

## Wavelet Transform and Wavelet Based Numerical Methods: an Introduction

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**Abstract:** Wavelet transformation is a new development in the area of applied mathematics. Wavelets are mathematical tools that cut data or functions or operators into different frequency components, and then study each component with a resolution matching to its scale. In this article, we have made a brief discussion on historical development of wavelets, basic definitions, formulations of wavelets and different numerical methods based on Haar and Daubechies wavelets for the numerical solution of differential equations, integral and integro-differential equations.

**Keywords:** wavelet transform; multi-resolution analysis; Daubechies wavelet; Haar wavelet; differential equation; integro-differential equation

### 1 Introduction

Wavelets are already recognized as a powerful new mathematical tool in signal and image processing, time series analysis, geophysics, approximation theory and many other areas. First of all, wavelets were introduced in seismology to provide a time dimension to seismic analysis, where Fourier analysis fails. Fourier analysis is ideal for studying stationary data (data whose statistical properties are invariant over time) but it is not well suited for studying data with transient events that can not be statistically predicted from the data past. Wavelets were designed with such non stationary data in mind. Their generality and strong results have quickly become useful to a number of disciplines. The wavelet transform has been perhaps the most exciting development in the decade to bring together researchers in several different fields such as signal processing, quantum mechanics, image processing, communications, computer science and mathematics - to name a few as in [1]. Today wavelet is not only the workspace in computer imaging and animation; they are also used by the FBI to encode its data base of million fingerprints. In future, scientists may put wavelet analysis for diagnosing breast cancer, looking for heart abnormalities, predicting the weather, signal processing, data compression, smoothing and image compression, fingerprints verification, DNA analysis, protein analysis, Blood-pressure, heart-rate and ECG analysis, finance, internet traffic description, speech recognition, computer graphics, and many others [10, 11, 12, 14]. Some applications of wavelet transform are described at the end of this paper. Wavelet analysis provides additional freedom as compared to Fourier analysis since the choice of atoms of the transform deduced from the analyzing wavelet is left to the user. Wavelet theory involves representing general functions in terms of simpler building blocks at different scale and positions. The fundamental idea behind the wavelet transform is to analyze according to scale.

The applications of wavelet theory in numerical methods for solving differential equations and integro-differential equations are roughly last two decades years old. In the early nineties, people were very optimistic because it seems that many nice properties of wavelets would automatically lead to efficient numerical method for differential equations. The reason for this optimism was the fact that many nonlinear partial differential equations (PDEs) have solutions containing local phenomena (e.g. formation of shock, hurricanes) and interactions between several scales (e.g. turbulence, particularly, atmospheric turbulence because there is motion on a continuous range of length scales). Such solutions can be well represented using wavelet bases because of its nice properties, for example compact support (locality in space) and vanishing moment (locality in scale).

Before, the most common numerical methods used for numerical solution of physical, chemical and biological problems were finite difference methods (FDM), finite volume methods (FVM), finite elements methods (FEM) and spectral

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methods. Moreover, FDM and FVM are an approximation to the differential equation while other methods are an approximation to its solution. As we noted earlier, spectral bases are infinitely differentiable, but have global support. On the other hand, bases functions used in finite difference or finite element methods have small compact support but poor continuity properties. Conclusively, spectral methods have good accuracy, but poor spatial localization, while FDM, FVM and FEM have good spatial localization but poor accuracy. Wavelet analysis when applied to above mentioned numerical methods seem to combine the advantage (spectral accuracy as well good localization) of all the methods using wavelet bases. The objective of this article is to provide the reader a sound understanding of the foundations of wavelet transforms and a comprehensive introduction to its type and numerical methods based on wavelet transform for the numerical solution of ordinary differential equations, partial differential equations and integro-differential equations. Our presentation aims at developing the insights and techniques that are most useful for attacking new problems. However, the matter presented in this article is available in different books and already published research articles but we summarize the important material in an effective manner which can serve as an introduction to new researchers and helpful both as learning tool and as a reference.

The organization of this paper is as follow

Section 2: In this section, we provide a brief history of wavelet transform, which tell us historic development of wavelet transform

Section 3: This section is consists of the definition of wavelet, wavelet transform. In this section, we also discuss about what continuous wavelet transform is and how we can formulate discrete wavelet transform from continuous wavelet transform and different families of wavelet transform.

