ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.13(2012) No.3,pp.259-266

A Differential Quadrature Algorithm for the Numerical Solution of the Second-Order One Dimensional Hyperbolic Telegraph Equation

Ram Jiwari[1] *, Sapna Pandit[2], R C Mittal[3]
[1]School of Mathematics and Computer Applications, Thapar University, Patiala
[2] Department of Mathematics, MNNIT Allahabad
[3]Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee
(Received 20 June 2011, accepted 16 January 2012)

Abstract: In this article, we proposed a numerical technique based on polynomial differential quadrature method (PDQM) to find the numerical solutions of one dimensional hyperbolic telegraph equation. The hyperbolic partial differential equations model the vibrations of structures (e.g., buildings, beams, and machines) and they are the basis for fundamental equations of atomic physics. The PDQM reduced the problem into a system of second order linear differential equation. Then, the obtained system is changed into coupled differential equations and lastly, RK4 method is used to solve the coupled system. The accuracy of the proposed method is demonstrated by three test examples. The numerical results are found to be in good agreement with the exact solutions. The whole computation work is done with help of software DEV C++ and MATLAB.

Keywords: telegraph equation; differential quadrature method; uniform grid points; Gauss-Lobatto-Chebyshev grid points; Runge-Kutta method

1 Introduction

In this article, we are dealing with the numerical approximation of the second-order one-dimensional linear hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2}(x,t) + 2\alpha \frac{\partial u}{\partial t}(x,t) + \beta^2 u(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) + g(x,t), \qquad (x,t) \in [a,b] \times [0,T], \tag{1.1a}$$

with initial conditions

$$u(x,0) = f_1(x), \tag{1.1b}$$

$$\frac{\partial u}{\partial t}(x,0) = f_2(x), \tag{1.1c}$$

and boundary conditions

$$u(a,t) = \phi_1(t), \qquad t \geq 0 \tag{1.1d}$$

$$u(b,t) = \phi_2(t), \qquad t \geq 0 \tag{1.1e}$$

where $\alpha > \beta \geq 0$ are known constants coefficients, $g,\ f_1, f_2, \phi_1, \phi_2$ are known functions and the function u is unknown.

Recently, it is found that telegraph equation is more suitable than ordinary diffusion equation in modeling reaction diffusion for such branches of sciences. The hyperbolic partial differential equations model the vibrations of structures (e.g., buildings, beams, and machines) and they are the basis for fundamental equations of atomic physics. The telegraph equation is important for modeling several relevant problems such as signal analysis [1], wave propagation [2], random walk theory [3], etc. The Eq. (1.1) also referred as models mixture between diffusion and wave propagation by introducing a term that accounts for effects of finite velocity to standard heat or mass transport equation [4]. However, Eq. (1.1) is commonly used in signal analysis for transmission and propagation of electrical signals [5] and also has applications in other fields [6]. So, the Eq. (1.1) has great importance in science and engineering.

In recent years, much attention has been given in the literature to the development of numerical schemes of the second order hyperbolic equations, see, for example [7-9]. The methods proposed in [7-9] are conditionally stable. After that,

* **Corresponding author**. *E-mail address*: ram1maths@gmail.com

IJNS.2012.06.15/603

Mohanty [10-11] proposed unconditionally stable schemes for the linear one dimensional hyperbolic equation (1.1). In [10], proposed a new technique to solve the linear one dimensional hyperbolic equation (1.1) which is unconditionally stable and is of second order accurate in both time and space components and in [11] the author developed a three level implicit unconditionally stable difference scheme for the equation (1.1) with variable coefficients.

