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Comparison of two Dimensional DTM and PDTM for solving Time-Dependent Emden-Fowler Type Equations

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Abstract: In this paper, we present the comparative study of two dimensional differential transform method and the projected differential transform method for solving Time-dependent Emden-Fowler type equations. Infact, the exact solutions can be obtained by known form of the series solutions in both the methods. Several illustrative examples are given to demonstrate the effectiveness of both the methods.

Keywords: two dimensional differential transform method; projected differential transform method; time-dependent Emden-Fowler type equations;wave type equation

1 Introduction

Many problems in mathematical physics and astrophysics related to the diffusion of perpendicular to the surface of parallel planes can be modeled by the heat equation [1],

$$u_{xx}+\frac{r}{x}u_x+af(x,t)g(u)+h(x,t)=u_t, 0<x\leq L, 0<t<T, r>0 \quad (1)$$

$$u(0,t)=\alpha, u_x(0,t)=0 \quad (2)$$

where α is a constant and $f(x,t)g(u)+h(x,t)$ is the nonlinear heat source,$u(x,t)$ is the temperature, and t is the dimensional time variable. For steady-state case, and $r=2$, $h(x,t)=0$, Eq. (1) becomes

$$u_{xx}+\frac{2}{x}u_x+af(x,t)g(u)=u_t, 0<x\leq L, 0<t<T, r>0 \quad (3)$$

which is known as Emden-Fowler equation where $f(x,t)$ and $g(u)$ are some given function of x and u respectively. When $f(x,t)=1$ and $a=1$, Eq(1.3) reduces to the Lane-Emden equation with specified $g(u)$ was used to model several phenomena in mathematical physics and astrophysics such as the theory of stellar structure, the thermal behaviour of a spherical cloud of a gas, isothermal gas sphere and theory of thermionic currents [2-4]. In this analysis, we also study the wave type equations with singular behaviour of the form

$$u_{xx}+\frac{r}{x}u_x+af(x,t)g(u)+h(x,t)=u_{tt}, 0<x\leq L, 0<t<T, r>0 \quad (4)$$

where $u(x,t)$ is the displacement of the wave at the position x and time t. The solution of the time-dependent Emden-Fowler equation as well as variety of linear, non-linear singular initial value problems in quantum mechanics and astrophysics is numerically challenging because of singularity behavior at the origin. The approximate solutions to the above problems were presented by Shawagfeh [5] and Wazwaz [6-8] using the adomain decomposition method, the homotopy perturbation method [9], the homotopy analysis method [10] and the variational iteration method [11]. In this paper, we studied the comparison between the two dimensional differential transform method and the projected differential transform method to solve the Time dependent Emden-Fowler type equations. Several illustrative examples are given to demonstrate the effectiveness of both the methods.

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The concept of differential transformation was first proposed by Zhou [12] in solving linear and nonlinear initial valued problems in electrical circuit analysis. The differential transform method obtains an analytical solution in the form of a polynomial. It is different from the traditional high order Taylor's series method, which requires symbolic competition of the necessary derivatives of the data functions. The Taylor series method is computationally taken long time for large orders. With this method, it is possible to obtain highly accurate results or exact solutions for differential equations.

2 Two-dimensional differential transform method

The differential transform method formalizes the Taylor's series in a totally different manner. With this technique the given differential equation and related initial conditions are transformed into a recurrence equation that finally leads to the solution of a system of algebraic equations as coefficients of a power series solution. The basic definitions and fundamental operations of the two dimensional differential transform are defined in [13-14] as follows: Consider a function of two variable $u(x,t)$ be analytic in the domain k and let $(x,t) = (x_0, t_0)$ in this domain.The function $u(x,t)$ is then represented by one series whose centre located at (x_0, t_0) The differential transform of the function $u(x,t)$ is the form

$$U(k,h) = \frac{1}{k!h!}\left[\frac{\partial^{k+h}u(x,t)}{\partial x^k \partial t^h}\right]_{(x_0,t_0)} \tag{5}$$

where $u(x,t)$ is the original function and $U(k,h)$ is the transformed function. The differential inverse transform of $U(k,h)$ is defined as

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U(k,h)(x-x_0)^k(t-t_0)^h \tag{6}$$

In a real application, and when (x_0, t_0) are taken as (0, 0), then the function $u(x,t)$ is expressed by a finite series and (2.2) can be written as

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} \frac{1}{k!h!}\left[\frac{\partial^{k+h}u(x,t)}{\partial x^k \partial t^h}\right]_{(0,0)} x^k t^h \tag{7}$$

In real applications when the general term of the series cannot be recognized, the following truncated series can be considered. Some fundamental operations for the standard differential transform method are presented in Table 1. It has been proved that the standard differential transform method is an efficient tool for solving many linear and non-linear problems [13-23,25-30].