Section 4: In this section methods based on Haar and Daubechies for solving differential and integral equation are described in detail.

Section 5: Many applications of wavelet transform is described in this section.

Section 6: Comparative study of methods with further direction of research and concluding remark are given below.

## 2 History of wavelet transform

The concept of “wavelets” or “ondelettes” started to appear in the literature only in the early 1980s. This new concept can be viewed as a synthesis of various ideas which originated from different disciplines including mathematics, physics and engineering. In 1982, Jean Morlet, a French geophysical engineer, did not plan to start a scientific revolution. He was merely trying to give geologists a better way to search for oil. Petroleum geologists usually locate underground oil deposits by making loud noise. Because sound waves travel through different materials at different speed, geologists can infer what kind of material lies under the surface by sending seismic wave into the ground and measure how quickly they rebound, it may be salt dome, which can trap a layer of oil underneath. Figuring out just how the geology translates into a sound wave is a tricky mathematical problem, and one that engineers traditionally solve with Fourier analysis. Unfortunately, seismic signal contains lots of transients-abrupt changes in the wave as it passes from one rock layer to another. Fourier analysis spreads out that spatial information all over the place.

Morlet, an engineer for Elf-Aqyitaine, developed his own way of analyzing the seismic signals creating components that were localized in space, which he called **wavelets of constant shape**. Later they were called **Morlet wavelets**. It was Alex Grossmann, a French theoretical physicist, who quickly recognized the importance of the Morlet wavelet transform which is something similar to coherent states formalism in quantum mechanics, and developed an exact inversion formula for the wavelet transform. In fact, wavelet transforms turned out to work better than Fourier transform because they are much less sensitive to small error in the computation. An error or an unwise truncation of the Fourier coefficients can turn a smooth signal into jumpy one or vice versa; wavelets avoid such disastrous consequences.

Morlet and Grossmann [2] used first time the word “wavelet”, in his research article which was published in 1984. Also, Yves Meyer [3] is one of the founders of wavelet theory and he is the first to realize the connection between Morlet wavelets. Earlier, in the work of Littlewood and Paley theory, Meyer went on to discover a new kind of wavelet, with a mathematical property called orthogonality that made the wavelet transform as easy to work with and manipulate as a Fourier transform. The next major achievement of wavelet analysis was due to Daubechies [4], who suggested a new construction of a ‘painless’ non orthogonal wavelet expansion. During 1985-86, further work of Meyer and Lemarie on the first construction of a smooth orthonormal wavelet basis on  $\mathbb{R}^n$  marked the beginning of their famous contributions to the wavelet theory. At the same time Stephans Mallat recognized that some quadratic mirror filters play an important role for the construction of orthogonal wavelet bases generalizing the Haar system. Meyer and Mallat realized that the orthogonal wavelet bases could be constructed systematically from a general formalism. Their collaboration culminated with the remarkable discovery by Mallat of a new formalism which is so called multiresolution analysis. It was also Mallat

who constructed the wavelet decomposition and reconstruction algorithms using the multiresolution analysis. Mallat brilliant work was the major source of many new developments in wavelets. A few months later, Battle and Lemarie independently proposed the construction of spline orthogonal wavelets with exponential decay. While reviewing two books on wavelet in 1993, Meyer state: "Wavelets are without doubt an exciting and intuitive concept. The concept brings with it a new way of thinking, which is absolutely essential and was entirely missing in previously existing algorithms".

Daubechies made a new remarkable contribution to wavelet theory by constructing families of compactly supported orthonormal wavelets with some degree of smoothness. Daubechies research article [4] had a tremendous positive impact on the study of wavelets and their diverse applications. This work significantly explained the connection between the continuous wavelets on  $\mathbb{R}$  and the discrete wavelets on  $\mathbb{Z}$  and  $\mathbb{Z}_n$ , where the latter has become useful for digital signal analysis. The idea of frames was introduced by Duffin and Schaeffer and subsequently studied in some detail by Daubechies. In spite of tremendous success, experts in wavelet theory recognized that it is difficult to construct wavelets that are symmetric, orthogonal and compactly supported. In order to overcome this difficulty, Cohen and other studied bi-orthogonal wavelets in some detail. Chui and Wang introduced compactly supported spline wavelets, and semi orthogonal wavelet analysis. On the other hand, Beylkin and other have successfully applied the multiresolution analysis generated by a completely orthogonal scaling function to study a wide variety of integral operator on  $L^2(\mathbb{R})$  by a matrix in a wavelet basis. This work culminated with the remarkable discovery of new algorithm in numerical analysis. Recently, there have also been significant applications of wavelet analysis to a wide variety of problems in many diverse fields including mathematics, Physics, medicine, computer science and engineering. Now we move on to definition of wavelet.