Recently, the one-dimensional linear hyperbolic equation (1.1) has been considered by Lakestani and Saray [12] and Dehghan and his associates [13-16]. In [12], the authors used interpolating scaling functions to find the numerical solutions of Eq. (1.1) while in [13] the authors proposed a numerical scheme to solve the one-dimensional hyperbolic telegraph equation using collocation points and approximating the solution using thin plate splines radial basis function. Mohebbi and Dehghan [14] combined a high-order compact finite difference scheme to approximate the spatial derivative and the collocation technique for time component to find the numerical solutions of one-dimensional linear hyperbolic equation. In [15], the authors used Chebyshev cardinal functions while in [16] proposed Chebyshev Tau method for numerical solutions of hyperbolic telegraph equation (1.1).

Differential quadrature method (DQM) has been successfully applied to solve various linear and nonlinear one and two dimensional partial differential equations. Korkmaz and Dag [18] solved Schrödinger equation by DQM. In [19-20], Mittal and Jiwari solved two dimensional Burgers' and Brusselator equations by DQM.

In the article, we proposed a numerical scheme based on polynomial differential quadrature method to find the numerical solutions of the telegraph equation (1.1) with initial conditions and Dirichlet boundary conditions. The PDQM reduced the problem into a system of second order linear differential equation. Then, the obtained system is changed into coupled differential equations and lastly, RK4 method is used to solve the coupled system. The accuracy and efficiency of the proposed method is demonstrated by three test examples. Finally, a comparison a made between the numerical solutions obtained by Chebyshev-Gauss-Lobatto grid points and the uniform grid points and finds that the Chebyshev-Gauss-Lobatto grid points give better accuracy and stable numerical solutions as comparison to the uniform grid points which agree with the theory of Shu given in his book [17].

2 Polynomial differential quadrature method

Differential quadrature method is a numerical technique for solving differential equations. By this method, we approximate the spatial derivatives of unknown function at any grid points using weighted sum of all the functional values at certain points in the whole computational domain. In this paper, we used one of the forms of differential quadrature method namely polynomial differential quadrature method (PDQM) to approximate the solution of the problem. Since the weighting coefficients are dependent only the spatial grid spacing, we assume N grid points $x_1 < x_2 < ... < x_N$ on the real axis. The grid points may be uniform and non-uniform. The differential quadrature discretization of the first and the second derivatives at a point x_i is given by the following equations

$$u_x\left(x_i,t\right)=\sum_{j=1}^{N} a_{ij}\,u\left(x_j,t\right), \qquad u_{xx}\left(x_i,t\right)=\sum_{j=1}^{N} b_{ij}\,u\left(x_j,t\right), \tag{2.1}$$

where a_{ij} and b_{ij} represent the weighting coefficients [17],$i = 1, 2, ..., N$. The following base functions are used to obtain weighting coefficients

$$g_k\left(x\right)=\frac{M\left(x\right)}{\left(x-x_k\right)\,M^{(1)}\left(x_k\right)},\;\; k=1,2,...,N \tag{2.2}$$

where

$$M\left(x\right)=\left(x-x_1\right)\left(x-x_2\right)...\left(x-x_N\right). \tag{2.3}$$

$$M^{(1)}(x_i)=\prod_{k=1,k\neq i}^{N}(x_i-x_k) \tag{2.4}$$

using the set of base functions given in equation (2.2), the weighting coefficients of the first order derivative are found as [17]

$$a_{ij}=\frac{M^{(1)}\left(x_i\right)}{\left(x_i-x_j\right)\,M^{(1)}\left(x_j\right)},\;\; k=1,2,...,N\,,\;\; i\neq j \tag{2.5}$$

$$a_{ii}=-\sum_{j=1,j\neq i}^{N} a_{ij}\,,\quad i=1,2,...,N \tag{2.6}$$

and for weighting coefficients of the second order derivative, the formula is [17]

$$b_{ij} = 2a_{ij}\left(a_{ii} - \frac{1}{x_i - x_j}\right), \quad i, j = 1, 2, ..., N, \ \ i \neq j \tag{2.7}$$

$$b_{ii} = -\sum_{j=1, j\neq i}^{N} b_{ij}\,, \quad i = 1, 2, ..., N \tag{2.8}$$

Similarly, Shu [17] proposed the weighting coefficients of the higher order derivative in the explicit form

