r^T + 2^J B_r\right).$$

Now the operational matrix of I_ϕ can be calculated as

$$I_\phi = \begin{bmatrix} M_r & B_r & \cdots & B_r \\ & \ddots & \ddots & \vdots \\ & & M_r & B_r \\ 0 & & & M_r \end{bmatrix}_{n,n}.$$

3 Description of numerical method

In this section, we solve singular IVPs of Emden-Fowler type (1.3), with the initial value (1.4), by using Legendre scaling functions.

For this purpose, we first assume

$$z(x) = g\left(y(x)\right), \qquad 0 \le x \le 1. \tag{3.13}$$

We now use Eq. (2.8), to approximate $z(x)$ as

$$z(x) = Z^T \Phi_J(x), \tag{3.14}$$

where Z is a $(n \times 1)$ unknown vector difined similarly to C in Eq. (2.9).

Let

$$y''(x) = U^T \Phi_J(x), \tag{3.15}$$

where U is a $(n \times 1)$ unknown vector and should be found.

By integrating from both sides of Eq. (3.15), we get

$$y'(x) - y'(0) = U^T \int_0^x \Phi_J(t)dt = U^T I_\phi \Phi_J(x). \tag{3.16}$$

Using the initial value (1.4) in Eq. (3.16), we get

$$y'(x) = U^T I_\phi \Phi_J(x). \tag{3.17}$$

Again by integrating from both sides of Eq. (3.17), we obtain

$$y(x)-y(0)=U^TI_\phi\int_0^x\Phi_J(t)dt=U^TI_\phi^2\Phi_J(x). \tag{3.18}$$

Using the initial value (1.4) in Eq. (3.18), we get

$$y(x)=U^TI_\phi^2\Phi_J(x)+\alpha. \tag{3.19}$$

Also the functions $f(x)$ and $k(x)$ in Eq. (1.3), using Eq. (2.8), can be approximated as

$$f(x)=F^T\Phi_J(x), \tag{3.20}$$

$$k(x)=K^T\Phi_J(x), \tag{3.21}$$

where F and K, are $n\times 1$ vectors defined as

$$F=\left[f_{J,0}^0,\dots,f_{J,0}^r|\dots|f_{J,2^J-1},\dots,f_{J,2^J-1}\right]^T,$$

$$K=\left[k_{J,0}^0,\dots,k_{J,0}^r|\dots|k_{J,2^J-1},\dots,k_{J,2^J-1}\right]^T.$$

The entries of the vectors F and K can be found similar to Eq. (2.11).

Applying Eqs. (3.14), (3.15), (3.17) and (3.19)-(3.21) in (1.3), we get

$$U^T\Phi_J(x)+\frac{\lambda}{x}U^TI_\phi\Phi_J(x)+aF^T\Phi_J(x)\Phi_J^T(x)Z-K^T\Phi_J(x)=0. \tag{3.22}$$

Suppose $x_{i+1}=\frac{i}{n-1}, i=0,...,n-1$, and

$$N_{i+1}=\Phi_J(x_{i+1})\Phi_J^T(x_{i+1}). \tag{3.23}$$

The block matrices N_{i+1} can be obtained by the following process.

Let $\hat{\Delta}_{i+1}$=$(\hat{\delta}_{j,k}^{i+1})$, be a $r\times r$ matrix defined as

$$\hat{\delta}_{1,1}^{i+1}=1,$$

$$\hat{\delta}_{1,2}^{i+1}=(2x_{i+1}-1)\sqrt{3},$$

$$\hat{\delta}_{1,l}^{i+1}=\left(\frac{(2l-3)(2x_{i+1}-1)}{l-1}\frac{\hat{\delta}_{1,l-1}^{i+1}}{\sqrt{2l-3}}-\frac{(l-2)}{l-1}\frac{\hat{\delta}_{1,l-2}^{i+1}}{\sqrt{2l-5}}\right)\sqrt{2l-1},\quad l=3,...,r,$$

and

$$\hat{\delta}_{m,\tilde{m}}^{i+1}=\hat{\delta}_{1,m}^{i+1}\hat{\delta}_{1,\tilde{m}}^{i+1},\qquad m=2,...,r,\ \tilde{m}=1,...,r.$$