### 3 Preliminaries of wavelet transform

In this section, we will discuss the preliminaries of wavelets which are useful to understand wavelet transform and its different classes as continuous and discrete wavelet transform.

#### 3.1 Wavelet

Wavelet may be seen as a complement to classical Fourier decomposition method. Suppose, a certain class of function is given and we want to find 'simple functions  $f_0, f_1, f_2, \dots$  such that

$$f(x) = \sum_{n=0}^{\infty} a_n f_n(x), \text{ for some coefficients, } a_n, \quad (1)$$

Wavelet is a mathematical tool used for representation of the type (1) for a wide class of function  $f$ .

Alternatively we can define wavelet as:

A wavelet means a small wave (the sinusoids used in Fourier analysis are big waves) and in brief, a wavelet is wave like oscillation with an amplitude that starts from zero, increase and then decrease back to zero. Equivalently, mathematical conditions for wavelet are:

$$\int_{-\infty}^{\infty} |\psi(t)|^2 dt < \infty; \quad (2)$$

$$\int_{-\infty}^{\infty} |\psi(t)| dt = 0; \quad (3)$$

$$\int_{-\infty}^{\infty} \frac{|\bar{\psi}(\omega)|^2}{|\omega|} d\omega < \infty, \quad (4)$$

where  $\bar{\psi}(\omega)$  is the Fourier Transform of  $\psi(\omega)$ . Equation (4) is called the admissibility condition.

#### 3.2 Wavelet transform

Wavelet [2] mean 'small wave'. So wavelet analysis is about analyzing signal with small duration finite energy functions. They transform the signal under investigation into another representation which converts the signal in a more useful form. This transformation of the signal is called wavelet transform. Wavelet transform have advantages over traditional Fourier transform for representing functions that have discontinuities and sharp peaks, and for accurately deconstructing

and reconstructing finite, non-periodic and non-stationary signals. Unlike Fourier transform, we have a different type of wavelets that are used in different fields. Choice of a particular wavelet depends on the type of application in hand. We manipulate wavelet in two ways. The first one is translation (change of position). We change the central position of the wavelet along the time axis. The second one is scaling. The wavelet transform is basically quantifies the local matching of the wavelet with the signal. If the wavelet matches with the signal well at a specific scale and location, then a large transform value is obtained. The transform value is then plotted in two-dimensional transform plane. The transform computed at various locations of the signal and for various scale of the wavelet. If the process is done in a smooth and continuous fashion, then transform is called continuous wavelet transform. If the scale and position are changed in discrete steps, the translation is called discrete wavelet transform. Note that in case of Fourier transform, spectrum is one-dimensional array of values whereas in wavelet transform, we get a two dimensional array of values. Also note that the spectrum depends on the type of wavelet used for analysis. Mathematically, we denote a wavelet as:

$$\psi_{a,b}(t) = \frac{1}{\sqrt{|a|}} \psi\left(\frac{t-b}{a}\right) \quad a, b \in \mathbb{R}, a \neq 0, \quad (5)$$

where  $b$  is location parameter and  $a$  is scaling parameter. For the function to be wavelet, it should be time limited. For a given scaling parameter  $a$ , we translate the wavelet by varying the parameter  $b$ .

We define wavelet transform as:

$$W(a, b) = \int_t f(t) \frac{1}{\sqrt{|a|}} \psi\left(\frac{t-b}{a}\right) dt \quad (6)$$

For every  $(a, b)$ , we have wavelet transform coefficient, if  $|a| < 1$ , then the wavelet in (6) is compressed version (smaller support in time domain) of the mother wavelet and corresponds mainly to higher frequencies. On the other hand, when  $|a| > 1$ , then  $\psi_{a,b}(t)$  has a larger time-width than  $\psi(t)$  and corresponds to lower frequencies. Thus, wavelets have time-widths adapted to their frequencies. This is the main reason for the success of Morlet [3] wavelets in signal processing and time-frequency signal analysis.