$$w_{ij}^{(r)} = r\left[a_{ij}w_{ii}^{(r-1)} - \frac{w_{ii}^{(r-1)}}{x_i - x_j}\right] \qquad \text{for } \ i \neq j, \quad i\,, j = 1, 2, ..., N; \quad r = 2, 3, ..., N-1 \tag{2.9}$$

$$w_{ii}^{(r)} = -\sum_{j=1, j\neq i}^{N} w_{ij}^{(r)} \qquad\qquad \text{for } \ i = j \tag{2.10}$$

where a_{ij} and $w_{ij}^{(r)}$ are the weighting coefficients of the first order derivative and rth order derivative respectively.

3 Numerical scheme for telegraph equation

The space derivatives in the one-dimensional linear hyperbolic equation (1.1) are approximated by the polynomial differential quadrature method. The Eq. (1.1) changed into the following form

$$\frac{d^2u}{dt^2}(x_i, t) + 2\alpha\frac{du}{dt}(x_i, t) + \beta^2 u(x_i, t) = \sum_{j=1}^{N} b_{ij}u(x_j, t) + g(x_i, t)\,, \qquad (x_i, t) \in [a, b] \times [0, T]\,, i = 1, 2, ..., N \tag{3.1a}$$

with initial conditions
$$u(x_i, 0) = f_1(x_i)\,, \tag{3.1b}$$
$$\frac{du}{dt}(x_i, 0) = f_2(x_i)\,, \tag{3.1c}$$
and boundary conditions
$$u(a, t) = \phi_1(t)\,, \qquad t \geq 0 \tag{3.1d}$$
$$u(b, t) = \phi_2(t)\,, \qquad t \geq 0 \tag{3.1e}$$

The system (3.1) is system of second order differential equations.
Now let

$$\frac{du}{dt}(x_i, t) = z(x_i, t)\,, \quad \text{then} \quad \frac{d^2u}{dt^2}(x_i, t) = \frac{dz}{dt}(x_i, t)\,, \tag{3.2}$$

Using the assumptions of Eq. (3.2) into the Eq. (3.1), the Eq. (3.1) changed as follow

$$\frac{du}{dt}(x_i, t) = z(x_i, t)\,, \tag{3.3a}$$

$$\frac{dz}{dt}(x_i, t) + 2\alpha\, z(x_i, t) + \beta^2 u(x_i, t) = \sum_{j=1}^{N} b_{ij}u(x_j, t) + g(x_i, t)\,, \qquad (x_i, t) \in [0, 1] \times [0, 1] \ i = 1, 2, ..., N, \tag{3.3b}$$

with initial conditions
$$u(x_i, 0) = f_1(x_i)\,, \tag{3.3c}$$
$$z(x_i, 0) = f_2(x_i)\,, \tag{3.3d}$$

and the boundary conditions (3.1d) and (3.1e).

The system (3.3) is coupled system of first order differential equations with initial and boundary conditions. The boundary conditions are handled by the PDQM and the coupled system of first order differential equations with initial conditions is solved by slandered RK4 method.

4 Selection of grid points and stability

In the numerical experiments, we have used the two types of grid points to find the numerical solutions of the Eq. (1.1)
(i) Uniform grid points

$$x_i = x_1 + i\,h, \quad i = 2, 3, ..., N \tag{4.1}$$

where $x_1 = a$ and $h = \frac{b-a}{N}$.
(ii) Chebyshev-Gauss-Lobatto grid points

$$x_i = a + \frac{1}{2}\left(1 - \cos\frac{((i-1)\,\pi)}{N-1}\right)(b-a), \qquad\qquad i = 1, 2, ..., N \tag{4.2}$$

The stability of the DQM depends on the eigen-values of differential quadrature discretization matrices. These eigen-values in turn very much depend on the distribution of grid points. It has been shown by Shu [17] in his book that the uniform grid points distribution does not give stable solution which we have also notice in our numerical experiments. The Tables 1-3 show that Chebyshev-Gauss-Lobatto grid points give better accuracy as comparison to uniform grid points. The uniform grid points do not work for time greater than $t = 1.5$ while nonuniform grid points work.

