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Quantization Dimension of Order ∞ for Random Self-conformal Measures

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Abstract: Let μ be a random self-conformal measure on $\mathbb{R}^d$ associated with a family of contractive conformal mappings $\{S_i\}_{i=1}^N$ and a probability vector $(p_i)_{i=1}^N$. When $\{S_i\}_{i=1}^N$ satisfies the strong open set condition, we determine the quantization dimension $D_\infty(\mu)$ and show that it coincides with the unique solution of the Bowen equation.

Keywords: quantization dimension; random self-conformal set; random self-conformal measure

1 Introduction

The quantization problem originates in the theory of signal processing and data compression. It was used by electrical engineers starting in the late 40's. The notion of quantization dimension was first introduced by Zador (see [9]). Mathematically, the main idea is to approximate a given probability measure by discrete probability measures with finite support. One can refer to Graf and Luschgy (see [6]) for details. Much of the previous work is based in L_r metrics $0 < r < \infty$ as a measure of the quantization error(see [2–4, 6, 7, 9]). Recently Zhu (see [12]) investigates the quantization dimension with respect to the geometric error ($r \to 0$) for self-conformal measures. In this paper we determine the quantization dimension for random self-conformal measures in case $r = \infty$.

Let $U \subset \mathbb{R}^d$ be a connected and open subset. For $0 < \gamma \leqslant 1$, denote by $Con_0^{1+\gamma}(U)$ the family of conformal diffeomorphisms $S : U \to S(U)$ for which there exists a constant C_S such that

$$|S^{'}(x) - S^{'}(y)| \leqslant C_S|x - y|^\gamma$$

for all $x, y \in U$, where $S^{'}(x)$ is the differential of S at x, and $|S^{'}(x)|$ is the operator norm of the differential.

Let $N \geqslant 2$ be a positive integer, we are given N conformal diffeomorphisms $S_i : U \to S_i(U)$, for all $i = 1, ..., N$. Consider the product space $\Omega_0 = (Con_0^{1+\gamma}(U))^N \times [0, 1]^N$, which is a separable metrical space with the Borel σ-algebra $\mathfrak{F}_0$.

By Patzschke (see [8]), a probability measure $\mathbb{P}_0$ on $(\Omega_0, \mathfrak{F}_0)$ is *a random conformal function system* if
i) there exists a compact connected subset $K \subset U$ with $K = \overline{\text{int}K}$ such that $S_i(\text{int}K) \subset \text{int}K$ and $S_i(U) \cap S_j(U) = \emptyset$ for $i, j = 1, \cdots, N, i \neq j$,
ii) there exist constants $C_0 \geq 0$, $0 < r_{\min} \leq r_{\max} < 1$ and $0 < p_{\min} \leq p_{\max} < 1$ such that $C_{S_i} \leq C_0$, $r_{\min} \leq |S^{'}(x)| \leq r_{\max}$ and $p_{\min} \leq p_i \leq p_{\max}$ for all $i = 1, \cdots, N$ and $\mathbb{P}_0$-almost all $(S_1, \cdots, S_N; p_1, \cdots, p_N)$, and
iii) $\int \sum\limits_{i=1}^N p_i \mathbb{P}_0(d(S_1, \cdots, S_N; p_1, \cdots, p_N)) = 1$.

For the random variable $(S_1, \cdots, S_N; p_1, \cdots, p_N)$, a unique compact random set $E \subset U$ is called a random self-conformal set associated to $\mathbb{P}_0$ if $E \stackrel{d}{=} \bigcup\limits_{i=1}^N S_i E_i$, where the E_i, $i = 1, \cdots, N$, are independent copies of E and independent of $(S_1, \cdots, S_N; p_1, \cdots, p_N)$. In analogy a random measure μ with supp$(\mu) = E$ is called a random self-conformal measure associated to $\mathbb{P}_0$ if $\mu \stackrel{d}{=} \sum\limits_{i=1}^N p_i \mu_i \circ S_i^{-1}$, where the μ_i, $i = 1, \cdots, N$, are independent samples of μ and independent of $(S_1, \cdots, S_N; p_1, \cdots, p_N)$. We say they are random self-similar sets and measures when S_i are similarities.