### 3.3 Continuous wavelet transform

Let  $f(x)$  be any square integrable function. Then the continuous wavelet transform  $W_\psi f$  of  $f \in L_2(\mathbb{R})$  with respect to  $\psi$  is defined as

$$W_\psi(b, a) = |a|^{-\frac{1}{2}} \int_{-\infty}^{\infty} f(x) \overline{\psi\left(\frac{x-b}{a}\right)} dx \quad (7)$$

where  $a$  and  $b$  are real and bar denotes the complex conjugation. Thus the wavelet transform is a function of two variables. There normalizing factor  $\frac{1}{\sqrt{|a|}}$  ensure that the energy stays the same for all  $a$  and  $b$ ; that is

$$\int_{-\infty}^{\infty} |\psi_{a,b}(x)|^2 dx = \int_{-\infty}^{\infty} |\psi(x)|^2 dx, \quad (8)$$

In order to reconstruct  $f$  from  $W_\psi f$ , we need to know the constant

$$C_\psi = \int_{-\infty}^{\infty} \frac{|\widehat{\psi}(x)|}{|\psi|} dx < \infty, \quad (9)$$

The finiteness of this constant (admissibility condition) restrict the class of  $L_2(\mathbb{R})$  function that can be used as wavelets. This implies

$$\int_{-\infty}^{\infty} \psi(x) dx = 0, \quad (10)$$

see [5] for more details.

With the constant  $C_\psi$ , we have the following reconstruction formula

$$f(x) = \frac{1}{C_\psi} \int_{\mathbb{R}^2} W_\psi(b, a) \overline{\psi\left(\frac{x-b}{a}\right)} \frac{dadb}{a^2}, \quad f \in L_2(\mathbb{R}). \quad (11)$$

Notice that the possibility of reconstruction is guaranteed by the admissibility condition. Now we move from CWT to discrete wavelet transform

### 3.4 Continuous to discrete wavelet transform

It is logical to wonder whether it is necessary to know  $C_\psi$  everywhere to construct a  $f$ . When the answer is negative, the use of a discrete subset seems a reasonable objective. The idea is as follows: we consider discrete subset of  $\mathbb{R}^{+*}$  and  $\mathbb{R}$ . Let us fix  $a_0 > 1$  and  $b_0 > 0$  and take  $a \in \{a_0^j\}_{j \in \mathbb{Z}}$  and  $b \in \{k a_0^j b_0\}_{j,k \in \mathbb{Z}}$ . Instead of using the family of wavelets:

$$\psi_{a,b}(t) = \frac{1}{\sqrt{|a|}} \psi\left(\frac{t-b}{a}\right) \quad a \in \mathbb{R}^{+*}, b \in \mathbb{R}. \tag{12}$$

For the discrete wavelet transform we use the family of wavelets indexed by  $Z$ :

$$\psi_{j,k}(x) = a_0^{-j/2} \psi(a_0^{-j}x - k b_0) \quad a_0 > 1, b_0 > 0 \text{ fixed for } j, k \in Z. \tag{13}$$

For  $f \in L^2$  we define the discrete wavelet transform of the function  $f$  by:

$$C_f(j, k) = \int_{\mathbb{R}} f(x) \overline{\psi_{j,k}(x)} dx = \langle f, \psi_{j,k} \rangle_{L^2}, \tag{14}$$

where  $j, k \in Z$ .

When value of  $a_0 = 2, b_0 = 1$  construct discrete wavelet transform as

$$\psi_{j,k}(x) = 2^{-j/2} \psi(2^{-j}x - k). \tag{15}$$

This is used in multiresolution analysis constituting an orthonormal basis for  $L^2(\mathbb{R})$ .

### 3.5 Wavelet families

A "wavelet system" consists of the scaling function  $\phi(x)$  and the wavelet function  $\psi(x)$ . In literature, several wavelets with different properties have been derived in references [3, 8], [6-12] and few of them are described below.