5 Numerical experiments

In this section, to illustrate the efficiency and accuracy of the numerical method, a set of numerical experiments have been carried out for solving the telegraph equation. The examples are chosen such that their exact solutions are known. The numerical computations have been done with the help of software DEV C++ and Matlab. In each example, we have calculated the absolute maximum error by the following formula

$$\text{Absolute Error} = \left\|u^{exact} - u^{num}\right\|_\infty = \max_j \left|u_j^{exact} - u_j^{num}\right|. \tag{5.1}$$

where u^{exact} and u^{num} denote the exact and numerical solution of the problem respectively.

Example 1 *In this example, we consider the telegraph equation (1.1) with domain* $[0,\ 1]$ *and the following initial and boundary conditions*

$$u\,(x,0) = \sinh(x),$$
$$\frac{\partial\,u}{\partial\,t}\,(x,0) = -2\,\sinh(x),$$
$$u\,(0,t) = 0,$$
$$u\,(1,t) = e^{-2t}\,\sinh(1), \quad t \geq 0,$$

and the function

$$g\,(x,t) = \left(3 - 4\,\alpha\, + \beta^2\right)\,e^{-2t}\,\sinh(x)$$

The exact solution is given by [14]

$$u\,(x,t) = e^{-2t}\,\sinh(x).$$

The maximum absolute errors of the example are given in the Table 1 for $\alpha = 20,\ \beta = 10$ *at different time by taking time step length* $\Delta t = 0.01,\ 0.001$ *and* 0.0001 *for uniform and non uniform grid points (Chebyshev-Gauss-Lobatto). Fig 1 and Fig 2 show the absolute errors obtained by uniform and non uniform grid points at* $t = 1.0$. *The Fig 7 shows the behavior of numerical solutions for* $\alpha = 50,\ \beta = 2$ *at different time.*

Example 2 *We consider the telegraph equation (1.1) with domain* $[0,\ 1]$ *and the following conditions*

$$u\,(x,0) = \sin(x),$$
$$\frac{\partial\,u}{\partial\,t}\,(x,0) = 0,$$
$$u\,(0,t) = 0,$$
$$u\,(1,t) = \cos(\,t)\,\sin(1), \quad t \geq 0,$$

$$g\,(x,t) = \; -2\,\alpha\,\sin\,(t)\,\sin(x) + \beta^2\,\cos\,(t)\,\sin\,(x)$$

The exact solution is given by [14]

$$u\,(x,t) = \cos\,(t)\,\sin(x).$$

Table 2 shows the maximum absolute errors of the example for $\alpha = 10,\ \beta = 5$ *at different time by taking time step length* $\Delta t = 0.01,\ 0.001\ and\ 0.0001$ *for uniform and non uniform grid points (Chebyshev-Gauss-Lobatto). Fig 3 and Fig 4 show the behavior of absolute errors obtained by uniform and non uniform grid points at* $t = 1.0$*. The Fig 8 shows the behavior of numerical solutions for* $\alpha = 20,\ \beta = 10$ *at different time.*

Example 3 *In this example, we consider the telegraph equation (1.1) with domain* $[0,\ 1]$ *and the following conditions*

$$u\,(x,0) = \tan(x/2),$$

$$\frac{\partial\,u}{\partial\,t}\,(x,0) = \frac{1}{2}\,\left(1 + \tan^2\,(x/2)\right),$$

$$u\,(0,t) = \tan(\,t/2),$$

$$u\,(1,t) = \tan\left(\frac{1+t}{2}\right),\quad t \geq 0,$$

$$g\,(x,t) = \;\alpha\;\left(1 + \tan^2\left(\frac{x+t}{2}\right)\right) + \beta^2\tan\left(\frac{x+t}{2}\right)$$

