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Integral Operator Defined by Convolution Product of Hypergeometric Functions

Maslina Darus *, Rabha W. Ibrahim
School of Mathematical Sciences, Faculty of science and Technology, Universiti Kebangsaan Malaysia
Bangi 43600, Selangor Darul Ehsan, Malaysia.
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Abstract: We define an integral operator on the class $\mathcal{A}$ of analytic functions in the unit disk involving $k-$th Hadamard product (convolution) of hypergeometric functions. This operator is a generalization to Noor integral operator for hypergeometric functions and Carlson- Shaffer convolution operator. New classes containing this operator are established.

Keywords: Hadamard product; Integral operator; Subordination; Superordination; Noor operator; Carlson-Shaffer operator; Jack's Lemma; Analytic function; Starlike function; Univalent function; Hypergeometric function.

1 Introduction and preliminaries

The theory of operators: differential and integral; has important roles not only in mathematics but also in physics, control systems, dynamical systems and engineering. It has wide applications in different fields of mathematics such as differential and integral equations, elliptic functions theory, mathematical physics and in computer sciences. In this work we proceed to derive integral operator in the unit disk, by employing the normal hypergeometric functions, and discuss some of its properties by using Jack's Lemma.

Let $\mathcal{H}$ be the class of functions analytic in U and $\mathcal{H}[a,n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$. Let $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in U.. \tag{1}$$

Let $\mathcal{A}$ be the class of analytic functions of the form (1). Given two functions $f, g \in \mathcal{A}$, $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ their convolution or Hadamard product $f(z) * g(z)$ is defined by $f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \ z \in U$. And for several functions $f_1(z), ..., f_m(z) \in \mathcal{A}$

$$f_1(z) * ... * f_m(z) = z + \sum_{n=2}^{\infty} (a_{1n}...a_{mn}) z^n, \ , \ z \in U.$$

For numbers a, b, c other than $0, -1, -2, ..$ the hypergeometric function ${}_2F_1(a, b; c, z)$ is defined by the infinite series

$${}_2F_1(a,b;c,z) = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + ... = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n}\frac{z^n}{n!},$$

while $(x)_n$ is the Pochhammer symbol defined by

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} 1, & n = 0 \\ x(x+1)...(x+n-1), & n = \{1, 2, ...\}. \end{cases}$$

* **Corresponding author**. *E-mail address*: halimgamil@yahoo.com

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Note that ${}_2F_1(a,b;c,z)$ converges absolutely for all $z \in U$ so that it represents an analytic function in U, then we obtain

$$z_2F_1(a,b;c,z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}} \frac{z^n}{(n-1)!}.$$

Assume that

$$\Phi(z) := \underbrace{z_2F_1(a,b;c,z) * ... * z_2F_1(a,b;c,z)}_{k-times} = z + \sum_{n=2}^{\infty} \Big[\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}\Big]^k z^n,$$

we introduce a function $[\Phi(z)]^{-1}$ given by

$$[\Phi(z)]^{-1} = \frac{z}{(1-z)^{\lambda+1}} = z + \sum_{n=2}^{\infty} \frac{(\lambda+1)_{n-1}}{(n-1)!} z^n, \quad \lambda > -1$$

and obtain the following operator $\Upsilon_{k,\lambda} : \mathcal{A} \to \mathcal{A}$

$$\Upsilon_{k,\lambda}(a,b,c)f(z) = [\Phi(z)]^{-1} * f(z), \quad \lambda > -1 \tag{2}$$

where $z \in U$, $f \in \mathcal{A}$ and

$$\Phi(z) * [\Phi(z)]^{-1} = z + \sum_{n=2}^{\infty} \Big[\frac{(c)_{n-1}(n-1)!}{(a)_{n-1}(b)_{n-1}}\Big]^k \Big[\frac{(\lambda+1)_{n-1}}{(n-1)!}\Big] z^n.$$

We have

$$\Upsilon_{k,\lambda}(a,b,c)f(z) = z + \sum_{n=2}^{\infty} \Big[\frac{(c)_{n-1}(n-1)!}{(a)_{n-1}(b)_{n-1}}\Big]^k \Big[\frac{(\lambda+1)_{n-1}}{(n-1)!}\Big] a_n z^n. \tag{3}$$

From (3) we have the following relations:

Lemma 1 *Let $f \in \mathcal{A}$. Then*

(i) $\Upsilon_{0,0}(a,b,c)f(z) = \Upsilon_{1,0}(a,1,a)f(z) = \Upsilon_{1,0}(1,b,b)f(z) = \Upsilon_{1,\lambda}(\lambda+1,b,b)f(z)$
$= \Upsilon_{1,\lambda}(a,\lambda+1,a)f(z) = f(z),$
(ii) $\Upsilon_{1,0}(2,b,b)f(z) = \Upsilon_{1,0}(a,2,a)f(z) = \Upsilon_{1,\lambda}(\lambda+1,2,1)f(z)$
$= \Upsilon_{1,\lambda}(2,\lambda+1,1)f(z) = \int_0^z \frac{f(t)}{t}dt,$
(iii) $z[\Upsilon_{k,\lambda}(a,b,c)f(z)]' = \Upsilon_{1,0}(1,1,2)f(z) = \Upsilon_{1,\lambda}(\lambda+1,1,2)f(z)$
$= \Upsilon_{1,\lambda}(1,\lambda+1,2)f(z).$