- Haar wavelet with orthogonal compact support
- Daubechies wavelet with different compact support
- Coiflet wavelet with different compact support
- Block spline semi-orthogonal wavelet
- Battele-Lemarie's wavelets
- Biorthogonal Wavelets of Cohen
- Shannon's wavelet and Meyer's wavelet
- Meyer wavelet is continuous and discrete transformation with infinite regularity
- Gaussian wavelets is discrete transformation with infinite regularity
- Mexican hat wavelet is continuous transformation with infinite regularity
- Morlet wavelet is continuous transformation with infinite regularity,

## 4 Numerical methods based on wavelet transformation:

As above mention, wavelet transformations are two types: continuous and discrete wavelet transformations. Due to improper integral used in continuous wavelet transformation, it is very difficult to find the numerical solution by using this transformation. So, all of the wavelets based numerical methods are based on discrete wavelet transformation to find the numerical solutions of differential equations and integro-differential equations. Most of the numerical methods are based on Daubechies wavelet with different compact support [7] and Haar wavelet with orthogonal compact support [6]. So, the methods are divided into the following two categories

- (i) Daubechies wavelet with different compact support based numerical methods.
- (ii) Haar wavelet with orthogonal compact support based numerical methods.

### 4.1 Daubechies wavelet based numerical methods for differential equations

In this sub-section, we describe the general numerical methods based on Daubechies wavelet which are used to solve the differential equations and existing in literature. Beginning from 1980s, wavelets have been used for solution of differential equations (PDE). The good features of this approach are possibility to detect singularities, irregular structure and transient phenomena exhibited by the analyzed equations. Most of the wavelet algorithms can handle easily periodic boundary conditions.

Before the explanation of Daubechies wavelets based numerical methods, we are interested to explain some definitions used in these methods.

#### 4.1.1 Compactly supported wavelets

The class of compactly supported wavelet bases was introduced by Daubechies [7]. They are an orthonormal bases for functions in  $L^2(R)$ . The construction of wavelet functions starts from building the scaling or dilation function,  $\varphi(x)$  and set of coefficients  $h_k, k \in Z$ , satisfies the two-scale relation or refinement equation,

$$\varphi(x) = \sum_{k=0}^{N-1} h_k \varphi(2x - k),$$

The wavelet function is

$$\psi(x) = \sum_{k=0}^{N-1} g_k \varphi(2x - k),$$

where  $g_k = (-1)^k h_{N-1-k}$ , and  $\int_{-\infty}^{\infty} \varphi(x) dx = 1$ .

#### 4.1.2 Vanishing movement

The wavelet is said to have  $M$  ( $M \in N$ ) vanishing movement if it verifies the following condition

$$\int_{-\infty}^{\infty} x^m \psi(x) dx = 0, \quad m = 0, 1 \dots M - 1,$$

where  $N = 2M$  for the Daubechies wavelets.

#### 4.1.3 Multiresolution analysis

The wavelet basis induces a multiresolution analysis [26] on  $L^2(R)$  i.e. the decomposition of the Hilbert space  $L^2(R)$  into a chain of closed subspaces

$$\dots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \dots$$

such that

$$\bigcup V_j = L^2(R)$$

and

$$\bigcap V_j = \{0\}.$$

By defining the  $W_j$  as an orthogonal complement of  $V_j$  and  $V_{j+1}$ ,

$$V_{j+1} = V_j \oplus W_j.$$

The space  $L^2(R)$  is represented as the direct sum of  $W_j$ 's as

$$L^2(R) = \oplus W_j.$$

On each fixed scale  $j$ , the wavelets  $\{\psi_{j,k}(x) = 2^{j/2} \psi(2^{j/2}x - k) : k \in Z\}$  form an orthonormal basis of  $W_j$  and the functions  $\{\phi_{j,k}(x) = 2^{j/2} \phi(2^{j/2}x - k) : k \in Z\}$  form an orthonormal basis of  $V_j$ . The coefficients  $H = \{h_k\}$  and  $G = \{g_k\}$  are quadrature mirror filters. Once the filter  $H$  has been chosen, it completely determines the function  $\phi$  and  $\psi$ .

#### 4.1.4 Connection coefficients

Any numerical scheme for solving differential equations must adequately represent the derivatives and non linearities of the unknown function. In the case of wavelet bases, these approximations give rise to certain  $L^2$  inner product of the basis functions. There derivatives and there translates, called the connection coefficients, which are defined as

$$\Lambda_{k_1, k_1, \dots, k_n}^{d_1, d_2, \dots, d_n} = \int_{-\infty}^{\infty} \prod_{j=1}^n \phi_{k_j}^{d_j}(x)$$

where  $d_i$  is the number of differentiation of scaling function  $\phi_i$  with respect to  $x$  which gives rise to the  $t$ -term connection coefficients. Chen et al. [15] have presented an efficient algorithm to calculate the connection coefficients in a bounded interval.