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IJNS.2012.04.15/599

Throughout β is assume to be locally finite in the sense that the number of points of α within any bounded subset of $\mathbb{R}^d$ is finite. The Vornoi region generated by $a \in \alpha$ is defined by

$$W(a|\alpha) = \{x \in \mathbb{R}^d : \|x-a\| = \min_{b\in\alpha} \|x-b\|\}.$$

The Vornoi regions $W(a|\alpha)$ are closed and star-shaped relative to a and the Voronoi diagram $\{W(a|\alpha) : a \in \alpha\}$ of α provides a covering of $\mathbb{R}^d$. A Borel measurable partition $\{A_a : a \in \alpha\}$ of $\mathbb{R}^d$ is called Vornoi partition with respect to α if

$$A_a \subset W(a|\alpha) \quad \text{for every } \ a \in \alpha.$$

Following the work of Graf and Luschgy (see [6]), we define the quantization dimension as following. Let μ be a random measure on $\mathbb{R}^d$. For $g : \mathbb{R}^d \to \mathbb{R}$ is Borel measurable, define

$$\|g\|_{\mu,\infty} = \|g\|_\infty = \inf\{c \geq 0 : \|g\| \leq c, \ \ \mu-\text{a.s.}\},$$

and

$$e_{n,\infty}(\mu) = \inf\{\|d_\alpha\|_\infty : \alpha \subset \mathbb{R}^d, 1 \leq \text{card}(\alpha) \leq n\},$$

where $d_\alpha(x) = d(x,\alpha) = \inf_{a\in\alpha} \|x-a\|$, $\|d_\alpha\|_\infty = \sup_{x\in\text{supp}(\mu)} d_\alpha(x)$.

Note that if supp(μ) is compact, then $e_{n,\infty}(\mu) < \infty$. It follows from the continuity of d_α that $\|d_\alpha\|_\infty = \sup_{x\in\text{supp}(\mu)} d_\alpha(x)$ and hence

$$e_{n,\infty}(\mu) = \inf\{\sup_{x\in\text{supp}(\mu)} \min_{a\in\alpha} \|x-a\| : \alpha \subset \mathbb{R}^d, 1 \leq \text{card}(\alpha) \leq n\}.$$

So if we define for a nonempty compact set $A \subset \mathbb{R}^d$,

$$e_{n,\infty}(A) = \inf\{\max_{x\in A} \min_{a\in\alpha} \|x-a\| : \alpha \subset \mathbb{R}^d, 1 \leq \text{card}(\alpha) \leq n\}$$

then $e_{n,\infty}(\mu) = e_{n,\infty}(A)$ for every probability measure μ with supp$(\mu) = A$. If the infimum is attained at some $\alpha \subset \mathbb{R}^d$ with $1 \leq \text{card}(\alpha) \leq n$, then the set α is called an *n-optimal set* for μ of order ∞. The collection of all the n-optimal sets is denoted by $C_{n,\,\infty}(A)$.

The *upper and lower quantization dimension* $\overline{D}_\infty(\mu)$, $\underline{D}_\infty(\mu)$ of μ are defined as

$$\overline{D}_\infty(\mu) := \limsup_{n\to\infty} \frac{\log n}{-\log e_{n,\infty}(\mu)}, \qquad \underline{D}_\infty(\mu) := \liminf_{n\to\infty} \frac{\log n}{-\log e_{n,\infty}(\mu)}$$

with probability one. If $\overline{D}_\infty(\mu) = \underline{D}_\infty(\mu)$, the common value is called the *quantization dimension* of μ and denoted by $D_\infty(\mu)$.

Proposition 1 *(Cf. Proposition 11.3. in [6]) If $0 \leq t < \underline{D}_\infty(\mu) < s$, then*

$$\lim_{n\to\infty} ne^t_{n,\infty}(\mu) = +\infty \ \textit{and} \ \liminf_{n\to\infty} ne^s_{n,\infty}(\mu) = 0.$$

Similar statements hold for the upper quantization dimension.