Remark 2 *For the incomplete beta function $\phi(a,c,z)$ we have $\phi(a,c,z) = z_2F_1(1,a;c,z)$, a convolution operator*

$$L(a,c) = \phi(a,c,z) * f(z), \tag{4}$$

was defined by Carlson and Shaffer (see [4]). Also for $k = 1$; $(\Upsilon_{1,\lambda}(a,b,c)f(z))$ the operator (3) reduced to Noor integral operator of hypergeometric functions (see [8]).

Let F and G be analytic functions in the unit disk U. The function F is $subordinate$ to G, written $F \prec G$, if G is univalent, $F(0) = G(0)$ and $F(U) \subset G(U)$. In general, given two functions $F(z)$ and $G(z)$, which are analytic in U, the function $F(z)$ is said to be subordinate to $G(z)$ in U if there exists a function $h(z)$, analytic in U with $h(0) = 0 \;\; and \;\; |h(z)| < 1 \;\; for \; all \; z \in U$ such that $F(z) = G(h(z)) \;\; for \; all \; z \in U$. Let $\phi : \mathbb{C}^2 \to \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the differential subordination $\phi(p(z)), zp'(z)) \prec h(z)$ then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, $p \prec q$. If p and $\phi(p(z)), zp'(z))$ are univalent in U and satisfy the differential superordination $h(z) \prec \phi(p(z)), zp'(z))$ then p is called a solution of the differential superordination. An analytic function q is called subordinant of the solution of the differential superordination if $q \prec p$. Let Φ be an analytic function in a domain containing $f(U)$, $\Phi(0) = 0$ and $\Phi'(0) > 0$ (see [3], [6,7]).

The function $f \in \mathcal{A}$ is called $\Phi-$like if $\Re\{\frac{zf'(z)}{\Phi(f(z))}\} > 0, \;\; z \in U$. This concept was introduced by Brickman [2] and established that a function $f \in \mathcal{A}$ is univalent if and only if f is $\Phi-$like for some Φ.

Definition 1 *Let Φ be analytic function in a domain containing $f(U)$, $\Phi(0) = 0$, $\Phi'(0) = 1$ and $\Phi(\omega) \neq 0$ for $\omega \in f(U) - 0$. Let $q(z)$ be a fixed analytic function in U, $q(0) = 1$. The function $f \in \mathcal{A}$ is called $\Phi-$like with respect to q if $\frac{zf'(z)}{\Phi(f(z))} \prec q(z),\ z \in U$.*

Lemma 3 (5) *Let $w(z)$ be analytic in U with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point z_0, then*

$$z_0 w'(z_0) = kw(z_0),$$

where k is a real number and $k \geq 1$.

Lemma 4 (1) *Let $k > 1$ and $\mu > -1$. Suppose that $p(z) \neq 0$ is analytic in U, such that $p(0) = 1$ and satisfies the condition*

$$\Big|\frac{zp'(z)}{p(z)} + \mu(p(z) - 1) - \frac{\mu + 1}{k^2 - 1}\Big| < \frac{k(\mu + 1)}{k^2 - 1}, \quad (z \in U).$$

Then

$$p(z) \prec \frac{k}{k + z} \quad i.e. \quad \Big|p(z) - \frac{k^2}{k^2 - 1}\Big| < \frac{k}{k^2 - 1}$$

and $k/(k + z)$ is the best dominant.

2 A class containing $\Upsilon_{k,\lambda}(a, b, c)$

In this section we consider a class involving the operator $\Upsilon_{k,\lambda}(a, b, c)$.

Now we define subclass of analytic functions containing the operator (3). Let $f \in \mathcal{A}$ then f is a member of the class $\mathcal{S}^*(\beta)$ if and only if

$$\frac{z[\Upsilon_{k,\lambda}(a, b, c)f(z)]'}{\Upsilon_{k,\lambda}(a, b, c)f(z)} \prec \frac{1 + z}{1 - \beta z}, \ \ (0 \leq \beta \leq 1).$$

The following result shows the sufficient condition for functions belonging to the class $\mathcal{S}^*(\beta)$.

Theorem 5 *If $f \in \mathcal{A}$ satisfies*

$$\Re\Big(\frac{z[\Upsilon_{k,\lambda}(a, b, c)f(z)]''}{[\Upsilon_{k,\lambda}(a, b, c)f(z)]'} + 1\Big) < \frac{\beta + 5}{2(1 - \beta)}, \ \ (z \in U) \tag{5}$$

for some $0 \leq \beta \leq 1$, then $f \in \mathcal{S}^*(\beta)$.