3 Projected differential transform method

However, there are difficulties in differential transform method while handling non-linear functions in two dimension. Let us consider the differential transform for $u^3(x,t)$ which involves four summations in the Table 1. Thus, it is necessary to have a lot of computational work to calculate such differential transform $U(k,h)$ for the large numbers (k,h). As we know that, differential transform method is based on the Taylor series for all variables. To avoid these difficulties, the projected differential transform method [24] considered the Taylor series of the function $u(x,t)$ with respect to the specific variable. Assume that the specific variable is the variable x. Then, we have the Taylor series expansion of the function u at $x = x_0$ as follows:

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k u(x,t)}{\partial x^k}\right]_{x=x_0} x^k \tag{8}$$

Definition 1 *The projected differential transform $U(k,t)$ of $u(x,t)$ with respect to the variable x at x_0 is defined by*

$$U(k,t) = \frac{1}{k!}\left[\frac{\partial^k u(x,t)}{\partial x^k}\right]_{x=x_0} \tag{9}$$

Definition 2 *The projected differential inverse transform $U(k,t)$ with respect to the variable x at $x_0$is defined by*

$$u(x,t) = \sum_{k=0}^{\infty} U(k,t)(x-x_0)^k \tag{10}$$

Since the projected differential transform method results from the Taylor's series of the function with respect to the specific variable, it is expected that the corresponding algebraic equation from the given problem is much simpler than the result obtained by the standard differential transform method.

Table 1: The fundamental operations for the two-dimensional transform method

Original function	Transformed function
$w(x,t)=u(x,t)\pm v(x,t)$	$W(k,h)=U(k,h)\pm V(k,h)$
$w(x,t)=\alpha u(x,t)$	$W(k,h)=\alpha U(k,h)$
$w(x,t)=\frac{\partial u(x,t)}{\partial x}$	$W(k,h)=(k+1)U(k+1,h)$
$w(x,t)=\frac{\partial u(x,t)}{\partial t}$	$W(k,h)=(h+1)U(k,h+1)$
$w(x,t)=u(x,t)v(x,t)$	$W(k,h)=\sum_{r=0}^{k}\sum_{s=0}^{h}U(r,h-s)V(k-r,s)$
$w(x,t)=x^m t^n$	$W(k,h)=k^m h^n\delta(k-m)\delta(h-n)=\begin{cases} k^m h^n, & k=m\,,h=n \\ 0, & \text{otherwise}\end{cases}$
$w(x,t)=\frac{\partial^{r+s}u(x,t)}{\partial x^r\partial t^s}$	$W(k,h)=(k+1)(k+2)\cdots(k+r)(h+1)(h+2)\cdots(h+s)U(k+r,h+s)$
$w(x,t)=e^{au(x,t)}$	$W(k,h)=\begin{cases} e^{aU(0,0)}, k=0,h=0 \\ a\sum_{r=0}^{k-1}\sum_{s=0}^{h}\frac{k-r}{k}U(k-r,s)W(r,h-s), & k\geq 1 \\ a\sum_{r=0}^{k}\sum_{s=0}^{h-1}\frac{h-s}{h}U(r,h-s)W(k-r,s), & h\geq 1\end{cases}$
$w(x,t)=ln(a+bu)$	$W(k,h)=\begin{cases} \ln(a+bU(0,0)), k=0,h=0 \\ \frac{b}{a}\left[U(k,h)-\sum_{r=0}^{k-1}\sum_{s=0}^{h}\frac{r+1}{k}U(k-r-1,s)W(r+1,h-s)\right], & k\geq 1 \\ \frac{b}{a}\left[U(k,h)-\sum_{r=0}^{k}\sum_{s=0}^{h-1}\frac{s+1}{h}U(r,h-s-1)W(k-r,s+1)\right], & h\geq 1\end{cases}$
$w(x,t)=u(x,t)v(x,t)z(x,t)$	$W(k,h)=\sum_{r=0}^{k}\sum_{q=0}^{k-r}\sum_{s=0}^{h}\sum_{p=0}^{h-s}U(r,h-s-p)V(q,s)Z(k-r-q,p)$

4 Comparison between differential transform method and projected differential transform method

In this section, we present the comparison of the differential transform method and the projected differential transform method for the Time dependent Emden-Fowler equations (1.1) - (1.2). First, let us consider the two dimensional differential transform for the Eq. (1.1) is

$$
\begin{aligned}
&\sum_{m=0}^{k}\delta(m-1)(k-m-1)(k-m-2)U(k-m-2,h)+r(k+1)U(k+1,h)\\
&+a\sum_{m=0}^{k}\sum_{s=0}^{h}\delta(m-1)F(k-m-1,h)G(k,h-s)+\sum_{m=0}^{k}\delta(m-1)H(k-m-1,h)\\
&=\sum_{m=0}^{k}\delta(m-1)(h+1)U(k-m,h+1)
\end{aligned}
\tag{11}
$$

where r, a are unknown constants. $F(k-m-1,h)$ is the differential transform for the function $f(x,t)$ and $G(k,h-s)$ is the differential transform for the non-linear function $e^{u(x,t)}$ given in Table 1 and $H(k-m-1,h)$ is the differential transform for the function $h(x,t)$. The transformed version for Eq. (1.2)

$$
U(0,h)=\begin{cases}\alpha, & h=0\\ 0, & \text{otherwise}\end{cases}\quad ,\quad U(1,h)=0, h=0,1,2,\cdots \tag{12}
$$

where α is unknown constant.