The exact solution is given by [14]

$$u\,(x,t) = \tan\left(\frac{x+t}{2}\right)$$

In Table 3, we present the maximum absolute errors of the example for $\alpha = 10,\ \beta = 5$ *at different time by taking time step length* $\Delta t = 0.01,\ 0.001\ and\ 0.0001$ *for uniform and non uniform grid points (Chebyshev-Gauss-Lobatto). The Figures 5-6 show the behavior of absolute errors obtained by uniform and non uniform grid points at time* $t = 1.0$*. The Fig 9 shows the behavior of numerical solutions* $\alpha = 20,\ \beta = 10$ *at different time.*

6 Conclusion

In this article, we presented a numerical technique for solving the second order one dimensional linear hyperbolic telegraph equation based on polynomial differential quadrature method. The technique is easy and very suitable for computer programming and provides numerical solutions closed to the exact solutions. Finally, a comparison a made via Tables 1-3 and Figures 1-6 between the absolute errors obtained by Chebyshev-Gauss-Lobatto grid points and the uniform grid points and finds that the Chebyshev-Gauss-Lobatto grid points give better accuracy and stable numerical solutions as comparison to the uniform grid points which agree with the theory of Shu given in his book [17].

References

[1] P.M. Jordan and A. Puri. Digital signal propagation in dispersive media. *J. Appl. Phys.*, 85(3) (1999): 1273-1282.

[2] V. H. Weston and S. He. Wave splitting of the telegraph equation in R3 and its application to inverse scattering. *Inverse Probl.*, 9 (1993): 789-812.

[3] J Banasiak and J R Mika. Singularly perturved telegraph equations with applications in the random walk theory. *J. Appl. Math. Stoch. Anal.*, 11(1) (1998):9-28.

[4] M. S. El-Azab and M. El-Gamel. A numerical algorithm for the solution of telegraph equations. *Appl. Math. Comput.* 190 (2007): 757-764.

[5] A.C. Metaxas and R.J. Meredith. Industrial microwave, Heating, Peter Peregrinus, London, 1993.

[6] G .Roussy and J.A .Pearcy. Foundations and industrial applications of microwaves and radio frequency fields, Wiley, New York, 1995.

[7] M. Dehghan. On the solution of an initial-boundary value problem that combines Neumann and integral condition for the wave equation. *Numer Methods Partial Differential Equation*, 21 (2005): 24-40.

[8] R. K. Mohanty, M. K. Jain and K. George. On the use of high order difference methods for the system of one space second order non-linear hyperbolic equations with variable coefficients. *J Comput Appl, Math*, 72(1996): 421-431.

[9] E. H. Twizell. An explicit difference method for the wave equation with extended stability range. BIT, 19 (1979): 378-383.
[10] R. K. Mohanty. An unconditionally stable difference scheme for the one-space dimensional linear hyperbolic equation. *Appl Math Lett*, 17 (2004): 101-105.
[11] R. K. Mohanty. An unconditionally stable finite difference formula for a linear second order one space dimensional hyperbolic equation with variable coefficients. *Appl Math Comput*, 165 (2005): 229-236.
[12] M. Lakestani and B N Saray. Numerical solution of telegraph equation using interpolating scaling functions. *Computers & Mathematics with Application*, 60 (7) (2010): 1964-1972.
[13] M. Dehghan and A. Shokri. A numerical method for solving the hyperbolic telegraph equation. *Numer Methods Partial Differential Equation*, 24(2008): 1080-1093.
[14] A. Mohebbi and M. Dehghan. High order compact solution of the one-space-dimensional linear hyperbolic equation. *Numer Methods Partial Differential Equation*, 24 (2008): 1222-1235.
[15] M Dehghan and M Lakestani. The use of Chebyshev cardinal functions for solution of the second-order one-dimensional telegraph equation. *Numer Methods Partial Differential Equation*, 931-938, DOI 10.1002/num.20382.
[16] A Saadatmandi and M Dehghan. Numerical solution of hyperbolic telegraph equation using the Chebyshev Tau method. *Numer Methods Partial Differential Equation*, 239-252, DOI 10.1002/num.20442.
[17] Chang Shu, Differential Quadrature and its Application in Engineering, Springer-Verlag London Ltd., Great Britain, 2000.
[18] A. Korkmaz and I. Dag. A differential quadrature algorithm for simulations of nonlinear Schrödinger equation. *Comput Math Appl*, 56 (2008): 2222-2234.
[19] R.C. Mittal and Ram Jiwari. Differential Quadrature Method for Two Dimensional Burgers' Equations. *Int. J. for Comput. Methods in Eng. Science and Mech*, 10 (2009): 450-459.
[20] R.C. Mittal and Ram Jiwari. Numerical solution of two dimensional reaction-diffusion Brusselator system. *Appl Math Comput.*, 12 (217) (2011):5404-5415.