If the strong separation condition (SSC) holds, i.e., if there is an open set O satisfying $S_i(O) \subset U$ for all i and $\min_{i\neq j} \inf\{d(S_i(x), S_i(y)) : x, y \in O\} > 0$. Graf and Luschgy (see [6]) proved that the quantization dimension (of order ∞) for a self-similar measure on $\mathbb{R}^d$ coincides with the similarity dimension of $\{S_i\}_{i=1}^N$ under the SSC. We say that $\mathbb{P}_0(or\mathbb{P})$ satisfies the strong open set condition (SOSC), if there is an open set O with
(i) $S_i(O) \subset O$ for all $i = 1, ..., N$ with probability one,
(ii) $S_i(O) \cap S_j(O) = \emptyset$ for all $i \neq j, i, j = 1, ..., N$ with probability one, and
(iii) $E \cap O \neq \emptyset$ with probability one.

In this paper, we will extend their result to random self-conformal measures. That is,

Theorem 2 *Let μ be the random self-conformal measure associated with $(S_1, \cdots, S_N; p_1, \cdots, p_N)$. Suppose that the SOSC is satisfied, then the quantization dimension $D_\infty(\mu)$ exists and equals the unique solution D of Bowen equation $P(D) = 0$.*

2 Definitions and notations

Let $\Sigma = \{1, \cdots, N\}^{\mathbb{N}}$ be the set of infinite sequences of integers from 1 to N, consider the shift operator $T : \Sigma \to \Sigma$, defined by $T(\sigma) = \sigma_2\sigma_3 \cdots$ for $\sigma = \sigma_1\sigma_2\sigma_3 \cdots \in \Sigma$. Denote by $\Sigma_n = \{1, \cdots, N\}^n$ the set of sequences of length n and by $\Sigma_* = \bigcup_{n=0}^{\infty} \Sigma_n$ the set of all sequences of finite length including $\emptyset$. For $\tau \in \Sigma_n$, denote by $|\tau| = n$ the length of the sequence. If $\tau \in \Sigma_*$ and $\sigma \in \Sigma_* \cup \Sigma$ let $\Delta_\tau\sigma = \tau * \sigma$ be the sequence consisting of the concatenation of both sequences. The truncation of a sequence to the first k entries is denoted by $\sigma|_k = \sigma_1 \cdots \sigma_k$ where $\sigma \in \Sigma$ or $\sigma \in \Sigma_n$ with $n \geq k$, and $\sigma^- = \sigma|_{k-1} = \sigma_1 \cdots \sigma_{k-1}$. We call τ a predecessor of σ and write $\tau \prec \sigma$ for $\tau \in \Sigma_*$ and $\sigma \in \Sigma_* \bigcup \Sigma$ if $\sigma|_{|\tau|} = \tau$. We say σ, τ are incomparable if neither $\tau \prec \sigma$ nor $\sigma \prec \tau$. Further, let $[\sigma] = \{\tau \in \Sigma : \sigma \prec \tau\}$ be the cylinder set of sequences starting with $\sigma, \sigma \in \Sigma_*$.

Next let $\Omega = \Omega_0^{\Sigma_*}$, $\mathfrak{F}$ be the product σ-algebra on Ω, and $\mathbb{P}$ be the product measure on Ω with $\mathbb{P}_0$ on each component, we say that $\mathbb{P}$ is a random conformal iterated function system. Thus we get the primary probability space $(\Omega, \mathfrak{F}, \mathbb{P})$. This space assigns to each finite $\tau \in \Sigma_*$ a random variable $(S_{\tau*1}, ..., S_{\tau*N}; p_{\tau*1}, ..., p_{\tau*N})$. Write the elements $\omega \in \Omega$ in the form

$$\omega(\tau) = (S_{\tau*1}, ..., S_{\tau*N}; p_{\tau*1}, ..., p_{\tau*N})$$

for $\tau \in \Sigma_*$. Define

$$\overline{S}_\tau = S_{\tau|_1} \circ \cdots \circ S_{\tau|_{|\tau|}}, \qquad \overline{p}_\tau = p_{\tau|_1} \cdots p_{\tau|_{|\tau|}}$$

and $\overline{S}_\emptyset = id, \overline{p}_\emptyset = 1$ for $\tau \in \Sigma_*$.