The projected differential transform with respect to the variable x for Eq. (1.1) is

$$
\begin{aligned}
&\sum_{m=0}^{k}\delta(m-1)(k-m-1)(k-m-2)U(k-m-2,t)+r(k+1)U(k+1,t)+a\sum_{m=0}^{k}F(m,t)G(k-m,t)\\
&+\sum_{m=0}^{k}\delta(m-1)H(k-m-1,t)=\sum_{m=0}^{k}\delta(m-1)(h+1)\frac{\partial U(k-m,t)}{\partial t}
\end{aligned}
\tag{13}
$$

where r, a are unknown constants. $F(m,t)$ is the differential transform for the function $f(x,t)$, $G(k-m,t)$is the differential transform for the non-linear function $e^{u(x,t)}$ given in Theorem 1 and $H(k-m-1,t)$ is the differential transform for the function $h(x,t)$.The transformed version for Eq. (1.2)

$$U(0,t)=\alpha,\quad U(1,t)=0 \tag{14}$$

where α is unknown constant.
Theorems on projected differential transform method for some non-linear functions are given below:

Theorem 1 *If $f(x,t)$ = $e^{u(x,t)}$, then $F(k,t)$ of $e^{u(x,t)}$ with respect to the variable x at $\widetilde{x}$ is*

$$F(k,t)=\begin{cases} e^{U(0,t)}, & k=0\\ \sum_{m=0}^{k-1}\frac{m+1}{k}U(m+1,t),F(k-m-1,t) & k\geq 1\end{cases}$$

Proof. Consider,

$$f(x,t)=e^{U(x,t)} \tag{15}$$

From the definition of transform,

$$F(0,t)=e^{U(0,t)} \tag{16}$$

Now, taking the differentiation of $f(x,t)$ = $e^{u(x,t)}$ w. r. t the variable x,

$$\frac{df(x,t)}{dx}=e^{u(x,t)}\frac{du(x,t)}{dt} \tag{17}$$

By taking the differential transform on both sides

$$(k+1)F(k+1,t)=\sum_{m=0}^{k}(m+1)U(m+1,t)F(k-m,t) \tag{18}$$

Replacing $k+1$ by k gives

$$F(k+1,t)=\sum_{m=0}^{k-1}\frac{(m+1)}{k}U(m+1,t)F(k-1-m,t),k\geq 1 \tag{19}$$

Combining (4.6) and (4.9), we obtain the recursive relationship for calculating the T-function of $f(x,t)$ = $e^{U(x,t)}$.

$$F(k,t)=\begin{cases} e^{U(0,t)}, & k=0\\ \sum_{m=0}^{k-1}\frac{m+1}{k}U(m+1,t),F(k-m-1,t) & k\geq 1\end{cases} \tag{20}$$

■

Corollary 2 *If $f(x,t)$ = $e^{u(x,t)}$, then $F(x,h)$ of $e^{u(x,t)}$ with respect to the variable t at $\widetilde{t}$ is*

$$F(x,h)=\begin{cases} e^{U(0,t)}, & k=0\\ \sum_{m=0}^{h-1}\frac{m+1}{h}U(x,h+1),F(x,h-m-1) & h\geq 1\end{cases}$$

Theorem 3 *If $f(x,t)$ = ln $u(x,t)$, then $G(k,t)$ of ln $u(x,t)$ with respect to the variable x at $\widetilde{x}$ is*

$$G(k,t)=\begin{cases} lnU(0,t), & k=0\\ \frac{U(1,t)}{U(0,t)}, & k=1\\ \frac{1}{U(0,t)}U(k,t)-\sum_{m=0}^{k-2}\frac{m+1}{k}G(m+1,t)U(k-m-1,t), & \\ & k\geq 2\end{cases}$$

Proof. Consider,

$$f(x,t)=lnU(x,t) \tag{21}$$

From the definition of transform,

$$F(0,t)=lnU(0,t) \tag{22}$$

Now, taking the differentiation of $f(x,t)$ = ln $u(x,t)$ w. r. t the variable x,

$$\frac{df(x,t)}{dx} = \frac{1}{u(x,t)}\frac{du(x,t)}{dt} \Rightarrow u(x,t)\frac{df(x,t)}{dx} = \frac{du(x,t)}{dx} \tag{23}$$

By taking the differential transform on both sides

$$\sum_{m=0}^{k}(k-m+1)F(k-m+1,t) = (k+1)U(k+1,t) \tag{24}$$

Replacing $k+1$ by k gives

$$\sum_{m=0}^{k-1}(k-m)F(k-m,t) = kU(k,t) \tag{25}$$

If $k=1$, then

$$F(1,t) = \frac{U(1,t)}{U(0,t)}, \tag{26}$$

If $k=2$, then

$$F(2,t) = \frac{U(2,t)}{U(0,t)} - \frac{U(1,t)}{2U(0,t)}F(1,t), \tag{27}$$

In general,

$$F(k,t) = \frac{1}{U(0,t)}\left[U(k,t) - \sum_{m=0}^{k-2}\frac{m+1}{k}F(m+1,t)U(k-m-1,t)\right], k \geq 2 \tag{28}$$

Combining (4.12), (4.16) and (4.18), we obtain the recursive relationship for calculating the T-function of $f(x,t)$ = $lnU(x,t)$.

