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The Control of the Stability of the Periodic Solution of Modified Chen System

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Abstract: Being based on the normal form theory, a normal form of the modified Chen system is obtained by conjugate operator method. With the design of a controller by Hopf bifurcation theorem, the relatively simple expressions of the stability coefficients d and l_1 are gained. Then, by appropriate changes of the parameter in the expressions, the stability of emerging period solutions is successfully controlled. Finally, numerical simulations are provided to verify the feasibility and effectiveness, so the result is mutually verified with the theoretical analyses and numerical simulations.

Keywords: hopf bifurcation; periodic solutions; control of the stability

1 Introduction

The bifurcation research in nonlinear system is one of the topic issues in the field of nonlinear science. In recent years, it has attracted increasing attention. At the same time, it has great application prospects in many fields, especially in controlling science, dynamics, information science, medical science, etc. With the study in a deep-going way in nonlinear system, bifurcation controlling further arouse people's wide concern. Actually, bifurcation refers to a phenomenon that when the system parameter reaches the critical value, the system behavior will vary suddenly. It may result in the break-down of the whole engineering system and cause inestimable loss. So, by adding appropriate controller in original system, investigators generally modify the dynamic action of the system near the bifurcation point. Different methodologies have been proposed to control bifurcations. Some authors have used feedback to stabilize a system with a bifurcation [1, 2]. Authors in ([3-8]) have used a kind of normal forms to analyze and control bifurcations, while the authors in [9] used center manifold theorem to analyze and control bifurcations.

For the control of the Hopf bifurcation, authors in ([6-9]) find a center manifold or a normal form through a change of coordinates and a feedback control input to control the Hopf bifurcation and the stability of emerging periodic solutions. But, meanwhile, the expressions for the stability coefficients of the Hopf bifurcation found in ([9, 10]) remain complicated.

In this letter, applying normal form theory to simplify appropriately and calculate, we obtain the relatively simple expressions of the stability coefficients which help us achieve the analysis and control of the stability of the emerging periodic solutions in the modified Chen system.

2 Preliminary results

2.1 Hopf Bifurcation

Theorem 1 *(Hopf Bifurcation Theorem) Suppose that the system* $\dot{x} = f(x,\mu)$, $\quad x \in R^n, \mu \in R$*, has an equilibrium point* (x_0, μ_0) *such that*

(A1)$D_x f(x_0,\mu_0)$ *has a simple pair of pure imaginary eigenvalues and no other eigenvalues with zero real parts.*

(A2) Let $\lambda(\mu)$, $\overline{\lambda}(\mu)$ *be the eigenvalues of* $D_x f(x_0,\mu_0)$ *which are imaginary at* $\mu=\mu_0$*, such that*

$$\frac{d}{d\mu}(Re(\lambda(\mu)))\,|_{\mu=\mu_0} = d \neq 0. \tag{1}$$

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Then there is a unique two-dimensional center manifold passing through $(x_0, \mu_0) \in R^n \times R$ and a smooth system of coordinates for which the Taylor expansion of degree two on the center manifold, in polar coordinates, is given by

$$\dot{r} = (d\mu + l_1 r^2)r, \quad \dot{\theta} = \omega + c\mu + br^2.$$

If $l_1 \neq 0$, then there is a surface of periodic solutions in the center manifold which has quadratic tangency with the eigenspace of $\lambda(\mu_0)$, $\bar{\lambda}(\mu_0)$ agreeing to second order with the paraboloid $\mu = -l_1 r^2/d$. If $l_1 < 0$, then these periodic solutions are stable, while if $l_1 > 0$, they are repelling limit cycles.

d is called the cross speed and l_1 the first Lyapunov coefficient, and both parameters are called the stability coefficients of the Hopf bifurcation. There exists an expression for bidimensional systems to find the first Lyapunov coefficient l_1.