Table 1: A comparison of maximum absolute errors of Example 1 for $\alpha = 20,\ \beta = 10$ at different time and $N = 21$.

	$\Delta t = 0.01$		$\Delta t = 0.001$		$\Delta t = 0.0001$	
T	Uniform Grid	Nonuniform grid	Uniform Grid	Nonunifrom grid	Uniform Grid	Nonunifrom grid
0.5	2.25640×10^{-4}	2.22066×10^{-4}	2.24300×10^{-5}	2.21712×10^{-5}	2.23874×10^{-6}	2.21693×10^{-6}
1.0	5.75205×10^{-3}	1.41294×10^{-4}	3.56696×10^{-4}	1.41832×10^{-5}	1.72404×10^{-5}	1.22688×10^{-6}
1.5	2.43143×10^{-2}	6.93111×10^{-5}	1.94934×10^{-2}	6.91111×10^{-6}	4.14630×10^{-3}	6.73102×10^{-7}
2.0	....	2.27829×10^{-5}	....	2.16147×10^{-6}	....	2.66660×10^{-7}
CPU Time	1 second		2 second		5 second	

Table 2: A comparison of maximum absolute errors of Example 2 for $\alpha = 10,\ \beta = 5$ at different time and $N = 21$.

	$\Delta t = 0.01$		$\Delta t = 0.001$		$\Delta t = 0.0001$	
T	Uniform Grid	Nonuniform grid	Uniform Grid	Nonunifrom grid	Uniform Grid	Nonunifrom grid
0.2	1.66760×10^{-4}	1.24976×10^{-4}	1.67216×10^{-5}	1.28551×10^{-5}	1.67170×10^{-6}	1.28705×10^{-6}
0.5	4.30081×10^{-3}	3.13646×10^{-4}	4.71302×10^{-4}	3.20793×10^{-5}	4.73467×10^{-5}	3.21072×10^{-6}
1.0	8.47480×10^{-3}	5.59863×10^{-4}	8.62796×10^{-4}	5.71454×10^{-5}	7.08569×10^{-4}	5.71910×10^{-6}
1.5	....	6.69671×10^{-4}	....	6.82827×10^{-5}	....	6.83347×10^{-6}
CPU Time	1 second		2 second		5 second	

Table 3: A comparison of maximum absolute errors of Example 3 for $\alpha = 20,\ \beta = 10$ at different time and $N = 21$.