Now suppose

$$\Delta_{r(\hat{i}-1)+\hat{j}}=\hat{\Delta}_{2^J(\hat{j}-1)+\hat{i}},\qquad \hat{i}=1,...,2^J,\ \hat{j}=1,...,r.$$

The matrices $N_{i+1}, i=0,1,...,n-1$ in Eq. (3.23), can be expressed as

$$\begin{cases} N_{i+1}=2^J.diag\left[\Delta_{i+1},O_{r,r},...,O_{r,r}\right], & i=0,...,r-1,\\ N_{i+1}=2^J.diag\left[O_{r,r},\Delta_{i+1},O_{r,r},...,O_{r,r}\right], & i=r,...,2r-1,\\ \quad\vdots & \\ N_{i+1}=2^J.diag\left[O_{r,r},...,O_{r,r},\Delta_{i+1}\right], & i=r2^{J-1},...,2^Jr-1,\end{cases} \tag{3.24}$$

where $O_{r,r}$ is $r\times r$ zero matrix.

Let

$$\Upsilon_{i+1}=\Phi_J(x_{i+1}),\quad i=0,1,...,n-1, \tag{3.25}$$

be $(n\times 1)$ vectors.

The entries of vectors Υ_{i+1} can be found in the following way.

Suppose $\hat{\Upsilon}_{i+1}$=$(\hat{v}_{j,k}), i = 0, 1, ..., n-1$, be $(1 \times r)$ vectors with the following entries:

$$\hat{v}_{1,l}^{i+1} = \delta_{1,l}^{i+1}, \qquad l = 1, \ldots, r.$$

Now it can be shown that

$$\begin{cases} \Upsilon_{i+1} = \left[\hat{\Upsilon}_{i+1}, O_r, ..., O_r\right]^T, & i = 0, ..., r-1, \\ \Upsilon_{i+1} = \left[O_r, \hat{\Upsilon}_{i+1}, O_r, ..., O_r\right]^T, & i = r, ..., 2r-1, \\ \quad \vdots & \\ \Upsilon_{i+1} = \left[O_r, ..., O_r, \hat{\Upsilon}_{i+1}\right]^T, & i = r2^{J-1}, ..., 2^J r - 1, \end{cases} \tag{3.26}$$

where O_r is $(1 \times r)$ zero vector. Using the Taylor series expansion, we have

$$y'(x) = y'(0) + y''(0)x + O(x^2). \tag{3.27}$$

By using Eq. (1.4) in Eq. (3.27), for small value of x, we can write

$$\frac{1}{x} y'(x) \approx y''(0). \tag{3.28}$$

For $x = 0$, Eq. (3.22) using Eq. (3.28), may be replaced as

$$(\lambda + 1)U^T \Phi_J(0) + aF^T \Phi_J(0)\Phi_J^T(0)Z - K^T \Phi_J(0) = 0. \tag{3.29}$$

Collocating Eq. (3.22) in point $x_{i+1}, i = 1, 2, \ldots, n-1$, we have

$$\begin{array}{c} U^T \Phi_J(x_{i+1}) + \lambda/x_{i+1} U^T I_\phi \Phi_J(x_{i+1}) + aF^T \Phi_J(x_{i+1}) \Phi_J^T(x_{i+1}) Z \\ -K^T \Phi_J(x_{i+1}) = 0, \quad i = 1, 2, \ldots, n-1. \end{array} \tag{3.30}$$