$$G(k,t) = \begin{cases} lnU(0,t), & k=0 \\ \frac{U(1,t)}{U(0,t)}, & k=1 \\ \frac{1}{U(0,t)}\left[U(k,t) - \sum_{m=0}^{k-2}\frac{m+1}{k}G(m+1,t)U(k-m-1,t)\right], & k \geq 2 \end{cases} \tag{29}$$

■

Corollary 4 *If $f(x,t)$ = $lnu(x,t)$, then $G(x,h)$ of $lnu(x,t)$ with respect to the variable t at $\widetilde{t}$ is*

$$G(x,h) = \begin{cases} lnU(x,h), & h=0 \\ \frac{U(x,h)}{U(x,h)}, & h=1 \\ \frac{1}{U(x,h)}\left[U(x,h) - \sum_{m=0}^{h-2}\frac{m+1}{h}G(x,m+1)U(x,k-m-1)\right], & h \geq 2 \end{cases}$$

5 Applications

In this section, we implemented the two dimensional differential transform method and the projected differential transform method for solving Time-dependent Emden-Fowler type equations, which have been widely discussed in the literature.

Example 1 *First we consider the following linear non-homogeneous equation [7]*

$$u_{xx} + \frac{2}{x}u_x - (5+4x^2)u = u_t + (6-5x^2-4x^4) \tag{30}$$

subject to

$$u(0,t) = e^t, u_x(0,t) = 0 \tag{31}$$

Differential transform method: The transformed version of (5.1) is

$$\sum_{m=0}^{k}\sum_{r=0}^{h}\delta(m-1,h-r)(k-m+1)(k-m+2)U(k-m+2,r) + 2(k+1)U(k+1,h)$$
$$-5\sum_{m=0}^{k}\sum_{r=0}^{h}\delta(m-1,h-r)U(k-m,r) - 4\sum_{m=0}^{k}\sum_{r=0}^{h}\delta(m-3,h-r)U(k-m,r)$$
$$= \sum_{m=0}^{k}\sum_{r=0}^{h}\delta(m-1,h-r)(r+1)U(k-m,r+1) + 6\delta(k-1,h) - 5\delta(k-3,h) - 4\delta(k-5,h) \tag{32}$$

The transformed versions of (5.2) is

$$U(0,h) = \frac{1}{h!}; U(1,h) = 0; h = 0,1,... \quad (33)$$

Substituting (5.4) into (5.3), we obtain the following $U(k,h)$ *values.*
$U(1,0) = 0, U(2,0) = 2, U(3,0) = 0, U(4,0) = \frac{1}{2}, ..., U(1,1) = 0, U(2,1) = 1, U(3,1) = 0, U(4,2) = \frac{1}{2}, ..., U(1,2) = 0, U(2,2) = \frac{1}{2}, , U(3,0) = 0, ...$
Substituting $U(k,h)$*'s in (2.2). We obtain the solution in the following form*

$$u(x,t) = x^2 + \left[1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + ...\right]\left[1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + ...\right] \quad (34)$$

Projected differential transform method: The modified transformed version of (5.1) w. r. t. x *is*

$$\sum_{m=0}^{k} \delta(m-1)(k-m+1)(k-m+2)U(k-m+2,t) + 2(k+1)U(k+1,t)$$
$$- \sum_{m=0}^{k} F(m,t)U(k-m,t) = \sum_{m=0}^{k} \delta(m-1)\frac{\partial U(k-m,t)}{\partial t} + H(k,t) \quad (35)$$

$$F(m,t) = \begin{cases} 5, & m=1 \\ 4, & m=3 \\ 0, & \text{other wise} \end{cases} \quad (36)$$

$$H(k,t) = \begin{cases} 6, & k=1 \\ -5, & k=3 \\ -4, & k=5 \end{cases} \quad (37)$$

The transformed versions of (5.2) is

$$U(0,t) = e^t, \frac{\partial U(0,t)}{\partial x} = 0 \quad (38)$$

Substituting (5.9) into (5.6)-(5.8) gives
$U(1,t) = U(3,t) = U(5,t) = ... = 0, U(2,t) = e^t + 1, U(4,t) = \frac{e^t}{2!}, U(6,t) = \frac{e^t}{3!}$
Substituting all $U(k,t)$*'s into (3.3). We obtained the solution in the following form*

$$u(x,t) = e^t + (e^t+1)x^2 + e^t\frac{x^4}{2!} + ... = x^2 + e^t\left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + ...\right) \quad (39)$$

and the analytic solution of the problems is given by $u(x,t) = x^2 + e^{t+x^2}$.
In Fig.1(a) and 1(b), the images of the four term approximate solution using differential transform method and projected differential transform method are given. Analytical solution is presented in Fig. 1(c).