We say the μ and E are random self-conformal sets and measures when S_i are conformal. The *similarity dimension* for this collection $\{S_i\}_{i=1}^N$ is defined as the unique positive solution s of the equation

$$\sum_{i=1}^{N} \mathbb{E}\mathrm{Lip}(S_i)^s = 1.$$

In this paper the role of similarity dimension is played by the unique solution s of the Bowen equation $P(D) = 0$, where the pressure $P(t)$ defined by

$$P(t) = \lim_{n\to\infty} \frac{1}{n} \log \sup_{x\in E} \sum_{\tau\in\Sigma_n} \mathbb{E}|\overline{S}_\tau^{'}(x)|^t,$$

for $t > 0$. It is immediate that the Hausdorff measure $\mathcal{H}^D(E) < \infty$.

A useful tool for the investigation of the random fractal is the notion of a finite maximal Markov antichain. We call a finite random subset $\Gamma(\omega) \subset \Sigma_*$ a finite maximal Markov antichain if
(i) every distinct words in $\Gamma(\omega)$ are incomparable,
(ii) every $\tau \in \Sigma$ has a predecessor in $\Gamma(\omega)$, and
(iii) $\{\omega \in \Omega : \tau \in \Gamma(\omega)\} \in \mathfrak{F}_{|\tau|}$ for all $\tau \in \Sigma_*$.

If $(q_1, \cdots, q_N)$ is an N-tuples random numbers with $\mathbb{E}\sum_{i=1}^{N} q_i = 1$ and $\Gamma(\omega)$ is a finite maximal Markov antichain then

$$\mathbb{E} \sum_{\sigma\in\Gamma(\omega)} q_\sigma = 1. \tag{1}$$

If $0 \leq \varepsilon \leq \min\{q_1, \cdots, q_N\}$ then

$$\Gamma(\varepsilon) = \{\sigma \in \Sigma_* : q_{\sigma^-} \geq \varepsilon > q_\sigma\}$$

is a finite maximal Markov antichain.

For every $\tau \in \Sigma_*$, write $\|\overline{S}_\tau^{'}\| := \sup\{|\overline{S}_\tau^{'}(x)| : x \in E\}$. As the same proof as Lemma 2.1 in [8], we have there is a constant $C \geq 1$ such that

$$|\overline{S}_\tau^{'}(x)| \leq C|\overline{S}_\tau^{'}(y)|, \qquad \text{for all } x, y \in E \text{ and all } \tau \in \Sigma_* \tag{2}$$

with probability one. This implies that

$$C^{-1}\|\overline{S}_\tau^{'}\| \le |\overline{S}_\tau^{'}(x)| \le \|\overline{S}_\tau^{'}\|, \qquad \text{for all } x \in E \text{ and all } \tau \in \Sigma_* \tag{3}$$

with probability one. By [8], Lemma 2.2, there is a constant $\widetilde{C} \ge C$ such that

$$\widetilde{C}^{-1}\|\overline{S}_\tau^{'}\|d(x,y) \le d(\overline{S}_\tau(x),\overline{S}_\tau(y)) \le \widetilde{C}\|\overline{S}_\tau^{'}\|d(x,y) \quad \text{for all } x,y \in K \text{ and all } \tau \in \Sigma_* \tag{4}$$

with probability one. And by [8], Corollary 2.3, we have for $\tau \in \Sigma_*$,

$$\widetilde{C}^{-1}\|\overline{S}_\tau^{'}\||K| \le |K_\tau| \le \widetilde{C}\|\overline{S}_\tau^{'}\||K| \tag{5}$$

with probability one, where $K_\tau := \overline{S}_\tau(K)$.

3 Proof of main result

In this section, we will prove Theorem 2. First we need some lemmas as following.

