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Reduced Differential Transform Method for Solving the Fornberg-Whitham Type Equation

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Abstract: In this paper, a general framework of the reduced differential transform method (RDTM) is presented for solving Fornberg-Whitham type equations. In this method, the solution is calculated in the form of convergent power series with easily computable components. Two test modeling problems from mathematical physics, original and modified Fornberg-Whitham equations are discussed to illustrate the effectiveness and the performance of the proposed method. The results show that the proposed iteration technique, without linearization or small perturbation, is very effective and convenient.

Keywords: Fornberg-Whitham equation; reduced differential transform method; initial value problem

1 Introduction

The Fornberg-Whitham equation [1] given as

$$u_t - u_{xxt} + u_x + uu_x = 3u_x u_{xx} + uu_{xxx}, \tag{1}$$

was first proposed for studying the qualitative behaviour of wave breaking [1]. In 1978, Fornberg and Whitham obtained a peaked solution $u(x,t) = Ae^{-\frac{1}{2}|x-\frac{4}{3}t|}$ with an arbitrary constant A [2]. Recently, there has been lots of work focusing on finding the travelling wave solutions to Eq. (1) [3]-[5]. Modifying the nonlinear term uu_x in (1) to u^2u_x, He et al. proposed in [6] the modified Fornberg-Whitham equation

$$u_t - u_{xxt} + u_x + u^2 u_x = 3u_x u_{xx} + uu_{xxx}. \tag{2}$$

Some peakon and solitary wave solutions for (2) were given, based upon the bifurcation theory and the method of phase portrait analysis.

In this paper, we will apply the Reduced differential transform method for solving the original and modified Fornberg-Whitham equations. Unlike the existing analytical methods such as the Adomian decomposition method [3], the homotopy perturbation method [3], [7] and the variational iteration method (VIM) [8], the RDTM [9]-[12] can give exact solution for the Fornberg-Whitham equations and provide converged approximate solutions for the noteworthy equation without linearization, discretization, perturbation, or the calculation of the complicated Adomian polynomials. Due to the above advantages and the simple implementation of the RDTM, we obtain approximate solutions to the Fornberg-Whitham type equations of high accuracy. Numerical experiments associated with two initial value problems are shown to verify the efficiency of the RDTM.

The paper is organized as follows. In Section 2, theoretical aspects of the method are discussed. In Section 3, several examples with analytical solutions in one and two dimensional cases will be given to show the effectiveness of the proposed method. Conclusions are given in Section 4.

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Table 1: Reduced differential transformation.

Function Form	Transformed Form
$u(x,t)$	$U_k(x) = \frac{1}{k!}[\frac{\partial^k}{\partial t^k}u(x,t)]_{t=0}$
$w(x,t) = u(x,t) \mp v(x,t)$	$W_k(x) = U_k(x) \mp V_k(x)$
$w(x,t) = \alpha u(x,t)$	$W_k(x) = \alpha U_k(x)$
$w(x,y) = x^m t^n$	$W_k(x) = x^m \delta(k-n)$
$w(x,y) = x^m t^n u(x,t)$	$W_k(x) = x^m U(k-n)$
$w(x,t) = u(x,t)v(x,t)$	$W_k(x) = \sum_{r=0}^{k} V_r(x)U_{k-r}(x) = \sum_{r=0}^{k} U_r(x)V_{k-r}(x)$
$w(x,t) = \frac{\partial^r}{\partial t^r}u(x,t)$	$W_k(x) = (k+1)...(k+r)U_{k+r}(x) = \frac{(k+r)!}{k!}U_{k+r}(x)$
$w(x,t) = \frac{\partial}{\partial x}u(x,t)$	$W_k(x) = \frac{\partial}{\partial x}U_k(x)$

2 Basic idea of RDTM

The basic definitions of RDTM are defined as follows.

Definition 1 *If function $u(x,t)$ is analytic and differentiated continuously with respect to time t and space x in the domain of interest, then let*

$$U_k(x) = \frac{1}{k!}[\frac{\partial^k}{\partial t^k}u(x,t)]_{t=0}, \tag{3}$$

where the t-dimensional spectrum function $U_k(x)$ is the transformed function.

In this paper, the lowercase $u(x,t)$ represents the original function while the uppercase $U_k(x)$ stands for the transformed function.