Consider the system

$$\dot{x} = Jx + F(x),$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad F(x) = \begin{pmatrix} F_1(x) \\ F_2(x) \end{pmatrix}, \quad F(0) = 0$, and $DF(0) = 0$

Then

$$l_1 = \frac{1}{16}(R_1 + R_2) \tag{2}$$

where

$$R_1 = F_{1x_1x_2}(F_{1x_1x_1} + F_{1x_2x_2}) - F_{2x_1x_2}(F_{2x_1x_1} + F_{2x_2x_2}) - F_{1x_1x_1}F_{2x_1x_1} + F_{1x_2x_2}F_{2x_2x_2},$$

$$R_2 = F_{1x_1x_1x_1} + F_{1x_1x_2x_2} + F_{2x_1x_1x_2} + F_{2x_2x_2x_2}.$$

There exists another way to express R_2. If $F(x) = \frac{1}{2}L(x,x) + \frac{1}{6}\phi(x,x,x) + ...$, where $L = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$, $\phi = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$ with $Q_i, C_{ij} \in R^{2\times 2}$, then

$$R_2 = tr(\phi) = tr(C_{11} + C_{22}), \tag{3}$$

2.2 Conjugate operator

Theorem 2 *: Suppose that L^* is the complex conjugate transposed matrix of L, where $L = D_x f(0)$ is a linear matrix, such that $H_n^k = B_n^k \oplus N(ad_L)$, where, H_n^k is the linear space which is made up of all the homogeneous polynomials with the same order k, $B_n^k = ad_L(H_n^k)$, $N(ad_L)$ is the nullspace of the linear operator ad_L.*

3 Statement of the controlled system

Consider a nonlinear system

$$\begin{cases} \dot{x} = -y + xz \\ \dot{y} = x - xz + cyz \\ \dot{z} = xy + z \end{cases}, \tag{4}$$

That, via a composed control input $G(\eta)u$, can be taken to the form

$$\begin{cases} \dot{x} = -y + xz + ux \\ \dot{y} = x - xz + cyz + uy \\ \dot{z} = xy + z \end{cases}, \tag{5}$$

where $\eta = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in R^3$ is the state, $u \in R$ is the new control input. $J = \begin{bmatrix} J_H & 0 \\ 0 & J_S \end{bmatrix}$ is the Jacabian of the linear part of the system, where $J_H = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $J_S = 1 \in R^{1\times 1}$ is Hurwitz. Suppose that $F(\eta) = \begin{vmatrix} xz \\ -xz + cyz \\ xy \end{vmatrix}$, then $F(0) = 0$, $DF(0) = 0$, F and G are smooth.

Then, we rewrite (5) as

$$\dot{\eta} = J\eta + F(\eta) + G(\eta)u \tag{6}$$

4 The calculation of normal form of the system

In this section, we mainly use conjugate operator method to find the normal form.

From theorem 4, we know that, as long as we find $N(ad_L)$, the normal form can be calculated.

Now, we may assume that $G_n^k = N(ad_L)$. Then, the calculation can be achieved by finding $N(\xi)$ in the following equation:

$$[DN(\xi)]\, L^*\xi - L^* N(\xi) = 0, N \in H_n^k. \tag{7}$$

In the system (5),

$$L = J = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad L^* = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Assumed that $\xi = (x_1, x_2, x_3)^T \in R^3$,$N(\xi) = (N_1(\xi), N_2(\xi), N_3(\xi)) \in H_3^3$,

Then, the equation (7) becomes

$$\begin{bmatrix} \frac{\partial N_1}{\partial x_1} & \frac{\partial N_1}{\partial x_2} & \frac{\partial N_1}{\partial x_3} \\ \frac{\partial N_2}{\partial x_1} & \frac{\partial N_2}{\partial x_2} & \frac{\partial N_2}{\partial x_3} \\ \frac{\partial N_3}{\partial x_1} & \frac{\partial N_3}{\partial x_2} & \frac{\partial N_3}{\partial x_3} \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} N_1(x) \\ N_2(x) \\ N_3(x) \end{bmatrix} = 0.$$