Example 2 *We consider the following non-homogeneous singular wave-type equation [7]*

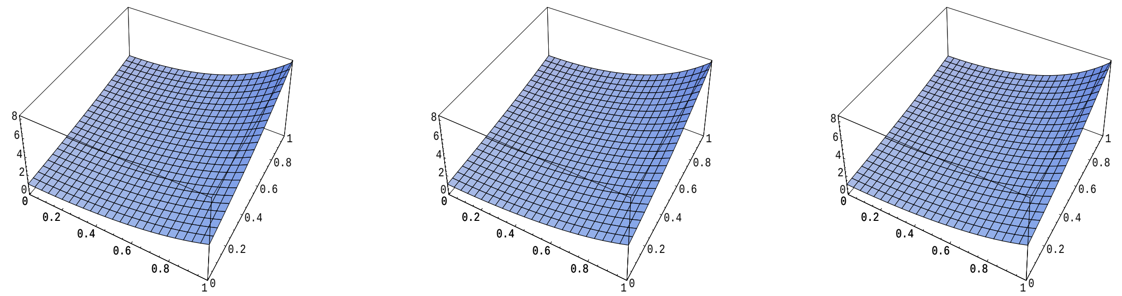
$$u_{xx} + \frac{2}{x}u_x - (5+4x^2)u = u_{tt} + (12x - 5x^3 - 4x^5) \quad (40)$$

subject to

$$u(0,t) = e^{-t}, u_x(0,t) = 0 \quad (41)$$

Differential transform method: The transformed version of (5.11) is

$$\sum_{m=0}^{k}\sum_{r=0}^{h} \delta(m-1,h-r)(k-m+1)(k-m+2)U(k-m+2,r) + 2(k+1)U(k+1,h)$$
$$-5\sum_{m=0}^{k}\sum_{r=0}^{h} \delta(m-1,h-r)U(k-m,r) - 4\sum_{m=0}^{k}\sum_{r=0}^{h} \delta(m-3,h-r)U(k-m,r)$$
$$= \sum_{m=0}^{k}\sum_{r=0}^{h} \delta(m-1,h-r)(r+1)(r+2)U(k-m,r+2) + 12\delta(k-2,h) - 5\delta(k-4,h) - 4\delta(k-6,h) \quad (42)$$



(a) Represents the numerical solution of $u(x,t)$ for different values of x and t using DTM

(b) Represents the numerical solution of $u(x,t)$ for different values of x and t using PDTM

(c) Exact Solution $u(x,t)$

Figure 1:

The transformed versions of (5.12) is

$$U(0,h) = \frac{(-1)^h}{h!}; U(1,h) = 0; h = 0, 1, ... \tag{43}$$

Substituting (5.14) into (5.13), we obtain the following $U(k,h)$values.
$U(0,0) = 1, U(1,0) = 0, U(2,0) = 1, U(3,0) = 1, U(4,0) = \frac{1}{2}, ..., U(1,1) = 0, U(2,1) = -1, U(3,1) = 0, U(4,1) = \frac{1}{2}, ..., U(1,2) = 0, U(2,2) = \frac{1}{2}, , U(2,3) = 0,, ...$
Substituting all $U(k,h)$'s in (2.2). We obtain the solution in the following form

$$u(x,t) = x^3 + \left[1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + ...\right]\left[1 - \frac{t}{1!} + \frac{t^2}{2!} - \frac{t^3}{3!} + ...\right] \tag{44}$$

Projected differential transform method: The modified transformed version of (5.11) w. r. t. x is

$$\sum_{m=0}^{k} \delta(m-1)(k-m+1)(k-m+2)U(k-m+2,t) + 2(k+1)U(k+1,t) - \sum_{m=0}^{k} F(m,t)U(k-m,t) = \sum_{m=0}^{k} \delta(m-1)\frac{\partial U(k-m,t)}{\partial t} + H(k,t) \tag{45}$$

$$F(m,t) = \begin{cases} 5, & m = 1 \\ 4, & m = 3 \\ 0, & \textit{other wise} \end{cases} \tag{46}$$

$$H(k,t) = \begin{cases} 12, & k = 2 \\ -5, & k = 4 \\ -4, & k = 6 \end{cases} \tag{47}$$

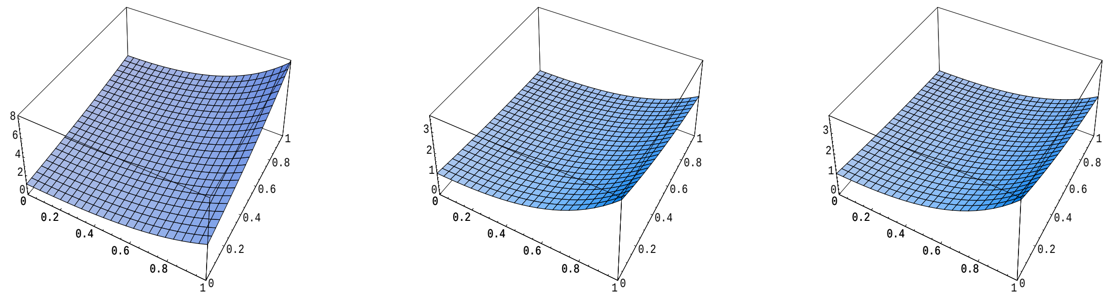
The transformed versions of (5.12) is

$$U(0,t) = e^{-t}, \frac{\partial U(0,t)}{\partial x} = 0 \tag{48}$$

Substituting (5.19) into (5.16)-(5.18) gives
$U(1,t) = 0, U(3,t) = 1, U(5,t) = 0, U(7,0) = 0, U(9,0) = 0, ...U(2,t) = e^{-t} + 1, U(4,t) = \frac{e^{-t}}{2!}$, $U(6,t) = \frac{e^{-t}}{3!}, ...$
Substituting all $U(k,t)$'s into (3.3). We obtained the solution in the following form

$$u(x,t) = x^3 + e^{-t}\left[1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + ...\right] \tag{49}$$

and the analytic solution of the problems is given by $u(x,t) = x^3 + e^{-t+x^2}$.
In Fig. 2(a) and 2(b), shows the four term approximate solution for the differential transform method and projected differential transform method are given. Analytical solution is presented in Fig. 2(c).