#### 4.1.5 Wavelet-Galerkin method

The differential equations are divided into two categories, first, ordinary differential equations and second, partial differential equations. The explanation of wavelet - galerkin method for the both types of equations is as follows

**(i) Ordinary differential equations (ODEs):** Many applications of mathematics require the numerical approximation of solutions of differential equations. In this sub-section we present Wavelet-Galerkin method for the solution of boundary value problems for ordinary differential equations. We consider the class of ordinary differential equation of the form

$$L u(x) = f(x), \quad x \in \Omega, \tag{16}$$

where  $L$  is differential operator and with some appropriate boundary conditions on  $u(x)$ .

The classical Galerkin methods have the disadvantage in solving the equation (16) since the stiffness matrix becomes ill conditioned as the problem size grows. To overcome this disadvantage, we use wavelets as basis functions in a Galerkin method. Then, the result is a linear system that is sparse because of the compact support of the wavelets, and that, after preconditioning, has a condition number independent of problem size because of the multiresolution structure.

In the wavelet Galerkin method we choose wavelet basis as weight functions. Having a multiresolution analysis,  $V_j, j \in Z$  with Daubechies compactly supported wavelet  $\psi(x)$ , one can use  $\psi_{j,k}(x)$  as the basis functions for the Galerkin method. We know that the set  $\{ \psi_{j,k}(x) = 2^j \psi(2^j x - k) : k \in Z \}$  forms an orthonormal basis of  $V_j$ .

By wavelets-Galerkin method, we approximate the solution of (16) in the following form

$$u(x) = \sum_{(j,k) \in \Lambda} a_{j,k} \psi_{j,k}(x), \tag{17}$$

where  $a_{j,k}$  are the scalars to be determined and  $\psi_{j,k}$  are orthonormal wavelets and satisfied the boundary conditions.

Substituting the trial solution (17) in the equation (16), we get

$$L \left( \sum_{(j,k) \in \Lambda} a_{j,k} \psi_{j,k} \right) - f(x) = R(x). \tag{18}$$

The method of weighted residuals  $R(x)$  minimizes by forcing it to zero in the domain  $\Omega$  using weight functions  $w_{l,m}(x)$  such that, for every weight function,

$$\sum_{(j,k) \in \Lambda} a_{j,k} \langle L\psi_{j,k}, \psi_{l,m} \rangle = \langle f, \psi_{l,m} \rangle \quad \forall (l,m) \in \Lambda, \tag{19}$$

for some finite set of indices  $\Lambda$ .

If the system of ODEs is linear, then we can write the system as a matrix equation of the form  $AX = b$ , where the vectors  $X = (a_{j,k})_{(j,k) \in \Lambda}$  and  $b = (b_{j,k})_{(j,k) \in \Lambda}$  are indexed by the pairs  $(j,k) \in \Lambda$  and the matrix

$A = [A_{l,m;j,k}]_{(l,m), (j,k) \in \Lambda}$  defined by

$$A_{l,m;j,k} = \langle l\psi_{j,k}, \psi_{l,m} \rangle \tag{20}$$

has its rows indexed by the pairs  $(l,m) \in \Lambda$  and its columns indexed by the pairs  $(j,k) \in \Lambda$ .

As suggested, we would like  $A$  to be sparse and have a low condition number. The system can be solved by some standard method like Gauss-elimination and we obtain stable solutions. If the system of ODEs is non-linear, then we

obtain a nonlinear system of equations and that system can be solved by some standard method like Newton-Raphson method.