We can obtain that

$$\begin{cases} x_2 \frac{\partial N_1}{\partial x_1} - x_1 \frac{\partial N_1}{\partial x_2} + x_3 \frac{\partial N_1}{\partial x_3} = N_2(x) \\ x_2 \frac{\partial N_2}{\partial x_1} - x_1 \frac{\partial N_2}{\partial x_2} + x_3 \frac{\partial N_2}{\partial x_3} = -N_1(x) \\ x_2 \frac{\partial N_3}{\partial x_1} - x_1 \frac{\partial N_3}{\partial x_2} + x_3 \frac{\partial N_3}{\partial x_3} = -N_3(x) \end{cases} \tag{8}$$

For $N \in H_3^3$, the solution of (8) is

$$N(X) = \begin{bmatrix} m\left(x_1^2 + x_2^2\right)(x_1 - x_2) \\ m\left(x_1^2 + x_2^2\right)(x_1 + x_2) \\ 0 \end{bmatrix}, \quad m \in R.$$

So, when $m = 1$, the normal form of system (4) is

$$\begin{cases} \dot{x}_1 = -x_2 + \left(x_1^2 + x_2^2\right)(x_1 - x_2) \\ \dot{x}_2 = x_1 + \left(x_1^2 + x_2^2\right)(x_1 + x_2) \\ \dot{x}_3 = x_3 \end{cases}. \tag{9}$$

Then, via the composed control input u, can be taken to the form

$$\begin{cases} \dot{x}_1 = -x_2 + \left(x_1^2 + x_2^2\right)(x_1 - x_2) + ux_1 \\ \dot{x}_2 = x_1 + \left(x_1^2 + x_2^2\right)(x_1 + x_2) + ux_2 \\ \dot{x}_3 = x_3 \end{cases} \tag{10}$$

5 Control of the stability of emerging periodic solutions

We know that there exists a change of coordinates $\eta = \xi + h(\xi)$ such that the original system is transformed into

$$\dot{\xi} = J\xi + f(\xi) + g(\xi)\, u, \tag{11}$$

where $f(\xi) = \begin{pmatrix} X^T X L_0 X + ... \\ X^T S_2 Y + \frac{1}{2} Y^T S_3 Y + ... \end{pmatrix}$, with $L_0 = \begin{pmatrix} \alpha_0 & -\beta_0 \\ \beta_0 & \alpha_0 \end{pmatrix}$ and $g(0) = G(0)$. see [2].

Consider (11), where $g(\xi) = b + M\xi + ...$, with $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, $b_1 \in R^2$, $b_2 \in R^1$, and $M = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix}$, with $M_1 \in R^{2\times 2}$, observe (9) and (11), it is easy to know that $b_1 = b_2 = 0$, $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Our goal in this section is to calculate the cross speed d and the first Lyapunov coefficient l_1. Furthermore we can verify the stability of the emerging periodic solutions and period trajectories.

For system (10), $b_1 = 0$ and $tr(M_1) \neq 0$, suppose $X_{12} = (x_1, x_2)^T$, $X_3 = x_3$, then, we can use the control law $u = \mu\alpha + \beta X_{12}^T X_{12}$,

Then, we can obtain the system

$$\dot{X}_{12} = J_H X_{12} + F_3 (X_{12}, \mu, X_3),$$
$$\dot{X}_3 = J_S X_3,$$

where, $F_3 (X_{12}, \mu, X_3) = X_{12}^T X_{12} L_0 X_{12} + \mu\alpha M_1 X_{12} + \beta M_1 X_{12} X_{12}^T X_{12}$,$L_0 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