(a) Represents the numerical solution of $u(x,t)$ for different values of x and t using DTM

(b) Represents the numerical solution of $u(x,t)$ for different values of x and t using PDTM

(c) Exact Solution $u(x,t)$

Figure 2:

Example 3 *Now, We consider the following non-homogeneous wave-type equation [7]*

$$u_{xx} + \frac{4}{x}u_x - (18x + 9x^4)u = u_{tt} - 2 - (18x + 9x^4)t^2 \tag{50}$$

subject to

$$u(x,0) = e^{x^3}, u_t(x,0) = 0 \tag{51}$$

Differential transform method: The transformed version of (5.21) is

$$\sum_{m=0}^{k}\sum_{r=0}^{h}\delta(m-1,h-r)(k-m+1)(k-m+2)U(k-m+2,r) + 4(k+1)U(k+1,h)$$
$$-18\sum_{m=0}^{k}\sum_{r=0}^{h}\delta(m-2,h-r)U(k-m,r) - 9\sum_{m=0}^{k}\sum_{r=0}^{h}\delta(m-5,h-r)U(k-m,r)$$
$$= \sum_{m=0}^{k}\sum_{r=0}^{h}\delta(m-1,h-r)(r+1)(r+2)U(k-m,r+2) - 2\delta(k-1,h) - 18\delta(k-2,h-2) - 9\delta(k-5,h-2) \tag{52}$$

Substituting the transformed versions of (5.22) in (5.23), we obtain the following $U(k,h)$values.
$U(0,2) = 1, U(1,2) = 0, U(2,2) = 0, U(3,2) = 0, U(4,0) = 0, ...$
Substituting all $U(k,h)$'s in (2.2). We obtain the solution in the following form

$$u(x,t) = t^2 + \left[1 + \frac{x^3}{1!} + \frac{x^6}{2!} + \frac{x^9}{3!} + ...\right] \tag{53}$$

Projected differential transform method: The modified transformed version of (5.21) w. r. t. t is

$$x\frac{\partial^2 U(x,h)}{\partial x^2} + 4\frac{\partial U(x,h)}{\partial x} - (18x^2 + 9x^5)U(x,h) = x(h+1)(h+2)U(x,h+2)$$
$$-2x - (18x^2 + 9x^5)\delta(h-2) \tag{54}$$

The transformed versions of (5.22) is

$$U(x,0) = e^{x^3}, \frac{\partial U(x,0)}{\partial t} = 0 \tag{55}$$

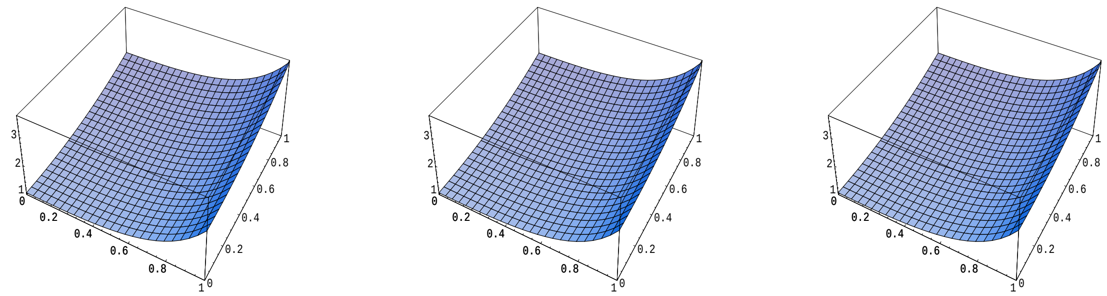
substituting (5.26) into (5.25), we obtain the following $U(k,t)$ values. $U(x,1) = U(x,3) = U(x,5) = ... = 0, U(x,2) = 1, U(x,4) = 0, ...$ *substituting all $U(x,h)$values into (3.3). We obtained the solution in the following form*

$$u(x,t) = t^2 + e^{x^3} \tag{56}$$

and the analytical solution is given by

$$u(x,t) = t^2 + e^{x^3} \tag{57}$$

The behaviour of the solution obtained by differential transform method and the projected differential transform method solutions are shown for different value of x and t in Fig. 3(a) and 3(b) respectively. Analytical solution is presented in Fig. 3(c).