Using Eqs.(3.23) and (3.25), in Eq. (3.30), we get

$$U^T \Upsilon_{i+1} + \lambda/x_{i+1} U^T I_\phi \Upsilon_{i+1} + aF^T N_{i+1} Z - K^T \Upsilon_{i+1} = 0, \quad i = 1, 2, \ldots, n-1. \tag{3.31}$$

Putting Eq. (3.19) in Eq. (3.13) and using Eq. (3.14) we have

$$g\left(U^T I_\phi^2 \Phi_J(x) + \alpha\right) = Z^T \Phi_J(x). \tag{3.32}$$

Collocating Eq. (3.32) in points $x_{i+1}, i = 0, 1, ..., n-1$, we get

$$g\left(U^T I_\phi^2 \Upsilon_{i+1} + \alpha\right) = Z^T \Upsilon_{i+1}, \quad i = 0, 1, \ldots, n-1, \tag{3.33}$$

also using Eqs. (3.23) and (3.25) in Eq. (3.29), we get

$$(\lambda + 1)U^T \Upsilon_1 + aF^T N_1 Z - K^T \Upsilon_1 = 0. \tag{3.34}$$

Equations (3.31), (3.33) and (3.34), generates a system of equations with $2n$ unknowns and equations, which can be solved to find Z and U. So the unknown function $y(x)$ may be found using Eq. (3.19).

4 Numerical example

In this section, we give some computational results of numerical experiments with method based on preceding sections, to support our theoretical discussion. In each case, we calculate the absolute error by the following formula

$$|f(x) - \hat{f}(x)|,$$

where $f(x)$ and $\hat{f}(x)$ are the exact and approximate solutions of the problem, respectively. Note that the nonlinear systems obtained by collocation method are solved by Newton method.

Example 1. Consider the following Emden-Fowler equation [13]:

$$y''(x) + 2/xy'(x) + y(x) = 0,$$

subject to the boundary conditions

$$y(0) = 1, \qquad y'(0) = 0.$$

The exact solution is

$$y(x) = sin(x)/x.$$

In Table 1, we represent the absolute errors obtained in solving the problem with different values of r and J.

Table 1. Absolute errors for y(x)

x	r=2 J=3	r=3 J=3	r=4 J=3
0.0	1.8×10^{-7}	3.0×10^{-8}	1.9×10^{-10}
0.1	1.2×10^{-7}	1.2×10^{-9}	5.5×10^{-11}
0.2	2.1×10^{-7}	6.6×10^{-9}	1.3×10^{-10}
0.3	5.6×10^{-7}	6.4×10^{-9}	3.1×10^{-10}
0.4	8.5×10^{-7}	1.1×10^{-8}	1.5×10^{-9}
0.5	3.3×10^{-8}	2.7×10^{-9}	6.4×10^{-10}
0.6	1.7×10^{-7}	1.2×10^{-8}	2.3×10^{-9}
0.7	7.6×10^{-8}	5.1×10^{-9}	1.7×10^{-9}
0.8	1.2×10^{-6}	4.7×10^{-9}	2.4×10^{-9}
0.9	1.4×10^{-6}	9.1×10^{-9}	6.3×10^{-9}
1.0	3.2×10^{-6}	1.9×10^{-8}	1.4×10^{-8}

Example 2. Consider the following Emden-Fowler equation:

$$y''(x) + 2/xy'(x) + x^3 ln(x) = 0,$$

subject to the boundary conditions

$$y(0) = 1, \qquad y'(0) = 0.$$

The exact solution is given by [13]

$$y(x) = 1 - 1/30\,(ln(x) - 11/30)\,x^5.$$

Figures 1 and 2, show the plot of error for $r = 3, J = 3$, and $r = 4, J = 4$, respectively.