In extended form we obtain

$$\begin{pmatrix} \dot{X}_{12} \\ \dot{\mu} \\ \dot{X}_3 \end{pmatrix} = \begin{pmatrix} J_H & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & J_S \end{pmatrix} \begin{pmatrix} X_{12} \\ \mu \\ X_3 \end{pmatrix} + \begin{pmatrix} F_3 (X_{12}, \mu, X_3) \\ 0 \\ 0 \end{pmatrix}.$$

From the center manifold theory, the extended system has a three-dimensional center manifold through the origin, characterized by $X_3 = h(X_{12}, \mu)$, with $h(0,0) = 0$, $Dh(0,0) = 0$, and the dynamics of the system on this manifold is given by

$$\dot{X}_{12} = J_H X_{12} + F_3 (X_{12}, \mu, h(X_{12}, \mu))$$

$$= J_H X_{12} + X_{12}^T X_{12} L_0 X_{12} + \mu\alpha M_1 X_{12} + \beta M_1 X_{12} X_{12}^T X_{12}. \quad (12)$$

We can find that it is not necessary to calculate the center manifold h because the relevant terms do not depend on it.

From (12), the dynamics of the system on this manifold is given by

$$\dot{X}_{12} = J_\mu X_{12} + C(X_{12}),$$

where $J_\mu = J_H + \alpha\mu M_1$, $C(X_{12}) = X_{12}^T X_{12} (L_0 + \beta M_1) X_{12}$

By the way, in this letter, the characteristic equation of J_μ is

$$(\lambda - \alpha\mu)^2 + 1 = 0.$$

Then, $\lambda = \alpha\mu \pm i$.

From (1), the cross speed is given by $d = \alpha$.

Then, using (2), we can know that the first Lyapunov coefficient $l_1 = 1 + \beta$.

Stability of the emerging periodic solutions and criticality of the Hopf bifurcation are as follows:

cross speed d	The first Lyapunov coefficient l_1	Stability of the emerging periodic solutions	Criticality of the Hopf bifurcation
+	-	stable	supercritical
-	-	stable	supercritical
+	+	unstable	subcritical
-	+	unstable	subcritical

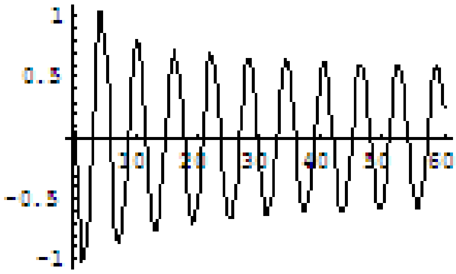
6 Numerical simulation

In uncontrolled system (4), $(0,0,0)$ is a Hopf bifurcation point. (Fig.1.a , Fig.1.b , Fig.1.c , Fig.1.d)

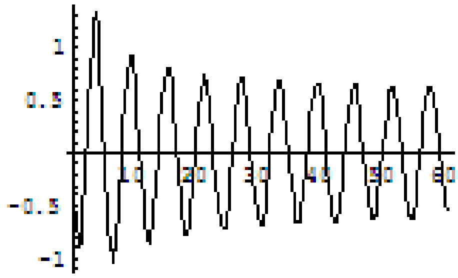
After adding the controller, in the controlled system (5), $(0,0,0)$ remains a Hopf bifurcation point. (Fig.2.a , Fig.2.b , Fig.2.c)

If $\alpha = 1$ and $\beta = -2$, then we can obtain $d = 1$ and $l_1 = -1$, that is, the system undergoes a supercritical Hopf bifurcation. (Fig.3)

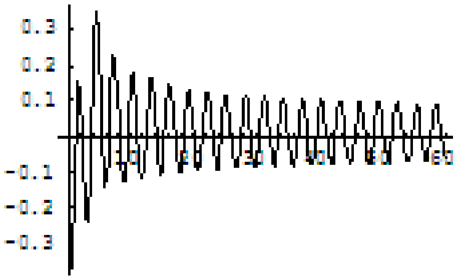
If $\alpha = 1$ and $\beta = 2$, then we can obtain $d = 1$ and $l_1 = 3$, that is, the system undergoes a subcritical Hopf bifurcation. (Fig.4)