(a) Represents the numerical solution of $u(x,t)$ for different values of x and t using DTM

(b) Represents the numerical solution of $u(x,t)$ for different values of x and t using PDTM

(c) Exact Solution $u(x,t)$

Figure 3:

Example 4 *Consider the following nonlinear time dependent equation [7]*

$$u_{xx} + \frac{5}{x}u_x + (24t + 16t^2x^2)e^u - 2x^2e^{\frac{u}{2}} = u_t \tag{58}$$

subject to

$$u(x,0) = 0, u_t(x,0) = -2x^2 \tag{59}$$

Differential transform method: The transformed version of (5.29) is

$$\begin{aligned} &\sum_{m=0}^{k}\sum_{r=0}^{h}\delta(m-1,h-r)(k-m+1)(k-m+2)U(k-m+2,r) + 5(k+1)U(k+1,h) \\ &+24\sum_{m=0}^{k}\sum_{r=0}^{h}\delta(m-1,h-r-1)F(k-m,r) + 16\sum_{m=0}^{k}\sum_{r=0}^{h}\delta(m-3,h-r-2)F(k-m,r) \\ &-2\sum_{m=0}^{k}\sum_{r=0}^{h}\delta(m-3,h-r)G(k-m,r) = \sum_{m=0}^{k}\sum_{r=0}^{h}\delta(m-1,h-r)(r+1)U(k-m,r+1) \end{aligned} \tag{60}$$

Where $F(k-m,r)$ is transformed version of e^u and $G(k-m,h)$ is the transformed version of $e^{\frac{u}{2}}$.

$$F(k,h) = \begin{cases} e^{U(0,0)}, & k=0, h=0 \\ \sum_{m=0}^{k-1}\sum_{r=0}^{h}\frac{k-r}{k}U(k-m,r)F(m,h-r), & k \geq 1 \\ \sum_{m=0}^{k}\sum_{r=0}^{h-1}\frac{h-r}{h}U(m,h-r)F(k-m,r), & h \geq 1 \end{cases} \tag{61}$$

$$G(k,h) = \begin{cases} e^{\frac{U(0,0)}{2}}, & k=0, h=0 \\ \frac{1}{2}\sum_{m=0}^{k-1}\sum_{r=0}^{h}\frac{k-r}{k}U(k-m,r)G(m,h-r), & k \geq 1 \\ \frac{1}{2}\sum_{m=0}^{k}\sum_{r=0}^{h-1}\frac{h-r}{h}U(m,h-r)G(k-m,r), & h \geq 1 \end{cases} \tag{62}$$

The transformed versions of (5.30) is

$$U(k,0) = 0, U(k,1) = \begin{cases} -2, & k=2 \\ 0, & \textit{other wise} \end{cases} \tag{63}$$

Substituting (2.34) $k=1, h=1$into (2.31). And also, substituting $k=0, h=0$into (5.32) and (5.33) to get the value of $F(0,0)$ and $G(0,0)$. Therefore, we have:
$F(0,0) = 1, G(0,0) = 1, F(1,0) = G(1,0) = 0, F(2,1) = -2, ..., U(1,2) = U(2,2) = U(3,2) = 0, U(4,2) = 1, U(5,2) = 0, ..., U(0,3) = U(1,3) = U(2,3) = ... = 0$
Following the same recursive procedure, we obtain
$U(4,2) = 1, U(6,3) = -\frac{2}{3}, U(8,4) = \frac{1}{2}, U(10,5) = -\frac{2}{5}, ...$
substituting all $U(k,h)$ values in (2.2). We obtain the solution in the following form

$$u(x,t) = -2x^2t + x^4t^2 - \frac{2x^6t^3}{3} + \frac{x^8t^4}{2} - \frac{2x^{10}t^5}{5} + ... \tag{64}$$

Projected differential transform method: The transformed version of (5.29) w. r. t. t is

$$x\frac{\partial^2 U(x,h)}{\partial x^2}+5\frac{\partial U(x,h)}{\partial x}+24x\sum_{m=0}^{h}\delta(m-1)F(x,h-m)$$
$$+16x^3\sum_{m=0}^{h}\delta(m-2)F(x,h-m)-2x^3G(x,h)=x(h+1)U(x,h+1) \tag{65}$$

$$F(k,h)=\begin{cases} e^{U(x,0)}, & h=0\\ \sum_{m=0}^{h-1}\frac{m+1}{h}U(x,m+1)F(x,k-m-1), & h\geq 1\end{cases} \tag{66}$$

$$G(k,h)=\begin{cases} e^{\frac{U(x,0)}{2}}, & h=0\\ \frac{1}{2}\sum_{m=0}^{h-1}\frac{m+1}{h}U(x,m+1)G(x,k-m-1), & h\geq 1\end{cases} \tag{67}$$

The transformed versions of (5.30) is

$$U(x,0)=0,U(x,1)=-2x^2 \tag{68}$$

substituting (5.39) into (5.36)-(5.38) gives $U(x,2)=x^4, U(x,3)=-\frac{2}{3}x^6, U(x,4)=\frac{x^8}{2},...$ *substituting all* $U(x,h)$ *values into (3.3). We obtained the solution in the following form*

$$u(x,t)=-2x^2t+x^4t^2-\frac{2x^6t^3}{3}+\frac{x^8t^4}{2}-\frac{2x^{10}t^5}{5}+... \tag{69}$$

and the analytical solution is given by $u(x,t)=-2ln(1+tx^2)$.