	$\Delta t = 0.01$		$\Delta t = 0.001$		$\Delta t = 0.0001$	
T	Uniform Grid	Nonuniform grid	Uniform Grid	Nonunifrom grid	Uniform Grid	Nonunifrom grid
0.2	8.33780×10^{-4}	7.33771×10^{-4}	8.42642×10^{-5}	7.13861×10^{-5}	8.25562×10^{-5}	7.10494×10^{-5}
0.5	9.33501×10^{-3}	9.23501×10^{-4}	2.16334×10^{-4}	9.33979×10^{-5}	1.81321×10^{-5}	2.20638×10^{-5}
1.0	1.71143×10^{-2}	1.67143×10^{-3}	1.71289×10^{-3}	1.68289×10^{-4}	9.87860×10^{-4}	1.71277×10^{-5}
1.5	....	5.03633×10^{-3}	....	5.02951×10^{-4}	....	5.02868×10^{-5}
CPU Time	1 second		2 second		5 second	

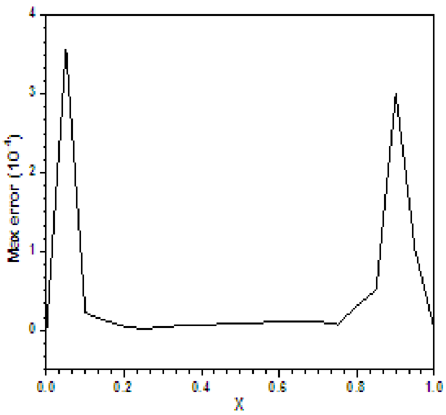

Figure 1: Absolute errors of Example 1 for $\alpha = 20,\ \beta = 10, N = 21$ and $\Delta t = 0.001$ at uniform grid points and $t = 1.0$.

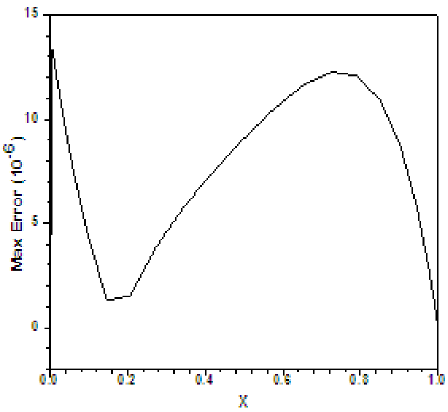

Figure 2: Absolute errors of Example 1 for $\alpha = 20,\ \beta = 10, N = 21$ and $\Delta t = 0.001$ at uniform grid points and $t = 1.0$.

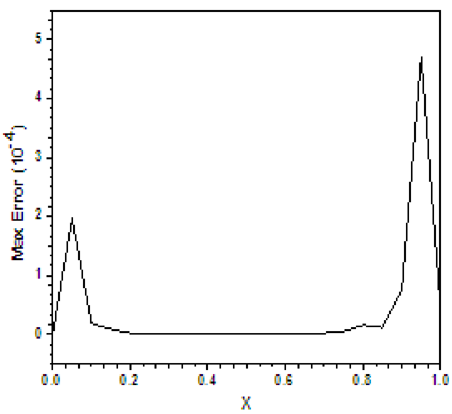

Figure 3: Absolute errors of Example 2 for $\alpha = 10,\ \beta = 5, N = 21$ and $\Delta t = 0.001$ at uniform grid points and $t = 1.0$.

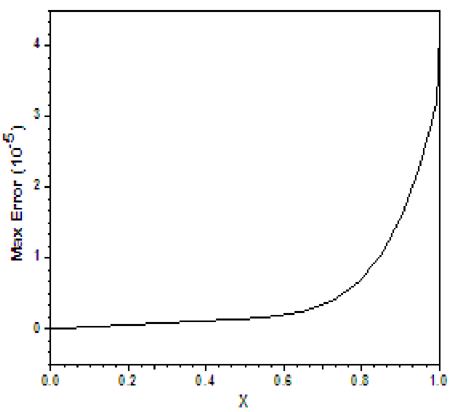

Figure 4: Absolute errors of Example 2 for $\alpha = 10,\ \beta = 5, N = 21$ and $\Delta t = 0.001$ at uniform grid points and $t = 1.0$.