**(ii) Partial differential equations (PDEs):** In this sub-section we will present Wavelet-Galerkin method for time dependent problems only. Singularities and sharp transitions in solutions of partial differential equations model important physical phenomena such as beam focusing in nonlinear optics, the formation of shock waves in compressible gas flow the formation of vortex sheets in high Reynolds number incompressible flows, etc. A characteristic feature of such phenomena is that the complex behavior occurs in a small region of space and intermittently in time. This makes them particularly hard to simulate numerically by solving the partial differential equations with conventional numerical methods. This drawback can be resolve by the wavelets methods. For the procedure of wavelet-Galerkin method for PDEs, we consider the following time dependent problem

$$u_t(x, t) + Lu(x, t) = f(x, t), \quad x \in \Omega, \quad t > 0, \tag{21}$$

where  $L$  is spatial operator and with some appropriate initial and boundary conditions on  $u(x, t)$ .

By wavelets-Galerkin method, we approximate the solution of (21) in the following form

$$u(x, t) = \sum_{(j,k) \in \Lambda} a_{j,k}(t) \psi_{j,k}(x) . \tag{22}$$

Substituting the trial solution (22) in the equation (21) and applying the orthogonality condition, we have

$$\sum_{(j,k) \in \Lambda} \frac{da_{j,k}(t)}{dt} \langle \psi_{j,k}, \psi_{l,m} \rangle + \sum_{(j,k) \in \Lambda} a_{j,k}(t) \langle L\psi_{j,k}, \psi_{l,m} \rangle = \langle f, \psi_{l,m} \rangle \quad \forall (l, m) \in \Lambda \tag{23}$$

The system (23) is a system of linear or nonlinear ODEs which can be solved by Runge- Kutta 4<sup>th</sup> order method. Linear system can be solved easily but, if the system is nonlinear it is very complicated to solve due to the product of integrals called connection coefficients. For example, consider a well known nonlinear PDEs Burger' equation

$$u_t(x, t) + uu_x = \varepsilon u_{xx}, \quad x \in \Omega, \quad t > 0. \tag{24}$$

Let the trial solution of partial differential equation (24) be

$$u(x, t) = \sum_{k=-N+1}^n a_k(t) \psi_{j,k}(x) , \tag{25}$$

where  $n = 2^j$  and  $\psi_{j,k}(x) = 2^j \psi(2^j x - k)$  and is a scaling function of Daubechies compactly supported wavelet with vanishing moment  $N/2$ . Now taking test function as  $\psi_{j,p}(x) = 2^j \psi(2^j x - p)$  and applying the Wavelet-Galerkin method we get a system of simultaneous differential equations of first order,we get

$$\frac{da_p(t)}{dt} = \varepsilon \sum_{k=-N+1}^n a_k(t) \Lambda_{k,p}^{2,0} - \sum_{k=-N+1}^n \sum_{m=-N+1}^n a_k(t) a_m(t) \Omega_{k,p}^{1,0}, \tag{26}$$

where  $\Lambda_{k,p}^{2,0} = \int_{\Omega} \ddot{\psi}_{j,k} \psi_{j,p} dx$  and  $\Omega_{k,p}^{1,0} = \int_{\Omega} \dot{\psi}_{j,k} \psi_{j,p} dx$  these are called connection coefficients.

**Remark 1 :** In wavelet-Galerkin methods, the treatment of nonlinearities is complicated which can be handled with the following techniques:

(i)Using the connection coefficients discussed in Sec. IIE (This approach is expensive due to the summation over multiple indices)

(ii)Using the quadrature formula [16]

(iii)Pseudo approach [17] (first map wavelet space to physical space, compute nonlinear term in physical space and then back to wavelet space, this approach is not very practical because it requires transformation between the physical space and wavelet space).

**4.1.6 Wavelet collocation method**

Collocation method involves numerical operators acting on point values (collocation points) in the physical space. Generally, wavelet collocation methods are created by choosing a wavelet and some kind of grid structure which will be computationally adapted. In effect, one obtains finite differences on nonuniform grid. The treatment of nonlinearities in wavelet collocation method is straightforward task due to collocation nature of algorithm. Moreover, proofs are easier with Galerkin methods, whereas implementation is more practical with collocation methods. For more details see [18,19].



**4.1.7 Wavelet-Taylor Galerkin method:**

The wavelet-Taylor Galerkin Method is similar to wavelet-Galerkin Method. In this method, first time derivatives are discretized by time stepping method such as forward difference, Crank-Nicolson, Leap-Frog etc. and than Taylors’ series method is applied. Finally, wavelet-Galerkin method is applied. The procedure of the wavelet-Taylor Galerkin Method has shown by taking a simple example of heat equation as follows:

Consider a heat equation

$$u_t = \alpha u_{xx} + f(x), \quad t > 0 \tag{27}$$

with some initial and boundary conditions and  $\alpha$  is positive constant.