Example 5 *Finally, we consider the following nonlinear time dependent homogeneous equation [7]*

$$u_{xx}+\frac{6}{x}u_x+(14t+x^4)u+4tulnu=u_{tt} \tag{70}$$

subject to

$$u(x,0)=1, u_t(x,0)=-x^2 \tag{71}$$

Differential transform method: The transformed version of (5.41) is

$$\sum_{m=0}^{k}\delta(m-1)(k-m+1)(k-m+2)U(k-m+2,h)+6(k+1)U(k+1,h)$$
$$+14\sum_{m=0}^{k}\sum_{r=0}^{h}\delta(m-1)\delta(h-r-1)U(k-m,r)+\sum_{m=0}^{k}\sum_{r=0}^{h}\delta(m-5)U(k-m,r)$$
$$+\sum_{m=0}^{k}\sum_{r=0}^{h}\delta(m-1,h-1)U(m,h-r)F(k-m,r)=\sum_{m=0}^{k}\delta(m-1)(h+1)(h+2)U(k-m,h+2) \tag{72}$$

Where $F(k-m,r)$ *is transformed version of* lnu *and it is given by*

$$F(k,h)=\begin{cases} lnU(0,0), & k=0,h=0\\ \frac{U(1,0)}{U(0,0)}, & k=1,h=0\\ \frac{U(0,1)}{U(0,0)}, & k=0,h=1\\ \frac{1)}{kU(0,0)}\left[kU(k,h)-\sum_{m=0}^{k-2}\sum_{r=0}^{h}U(k-m-1,r)(m+1)F(m+1,h-r)\right], & k\geq 2\\ \frac{1)}{hU(0,0)}\left[hU(k,h)-\sum_{m=0}^{k}\sum_{r=0}^{h-2}U(k-m-1,r)(h-r+1)F(m,h-r+1)\right], & h\geq 2\end{cases} \tag{73}$$

The transformed versions of (5.42) is

$$U(k,0)=\begin{cases}1, & k=0\\ 0, & other\ wise\end{cases}, U(k,1)=\begin{cases}-1, & h=2\\ 0, & other\ wise\end{cases} \tag{74}$$

substituting (5.45) k = 1, h = 1 into (5.43). And also, substituting k = 0, h = 0 into (5.44) to get the value of $F(0,0)$. Therefore, we have: $F(0,0) = 0, F(1,0) = 0, F(0,1) = 0, ...\ U(0,2) = U(1,2) = U(2,2) = U(3,2) = 0, U(4,2) = \frac{1}{2}, U(5,2) = 0, ...,$
Following the same recursive procedure, we obtain $U(6,3) = -\frac{1}{6}, U(8,4) = \frac{1}{24}, ...$
substituting all $U(k,h)$ values in (2.2). We obtain the solution in the following form

$$u(x,t) = 1 - x^2 t + \frac{x^4 t^2}{2!} - \frac{x^6 t^3}{3!} + \frac{x^8 t^4}{4!} - ... \quad (75)$$

Projected differential transform method: The modified transformed version of (5.41) w. r. t.t is

$$x\frac{\partial^2 U(x,h)}{\partial x^2} + 6\frac{\partial U(x,h)}{\partial x} + 14x\sum_{m=0}^{h}\delta(m-1)U(x,h-m) + x^5 U(x,h)$$
$$+4x\sum_{m=0}^{h}\sum_{n=0}^{m}\delta(m-1)U(x,h-m)G(x,m-n) = x(h+1)(h+2)U(x,h+2) \quad (76)$$

Where $G(x,m-n)$ is the transformed function of lnu

$$G(x,h) = \begin{cases} lnU(x,0), & h=0 \\ \frac{U(x,1)}{U(x,0)}, & h=1 \\ \frac{1}{U(x,0)}\left[U(x,h)\sum_{m=0}^{h-2}F(x,m+1)U(x,k-1-m)\right], & h\geq 2 \end{cases} \quad (77)$$

The transformed versions of (5.42) is

$$U(x,0) = 1, U(x,1) = -x^2 \quad (78)$$

substituting (5.49) into (5.47)-(5.48). We obtain the following $U(k,h)$ values. $U(x,2) = \frac{x^4}{2!}, U(x,3) = -\frac{x^6}{3!}, U(x,4) = \frac{x^8}{4!}, ...$ substituting all $U(x,h)$ values into (3.3). We obtained the closed form series solution as

$$u(x,t) = 1 - x^2 t + \frac{x^4 t^2}{2!} - \frac{x^6 t^3}{3!} + \frac{x^8 t^4}{4!} - ... \quad (79)$$

and the analytical solution is given by $u(x,t) = e^{-tx^2}$.

6 Conclusions

In this study, we implemented the two-dimensional differential transform method and projected differential transform method for time-dependent Emden-Fowler type equations. Differential transform method is an effective tool for solving linear and nonlinear problems. But, it also faces some difficulties while constructing recursive equation for the nonlinear problems and it requires an expensive computational cost to solve the algebraic recursive equation. The proposed projected differential transform method for the specific variable can obtain the simple recursive equation. Thus, it is concluded that projected differential transform method enhances the effectiveness of the computational work when compared with the differential transform method.

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