Definition 2 *The differential inverse transform of $U_k(x)$ is defined as follows:*

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k. \tag{4}$$

Then combining equation (3) and (4) we can write

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}[\frac{\partial^k}{\partial t^k}u(x,t)]_{t=0}\, t^k. \tag{5}$$

From the above definitions, it is easy to verify that the concept of the RDTM is obtained from the power series expansion. For illustration of the proposed method, we write the Fornberg-Whitham equation (1) in the standard operator form

$$L(u(x,t)) + R(u(x,t)) + N(u(x,t)) = g(x,t), \tag{6}$$

with initial condition

$$u(x,0) = f(x), \tag{7}$$

where $L = \frac{\partial}{\partial t}$, $R = \frac{\partial^3}{\partial x^2 \partial t^1}$, is a linear operator which has partial derivatives, $N(u(x,t)) = uu_x$ is a nonlinear term and $g(x,t)$ is an inhomogeneous term. According to the RDTM and Table 1, we can construct the following iteration formula:

$$(k+1)U_{k+1}(x) = G_k(x) - N(U_k(x)) - R(U_k(x)), \tag{8}$$

where $R(U_k(x))$, $N(U_k(x))$ and $G_k(x)$ are the transformations of the functions $R(u(x,t))$, $N(u(x,t))$ and $g(x,t)$ respectively.

From initial condition (7), we have

$$U_0(x) = f(x). \tag{9}$$

Substituting (9) into (8) and by a straight forward iterative calculations, we get the following $U_k(x)$ values. Then the inverse transformation of the set of values $\{U_k(x)\}_{k=0}^{n}$ gives approximation solution in the following form

$$\tilde{u_n}(x,t) = \sum_{k=0}^{n} U_k(x)t^k, \tag{10}$$

where n is order of approximation solution. Therefore, the exact solution of problem is given by

$$u(x,t) = \lim_{n\to\infty} \tilde{u_n}(x,t). \tag{11}$$

3 Numerical results

In order to assess the advantages and the accuracy of RDTM for solving Fornberg-Whitham equations, we consider the following examples:

Example 1 *Consider the Fornberg-Whitham equation (1) with the following initial condition:*

$$u(x,0) = e^{\frac{1}{2}x}. \tag{12}$$

As a starting point for the solution procedure, we first take the RDTM of (1) by using Table 1, and obtain the following equation

$$(k+1)U_{k+1}(x) - (k+1)\frac{\partial^2}{\partial x^2}U_{k+1}(x) = -\frac{\partial}{\partial x}U_k(x) - \sum_{r=0}^{k} U_{k-r}(x)\frac{\partial}{\partial x}U_r(x) +$$

$$\sum_{r=0}^{k} U_{k-r}(x)\frac{\partial^3}{\partial x^3}U_r(x) + 3\sum_{r=0}^{k} U_{k-r}(x)\frac{\partial^2}{\partial x^2}U_r(x). \tag{13}$$

From the initial condition (12) we write

$$U_0(x) = e^{\frac{1}{2}x}. \tag{14}$$

It is clear that equation (13) is a linear partial differential equation of order two. Solving (13) with the initial condition (14), we successively achieve values $U_k(x)$ as follows

$$U_1(x) = -\frac{2}{3}e^{\frac{1}{2}x},$$
$$U_2(x) = \frac{2}{9}e^{\frac{1}{2}x},$$
$$U_3(x) = -\frac{4}{81}e^{\frac{1}{2}x},$$
$$\ldots$$
$$\ldots$$

Finally the differential inverse transform of $U_k(x)$ gives

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k. \tag{15}$$

We, therefore, obtain

$$u(x,t) = e^{\frac{1}{2}x - \frac{2}{3}t}, \tag{16}$$

which is the exact solution [2] of this problem.

Example 2 *The initial value problem considered is that of the modified Fornberg-Whitham equation (2) with the following initial condition:*

$$u(x,0) = \frac{3}{4}(\sqrt{15}-5)\text{sech}^2(cx), \tag{17}$$

with a constant $c = \frac{1}{20}\sqrt{10(5-\sqrt{15})}$.