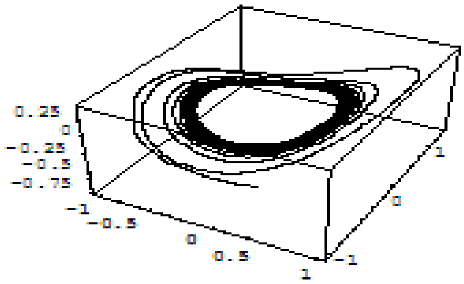
(a) The time series of $x(t)$ before controlling.



(b) The time series of $y(t)$ before controlling.

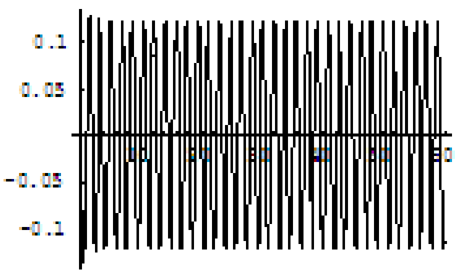


(c) The time series of $z(t)$ before controlling.

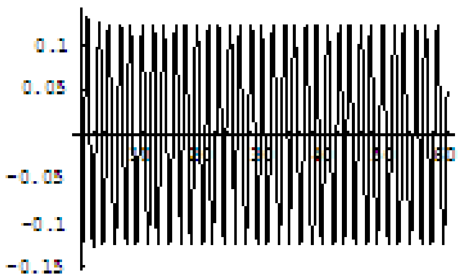


(d) The original system, initial point $(0, 0, 0)$.

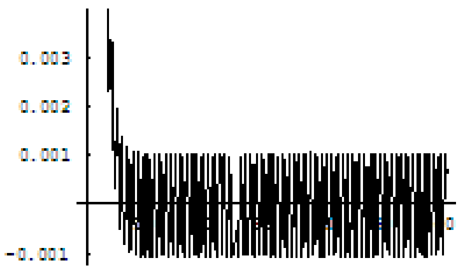
Figure 1: The uncontrolled system (4)



(a) The time series of $x(t)$ after controlling.



(b) The time series of $y(t)$ after controlling.



(c) The time series of $z(t)$ after controlling.

Figure 2: The controlled system (5)

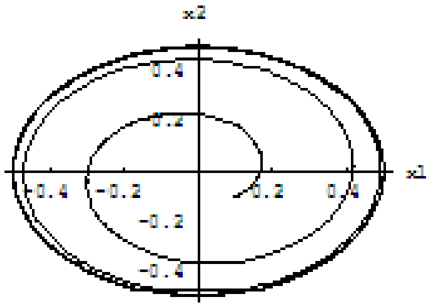


Figure 3: The supercritical Hopf bifurcation for $\mu = 0.25$.

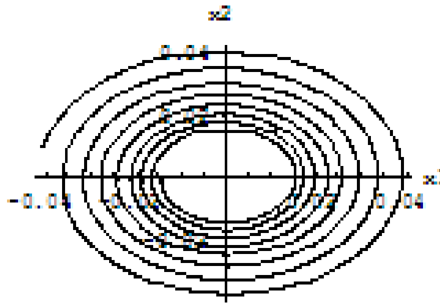


Figure 4: The supercritical Hopf bifurcation for $\mu = 0.02$.

7 Conclusions

In this paper, a normal form was introduced to simplify the expressions of the stability coefficients in the modified Chen system. At first, we found a normal form of the system by conjugate operator method. Then, after adding a control law by Hopf bifurcation theorem, in the controlled system, we obtain the simple expressions of the cross speed d and the first Lyapunov coefficient l_1, such that we can analyze and control the stability of the emerging periodic solutions. At last, the simulation results showed the validity and feasibility of the results.

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