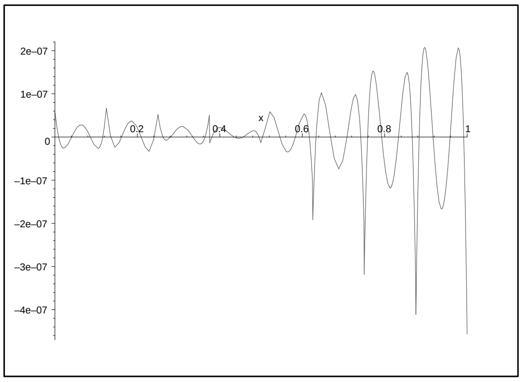

Fig.1. Error for example 2 with r=3 and J=3

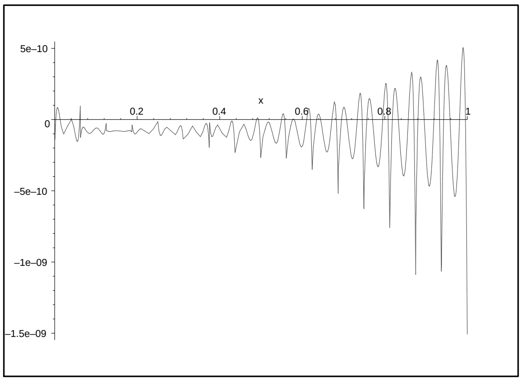

Fig.2. Error for example 2 with r=4 and J=4.

Example 3. Now we consider the following non-linear homogeneous Emden-Fowler equation [13]:

$$y''(x) + 8/xy'(x) + 18y(x) + 4y(x)ln(y(x)) = 0,$$

subject to the boundary conditions

$$y(0) = 1, \qquad y'(0) = 0.$$

The exact solution is

$$y(x) = exp(-x^2).$$

In Table 2, we represent the absolute errors obtained in solving the problem with different values of r and J.

Table 2. Absolute errors for y(x)

x	r=2 J=3	r=3 J=3	r=4 J=3
0.0	1.0×10^{-5}	1.7×10^{-6}	7.8×10^{-9}
0.1	6.2×10^{-6}	7.0×10^{-7}	8.0×10^{-10}
0.2	1.6×10^{-6}	2.7×10^{-7}	1.4×10^{-8}
0.3	3.0×10^{-5}	1.1×10^{-7}	4.6×10^{-10}
0.4	3.8×10^{-5}	3.4×10^{-7}	2.0×10^{-8}
0.5	2.0×10^{-6}	2.2×10^{-7}	2.4×10^{-8}
0.6	8.2×10^{-6}	1.9×10^{-7}	1.7×10^{-8}
0.7	4.0×10^{-6}	4.4×10^{-7}	7.6×10^{-9}
0.8	2.3×10^{-5}	5.0×10^{-7}	5.9×10^{-9}
0.9	2.2×10^{-5}	1.6×10^{-7}	3.5×10^{-8}
1.0	4.1×10^{-5}	1.3×10^{-6}	5.0×10^{-8}

Example 4. We consider the following linear, non-homogeneous Emden-Fowler equation:

$$y''(x) + 8/xy'(x) + xy(x) = x^5 - x^4 + 44x^2 - 30x,$$

subject to the boundary conditions

$$y(0) = 0, \qquad y'(0) = 0.$$

The exact solution is given by [13]

$$y(x) = x^4 - x^3.$$

Table 3, shows the absolute errors using the method introduced in previous section for different values of r and J. Also Figure 3, shows the plot of error with $r = 4$, and $J = 4$.