Lemma 3 *Let $A \subset \mathbb{R}^d$ be a nonempty compact set and let $S : \mathbb{R}^d \to \mathbb{R}^d$ be a conformal mapping, then*

$$\widetilde{C}^{-1}\|S^{'}\|e_{n,\infty}(A) \le e_{n,\infty}\big(S(A)\big) \le \widetilde{C}\|S^{'}\|e_{n,\infty}(A)$$

with probability one.

Proof. This is obvious by the definition of $e_{n,\infty}(A)$ and (5). ■

Lemma 4 *Let Γ be a finite maximal Markov antichain. Then for all $n \ge |\Gamma|$,*

$$e_{n,\infty}(\mu) \le \min\big\{\max_{\sigma\in\Gamma}\widetilde{C}\|\overline{S}_\sigma^{'}\|e_{n_\sigma,\infty}(\mu) : n_\sigma \ge 1, \sum_{\sigma\in\Gamma} n_\sigma \le n\big\}$$

with probability one.

Proof. For $n_\sigma \ge 1$ with $\sum\limits_{\sigma\in\Gamma} n_\sigma \le n$, let $\alpha_\sigma \in C_{n_\sigma,\infty}(S_\sigma(E))$ and let $\alpha = \bigcup\limits_{\sigma\in\Gamma}\alpha_\sigma$. Then $|\alpha| \le n$. Since $\text{supp}(\mu\circ S_\sigma^{-1}) = S_\sigma(E)$, by Lemma 6, we have

$$\begin{aligned}
e_{n,\infty}(\mu) &= e_{n,\infty}(E) \le \max_{x\in E}\min_{a\in\alpha}\|x-a\| \\
&= \max_{\sigma\in\Gamma}\max_{x\in S_\sigma(E)}\min_{a\in\alpha}\|x-a\| \le \max_{\sigma\in\Gamma}\max_{x\in S_\sigma(E)}\min_{a\in\alpha_\sigma}\|x-a\| \\
&= \max_{\sigma\in\Gamma} e_{n_\sigma,\infty}(S_\sigma(E)) \le \max_{\sigma\in\Gamma}\widetilde{C}\|\overline{S}_\sigma^{'}\|e_{n_\sigma,\infty}(E) \\
&= \max_{\sigma\in\Gamma}\widetilde{C}\|\overline{S}_\sigma^{'}\|e_{n_\sigma,\infty}(\mu)
\end{aligned}$$

with probability one. The lemma is proved. ■

Hence we obtain the following theorem.

Theorem 5 *Let D be a the unique solution of the Bowen equation $P(D) = 0$. Then*

$$\limsup_{n\to\infty} ne_{n,\infty}(\mu)^D < +\infty,$$

in particular, the upper quantization dimension $\overline{D}_\infty(\mu)$ of μ is less than or equal to D.

Proof. Let $q_\sigma = \|S_\sigma^{'}\|^D$ and $\varepsilon_0 = \min\{q_1,\cdots,q_N\}$. Let $m,n \in \mathbb{N}$ be arbitrary with $\frac{m}{n} < \varepsilon_0^2$. Set $\varepsilon = \varepsilon_0^{-1}\frac{m}{n}$ and $\Gamma(\varepsilon) = \{\sigma \in \Sigma_* : q_{\sigma^-} \ge \varepsilon > q_\sigma\}$. It is easy to show $\Gamma(\varepsilon)$ is a maximal Markov finite antichain. It follows by (1) that

$$1 = \mathbb{E}\sum_{\sigma\in\Gamma(\varepsilon)} q_\sigma = \mathbb{E}\sum_{\sigma\in\Gamma(\varepsilon)} q_{\sigma^-}q_{\sigma_{|\sigma|}} \ge \varepsilon\varepsilon_0|\Gamma(\varepsilon)|,$$

hence

$$|\Gamma(\varepsilon)| \leq (\varepsilon\varepsilon_0)^{-1} = \frac{m}{n}.$$

By Lemma 4, choosing $n_\sigma = m \geq 1$,

$$\begin{aligned} e_{n,\infty}(\mu)^D &\leq \mathbb{E} \max_{\sigma\in\Gamma(\varepsilon)} \widetilde{C}^D \|\overline{S}'_\sigma\|^D e_{m,\infty}(\mu)^D \leq \widetilde{C}^D \varepsilon e_{m,\infty}(\mu)^D \\ &= \widetilde{C}^D \varepsilon_0^{-1} \frac{n}{m} e_{m,\infty}(\mu)^D, \end{aligned}$$

and, hence,

$$n e_{n,\infty}(\mu)^D \leq \widetilde{C}^D \varepsilon_0^{-1} m e_{m,\infty}(\mu)^D.$$