Proof. Let $w(z)$ defined by

$$\frac{z[\Upsilon_{k,\lambda}(a, b, c)f(z)]'}{\Upsilon_{k,\lambda}(a, b, c)f(z)} = \frac{1 + w(z)}{1 - \beta w(z)}, \ \ (1 \neq \beta w(z)). \tag{6}$$

Then $w(z)$ is analytic in U. Further, from (6) we observe that $w(0) = 0$. Also it follows that

$$\begin{aligned}
\frac{z[\Upsilon_{k,\lambda}(a, b, c)f(z)]''}{[\Upsilon_{k,\lambda}(a, b, c)f(z)]'} + 1 &= \frac{(1 + \beta)zw'(z) + (1 + w(z))^2}{(1 + w(z))(1 - \beta w(z))} \\
\Rightarrow \Re\Big\{\frac{z[\Upsilon_{k,\lambda}(a, b, c)f(z)]''}{[\Upsilon_{k,\lambda}(a, b, c)f(z)]'} + 1\Big\} &= \Re\Big\{\frac{(1 + \beta)zw'(z) + (1 + w(z))^2}{(1 + w(z))(1 - \beta w(z))}\Big\} \\
&< \frac{\beta + 5}{2(1 - \beta)}
\end{aligned}$$

Now we proceed to prove that $|w(z)| < 1$. Suppose that there exists a point $z_0 \in U$ such that

$$max_{|z| \leq |z_0|}|w(z)| = |w(z_0)| = 1. \tag{7}$$

Then, using the Lemma 1.2 and assuming $w(z_0) = e^{i\theta}$ and $z_0 w'(z_0) = ke^{i\theta},\ k \geq 1$ yields

$$\begin{aligned}
\Re\Big\{\frac{z_0[\Upsilon_{k,\lambda}(a,b,c)f(z_0)]''}{[\Upsilon_{k,\lambda}(a,b,c)f(z_0)]'}+1\Big\} &= \Re\Big\{\frac{(1+\beta)z_0w'(z_0)+(1+w(z_0))^2}{(1+w(z_0))(1-\beta w(z_0))}\Big\}\\
&= \Re\Big\{\frac{(1+\beta)ke^{i\theta}+(1+e^{i\theta})^2}{(1+e^{i\theta})(1-\beta e^{i\theta})}\Big\}\\
&= \Re\Big\{\frac{(1+\beta)k+4}{2(1-\beta)}\Big\}\\
&\geq \frac{\beta+5}{2(1-\beta)}.
\end{aligned}$$

Thus we have

$$\Re\Big(\frac{z[\Upsilon_{k,\lambda}(a,b,c)f(z)]''}{[\Upsilon_{k,\lambda}(a,b,c)f(z)]'}+1\Big) \geq \frac{\beta+5}{2(1-\beta)},\ \ (z\in U)$$

which contradicts the hypothesis (5). Therefore, we conclude that $|w(z)| < 1$ for all $z \in U$ implies

$$\frac{z[\Upsilon_{k,\lambda}(a,b,c)f(z)]'}{\Upsilon_{k,\lambda}(a,b,c)f(z)} \prec \frac{1+z}{1-\beta z},\ \ (0\leq\beta\leq 1). \tag{8}$$

This completes the proof of the theorem. ■

Corollary 6 *If $f \in \mathcal{S}^*(0)$ then*

$$\Big|\frac{z[\Upsilon_{k,\lambda}(a,b,c)f(z)]'}{[\Upsilon_{k,\lambda}(a,b,c)f(z)]}-1\Big| < 1$$

and hence $\Upsilon_{k,\lambda}(a,b,c)$ is starlike.

By putting $k = \lambda = 0$ in Corollary 2.1, we have the following result:

Corollary 7 *If $f \in \mathcal{S}^*(0)$ then*

$$\Big|\frac{zf(z)'}{f(z)}-1\Big| < 1$$

and hence $f(z)$ is starlike.

By setting $\mu = 0$ in Lemma 1.3, we have the following result

Theorem 8 *If $f \in \mathcal{A}$ with $\Upsilon_{k,\lambda}(a,b,c)f(z) \in \mathcal{S}^*(\beta)$ then there is a constant k such that*

$$\Upsilon_{k,\lambda}(a,b,c)f(z) \prec \frac{k}{k+z} \quad i.e. \quad \Big|\Upsilon_{k,\lambda}(a,b,c)f(z)-\frac{k^2}{k^2-1}\Big| < \frac{k}{k^2-1}$$

and $k/(k+z)$ is the best dominant.

Note that many other integral operators are studied for different classes and different properties, which can be found in ([9]-[14]).

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