Employing RDTM, we attain

$$(k+1)U_{k+1}(x) - (k+1)\frac{\partial^2}{\partial x^2}U_{k+1}(x) = -\frac{\partial}{\partial x}U_k(x) - \sum_{r=0}^{k}\sum_{s=0}^{r}U_{k-r}(x)U_{r-s}(x)\frac{\partial}{\partial x}U_s(x)$$

$$+\sum_{r=0}^{k}U_{k-r}(x)\frac{\partial^3}{\partial x^3}U_r(x) + 3\sum_{r=0}^{k}U_{k-r}(x)\frac{\partial^2}{\partial x^2}U_r(x). \tag{18}$$

From the initial condition (17) we write

$$U_0(x) = \frac{3}{4}(\sqrt{15}-5)\text{sech}^2(cx). \tag{19}$$

Solving (18) with the initial condition (19), we obtain

$$U_1(x) = -\frac{105}{8}x + \frac{27\sqrt{15}}{8}x + \frac{31}{8}x^3 - \sqrt{15}x^3...,$$
$$U_2(x) = \frac{465}{8} - 15\sqrt{15} - \frac{825}{16}x^2 + \frac{213\sqrt{15}}{16}x^2 + ...,$$
$$U_3(x) = 305x - \frac{315\sqrt{15}}{4}x - \frac{36805}{192}x^3 + \frac{9503}{64}\sqrt{\frac{5}{3}}x^3...,$$
...

...

Finally the differential inverse transform of $U_k(x)$ *gives*

$$u(x,t) = \sum_{k=0}^{\infty}U_k(x)t^k, \tag{20}$$

Therefore, we obtain

$$u(x,t) = \frac{3}{4}(\sqrt{15}-5)\text{sech}^2(c(x-(5-\sqrt{15})t)), \tag{21}$$

which is the exact solution [6] of this problem.

4 Conclusion

In this paper, we proposed an analytical approach based on the reduced differential transformation for solving the initial value problems associated with the Fornberg-Whitham type equations. The numerical results showed that the RDTM performed well for the problems considered. The presented method reduces the computational difficulties of the other methods and all the calculations can be made simple manipulations. On the other hand the results are quite reliable. The Reduced differential transform method needs less work in comparison with the traditional methods. Therefore, this method can be applied to many complicated linear and non-linear problems and does not require linearization, discretization or perturbation.

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References

[1] G. B. Whitham. Variational methods and applications to water wave. *Proc. R. Soc. Lond. Ser. A*, 299(1967):6-25.

[2] B. Fornberg, G.B. Whitham. A numerical and theoretical study of certain nonlinear wave phenomena, Philos.*A Trans. R. Soc. Lond. Ser. A* , 289(1978):373-404.

[3] F. Abidi, K. Omrani, The homotopy analysis method for solving the Fornberg-Whitham equation and comparison with Adomians decomposition method, Comput. Math. Appl., 59(2010):2743–2750.

[4] J. Yin, L. Tian, X Fan. Classification of travelling waves in the Fornberg-Whitham equation. *J. Math. Anal. Appl*, 368(2010):133–143.

[5] J. Zhou, L. Tian. A type of bounded traveling wave solutions for the Fornberg-Whitham equation.*J. Math. Anal. Appl* ,346(2008):255-261.

[6] B. He, Q. Meng, S. Li. Explicit peakon and solitary wave solutions for the modified Fornberg-Whitham equation. *Appl. Math. Comput* , 217(2010):1976–1982.

[7] P. K. Gupta, M. Singh. Homotopy perturbation method for fractional Fornberg-Whitham equation.*Comput. Math. Appl* , 61 (2011):250-254.

[8] J. Lu. An analytical approach to the Fornberg-Whitham type equations by using the variational iteration method.*Comput. Math. Appl* , 61(2011):2010-2013.

[9] Y. Keskin, G. Oturanc. Reduced Differential Transform Method for Partial Differential Equations.*International Journal of Nonlinear Sciences and Numerical Simulation* , 10(6)2009:741–749.

[10] Y. Keskin, G. Oturanc. Numerical simulations of systems of PDEs by reduced differential transform method.*Communications in Nonlinear Science and Numerical Simulations* (In Press).

[11] Y. Keskin, G. Oturanc. Reduced Differential Transform Method for fractional partial differential equations. *Nonlinear Science Letters A* , 1(2)2010:61–72.

[12] Y. Keskin, Ph.D Thesis, Selcuk University, 2010 (in Turkish).

[13] G. E. Pukhov. Computational structure for solving differential equations by Taylor transformations.*Cybernetics and Systems Analysis*, 14(3)(1978): 383–390.