For fixed m this holds for all but finitely many n and yields

$$\limsup_{n\to\infty} n e_{n,\infty}(\mu)^D < +\infty,$$

The remaining statement of this theorem follows from Proposition 1. ■

Remark 6 *The proof of the preceding theorem shows that*

$$\limsup_{n\to\infty} n e_{n,\infty}(\mu)^D \leq \mathbb{E}\max\{\|S'_1\|^{-D}, \cdots, \|S'_N\|^{-D}\} \inf_{m\in\mathbb{N}} m e_{m,\infty}(\mu)^D.$$

Next we will show the following theorem.

Theorem 7 *Let $\{S_i\}_{i=1}^N$ satisfy the SOSC and let D be a the unique solution of the Bowen equation $P(D) = 0$. Then*

$$\liminf_{n\to\infty} n e_{n,\infty}(\mu)^D > 0,$$

in particular, the lower quantization dimension $\underline{D}_\infty(\mu)$ of μ is greater than or equal to D.

Remark 8 *By the fact $\|\cdot\|_r \leq \|\cdot\|_s$ for $1 \leq r \leq s \leq \infty$, we know $e_{n,r}(\mu) < e_{n,s}(\mu)$. So the preceding theorem is equivalent to*

$$\liminf_{n\to\infty} n e_{n,1}(\mu)^D > 0. \tag{6}$$

To prove (6), we need the following lemmas.

Lemma 9 *There exists a constant $c > 0$ with*

$$\mu(B(a,r)) \leq cr^D$$

for every $a \in E$ and $r > o$ with probability one under the SOSC, where $(B(a,r))$ is an open ball.

Proof. By [1], we know the Hausdorff measure $\mathcal{H}^D(E) > 0$ under the SOSC. Also with $\mathcal{H}^D(E) < \infty$. ■

Lemma 10 *Assume that there is a constant $c > 0$ with*

$$\mu(B(a,r)) \leq cr^D$$

for every $a \in E$ and $r > o$ with probability one. Then there exists a constant $c_1 > 0$ with

$$\mathbb{E}\int_B \|x-a\| d\mu(x) \geq c_1 \mu(B)^{1+\frac{1}{D}} \tag{7}$$

for all $a \in \mathbb{R}^d$ and all Borel sets $B \subset \mathbb{R}^d$ with probability one.

Proof. Let $a \in \mathbb{R}^d$ and let B be a Borel subset of $\mathbb{R}^d$. If $\mu(B) = 0$ then the conclusion (7) obvious holds. We assume $\mu(B) > 0$ and set $r_B = \inf\{r > 0 : \mu(B(a,r)) \geq \frac{1}{2}\mu(B)\}$. Since $\lim_{r\to\infty} \mu(B(a,r)) = 1 \geq \mu(B)$, there is an $r > 0$ with $\mu(B(a,r)) \geq \frac{1}{2}\mu(B)$. Hence, $r_B < \infty$.

For $r > r_B$, it follows by Lemma 9 that $cr^D \geq \mu(B(a,r)) \geq \frac{1}{2}\mu(B)$ which implies

$$cr_B^D \geq \frac{1}{2}\mu(B). \tag{8}$$

For $r < r_B$ we have $\mu(B(a,r)) < \frac{1}{2}\mu(B)$.