Table 3. Absolute errors for y(x)

x	r=2 J=3	r=3 J=3	r=4 J=3
0.0	7.7×10^{-5}	3.5×10^{-6}	1.6×10^{-10}
0.1	2.3×10^{-5}	1.4×10^{-6}	4.8×10^{-11}
0.2	3.1×10^{-5}	8.1×10^{-7}	6.6×10^{-11}
0.3	4.2×10^{-5}	8.1×10^{-7}	2.0×10^{-10}
0.4	6.5×10^{-5}	1.4×10^{-6}	2.4×10^{-9}
0.5	1.5×10^{-5}	3.2×10^{-7}	1.2×10^{-9}
0.6	1.0×10^{-6}	1.4×10^{-6}	3.6×10^{-9}
0.7	4.6×10^{-6}	8.0×10^{-7}	2.0×10^{-9}
0.8	9.8×10^{-5}	7.9×10^{-7}	2.2×10^{-9}
0.9	1.2×10^{-4}	1.5×10^{-6}	1.7×10^{-9}
1.0	2.8×10^{-4}	3.4×10^{-6}	6.0×10^{-9}

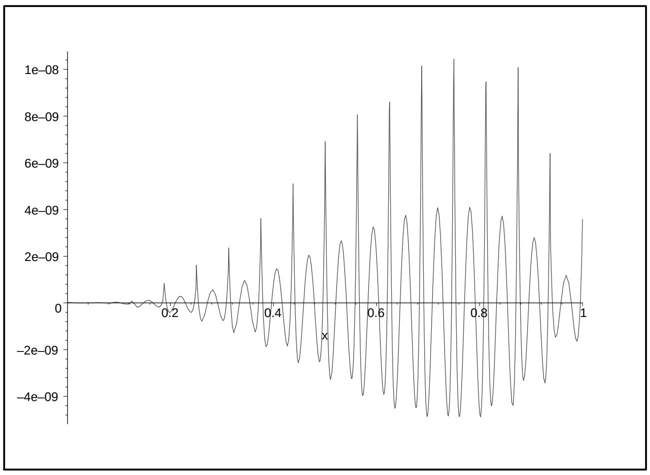

Fig.3. Error for example 4 with r=4 and J=4

Example 5. Finally we consider the following non-linear, homogeneous Emden-Fowler equation [13]:

$$y''(x) + 5/xy'(x) + 8\left(exp(y(x)) + 2e(y(x)/2)\right) = 0, \tag{4.35}$$

subject to the boundary conditions

$$y(0) = 0, \qquad y'(0) = 0.$$

The exact solution is

$$y(x) = -2ln(1 + x^2).$$

Figures 4 and 5, show the plot of error for $r = 2, J = 3$, and $r = 3, J = 3$, respectively.

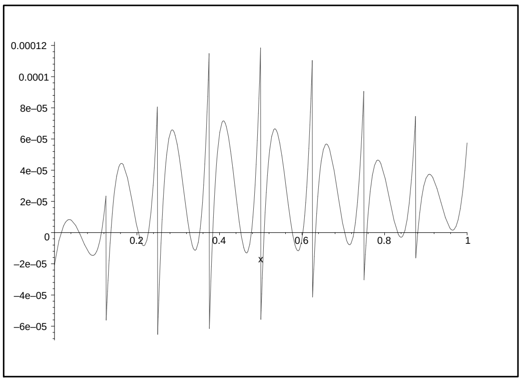

Fig.4. Error for example 5 with r=2 and J=3

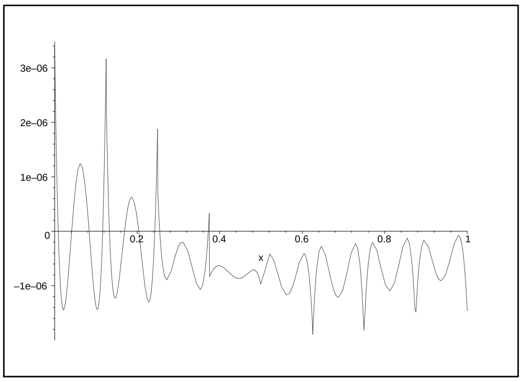

Fig.5. Error for example 5 with r=3 and J=3.

5 Conclusion

In this paper, we presented a numerical scheme for solving the Emden-Fowler equation. The Legendre scaling functions on interval [0, 1] was employed. The obtained results showed that this approach can solve the problem effectively.

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