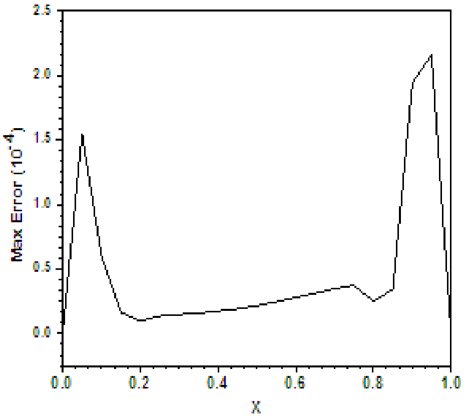

Figure 5: Absolute errors of Example 3 for $\alpha = 20,\ \beta = 10, N = 21$ and $\Delta t = 0.001$ at uniform grid points and $t = 1.0$.

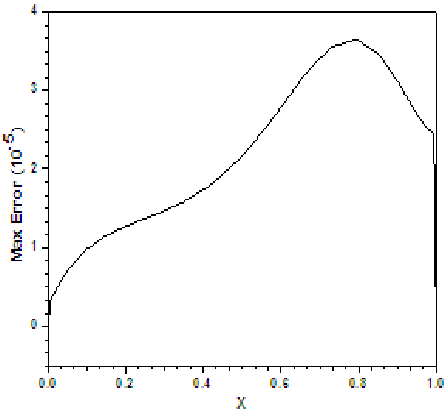

Figure 6: Absolute errors of Example 3 for $\alpha = 20,\ \beta = 10, N = 21$ and $\Delta t = 0.001$ at uniform grid points and $t = 1.0$.

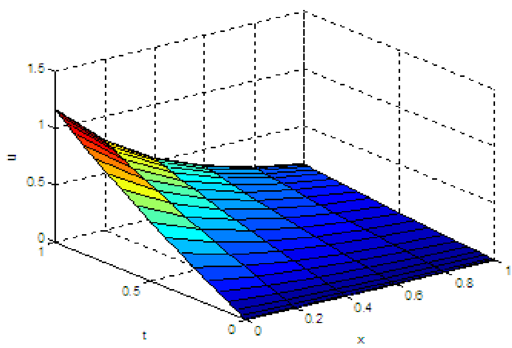

Figure 7: Numerical solutions of Example 1 for $\alpha = 50,\ \beta = 2.0$ and $\Delta t = 0.001$ at different times.

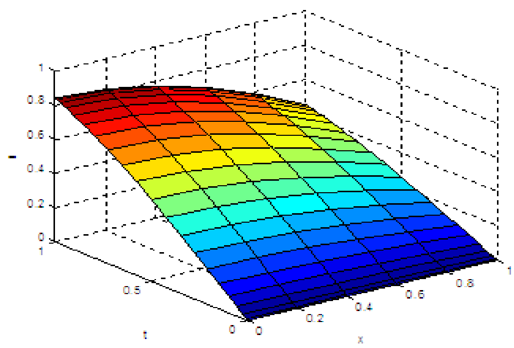

Figure 8: Numerical solutions of Example 2 for $\alpha = 20,\ \beta = 10$ and $\Delta t = 0.001$ at different times.

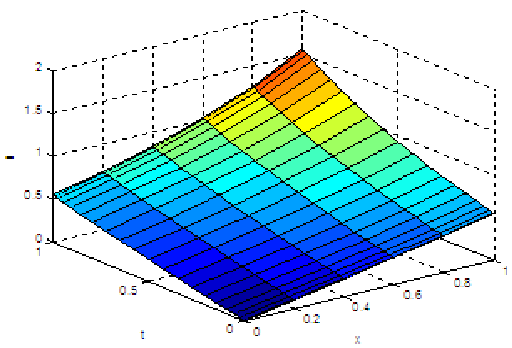

Figure 9: Numerical solutions of Example 3 for $\alpha = 20,\ \beta = 10$ and $\Delta t = 0.001$ at different times.