If $(r_n)_{n\in\mathbb{N}}$ is any increasing sequence with $r_n < r_B$ and $\lim_{n\to\infty} r_n = r_B$, we deduce from $B(a,r_B) = \bigcup_{n\in\mathbb{N}} B(a,r_n)$ that

$$\mu(B(a,r_B)) = \lim_{n\to\infty} \mu(B(a,r_n)) \leq \frac{1}{2}\mu(B). \tag{9}$$

Using (8) and (9) we get

$$\begin{aligned}\mathbb{E}\int_B \|x-a\|\mathrm{d}\mu(x) &\geq \mathbb{E}\int_{B\setminus B(a,r_B)} \|x-a\|\mathrm{d}\mu(x) \geq r_B\mu(B\setminus B(a,r_B))\\ &\geq r_B(\mu(B) - \mu(B(a,r_B))) \geq \frac{1}{2}r_B\mu(B)\\ &\geq \frac{1}{2}(\frac{1}{2c})^{\frac{1}{D}}\mu(B)^{1+\frac{1}{D}}.\end{aligned}$$

Thus let $c_1 = \frac{1}{2}(\frac{1}{2c})^{\frac{1}{D}}$, the lemma is proved. ∎

Corollary 11 *Assume that there is a constant* $c > 0$ *with*

$$\mu(B(a,r)) \leq cr^D$$

for every $a \in E$ *and* $r > o$ *with probability one. Then there exists a constant* $c_2 > 0$ *such that for every* μ*-packing* $\{B_1,\cdots,B_n\}$ *in* $\mathbb{R}^d$ *with* $\mu(\mathbb{R}^d \setminus \bigcup_{i=1}^n B_i) = 0$ *and all* $a_1,\cdots,a_n \in \mathbb{R}^d$*:*

$$n\big(\sum_{i=1}^n \mathbb{E}\int_{B_i} \|x-a_i\|d\mu(x)\big)^D \geq c_2$$

with probability one.

Proof. Set $p = 1 + \frac{1}{D}$ and let $q = 1 + D$. Then we have

$$\frac{1}{p} + \frac{1}{q} = 1, \text{ and } p > 1.$$

Hölder's inequality yields

$$\begin{aligned}S &:= \sum_{i=1}^n 1\big(\mathbb{E}\int_{B_i} \|x-a_i\|\mathrm{d}\mu(x)\big)^{\frac{D}{1+D}}\\ &\leq \big(\sum_{i=1}^n 1^q\big)^{\frac{1}{q}}\big(\sum_{i=1}^n \big(\mathbb{E}\int_{B_i} \|x-a_i\|\mathrm{d}\mu(x)\big)^{\frac{D}{1+D}p}\big)^{\frac{1}{p}}\\ &= n^{\frac{1}{q}}\big(\sum_{i=1}^n \mathbb{E}\int_{B_i} \|x-a_i\|\mathrm{d}\mu(x)\big)^{\frac{1}{p}}.\end{aligned}$$

This implies

$$S^q \leq n\big(\sum_{i=1}^n \mathbb{E}\int_{B_i} \|x-a_i\|\mathrm{d}\mu(x)\big)^D.$$

Using Lemma 11 we get for the constant $c_2 > 0$ of that proposition

$$S := \sum_{i=1}^{n} \big(\mathbb{E}\int_{B_i} \|x - a_i\| \mathrm{d}\mu(x)\big)^{\frac{D}{1+D}}$$
$$\geq \sum_{i=1}^{n} (c_1)^{\frac{D}{1+D}} \mu(B_i) = (c_1)^{\frac{D}{1+D}}.$$

Thus, the corollary holds, if we set $c_2 = (c_1)^D$. ■

Proof of Theorem 7 Let $\alpha_n \in C_{n,1}$ and let $\{A_a : a \in \alpha_n\}$ be a Voronoi partition of $\mathbb{R}^d$ with respect to α_n. By Corollary 3.1 we have

$$ne_{n,1}(\mu)^D = n\big(\sum_{a\in\alpha_n} \mathbb{E}\int_{A_a} \|x - a\| \mathrm{d}\mu(x)\big)^D \geq c_2 > 0.$$

This implies Remark 8, the remaining statement of Theorem 7 follows from Proposition 1. The theorem is proved.

So we can prove Theorem 1 immediately.

Proof of Theorem 2 Combing Theorem 5 and Theorem 7, Theorem 2 is obvious.

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