Discretized time by time stepping forward difference method, we obtain

$$u_t^n = \frac{u^{n+1} - u^n}{\delta t} = \alpha u_{xx}^n + f(x) \tag{28}$$

On using the difference approximation to  $u_t$  at time level  $n$  by forward time Taylor series expansion, including second and third time derivatives, we get

$$u_t^n = \frac{u^{n+1} - u^n}{\delta t} - \frac{\delta t}{2} u_{tt}^n + \frac{\delta t^2}{6} u_{ttt}^n - O(\delta t^3). \tag{29}$$

On successive differentiation of equation (27) with respect to  $t$ , we get

$$u_{tt} = \alpha^2 u_{xxxx} + \alpha f''(x) \text{ and } u_{ttt} = \alpha^2 (u_t)_{xxxx}. \tag{30}$$

Using (29) and (30) in semi-discrete equation (28), we get the following equation

$$\frac{u^{n+1} - u^n}{\delta t} - \frac{\alpha^2 \delta t^2}{6} (u_t)_{xxxx}^n = \alpha u_{xx}^n + \frac{\alpha^2 \delta t}{2} u_{xxxx}^n + \alpha \frac{\delta t}{2} f''(x) + f(x). \tag{31}$$

In equation (31), replace  $u_t^n$  by  $\frac{u^{n+1} - u^n}{\delta t}$  so that the semi-discrete equation (31) transforms into the following generalized Euler time stepping method

$$\left(1 - \frac{\alpha^2 \delta t^2}{6} \partial_{xxxx}\right) \left(\frac{u^{n+1} - u^n}{\delta t}\right) = \alpha u_{xx}^n + \frac{\alpha^2 \delta t}{2} u_{xxxx}^n + \alpha \frac{\delta t}{2} f''(x) + f(x). \tag{32}$$

Now the semi-discrete equation (32) can be solved by using Wavelet-Galerkin method.

Kumar and Mehra [22-25] have discussed the Wavelet-Taylor Galerkin method for parabolic and other type of partial differential equations.

**4.2 Haar wavelet**

Wavelet transform or wavelet analysis is a recently developed mathematical tool for many problems. One of the popular families of wavelet is Haar wavelets. Haar function is in fact the Daubechies wavelet of order 1. Due to its simplicity, Haar wavelet had become an effective tool for solving many problems arises in many branches of sciences. Haar functions have been used from 1910. It was introduced by the Hungarian mathematician Alfred Haar [6]. The Haar function is an odd rectangular pulse pair, is the simplest and oldest orthonormal wavelet with compact support. There are different definitions of Haar function and various generalizations have been used and published. Haar functions appear very attractive in many applications as for example, image coding, edge extraction, and binary logic design [16-18]. Haar showed that certain square wave function could be translated and scaled to create a basis set that span  $L^2$ . A year later, it was seen that the system of Haar is a particular wavelet system. If we choose scaling function to have compact support over  $0 \leq x < 1$ , then the Haar wavelet family for  $x \in [0, 1)$  is defined as

$$h_i(x) = \begin{cases} 1 & x \in [\xi_1, \xi_2), \\ -1 & x \in [\xi_2, \xi_3), \\ 0 & \text{otherwise} \end{cases} \tag{33}$$

where

$$\xi_1 = \frac{k}{m}, \quad \xi_2 = \frac{k + 0.5}{m}, \quad \xi_3 = \frac{k + 1}{m} \tag{34}$$

In the above definition the integer  $m = 2^j, j = 0, 1 \dots, J$  indicates the level of wavelet and integer  $k = 0, 1 \dots, m - 1$  is the translation parameter. Maximal level of resolution is  $J$ . The index  $i$  in the equation (33) is calculated from the formula  $i = m + k + 1$ . In the case of minimal values  $m = 1, k = 0$ , we have  $i = 2$ . The maximum value of  $i$  is  $i = 2M = 2^{J+1}$ . For  $i = 1$ , the function  $h_1(x)$  is scaling function for the family of the Haar wavelets which is defined as

$$h_1(x) = \begin{cases} 1 & x \in [0, 1) \\ 0 & otherwise \end{cases} \quad (35)$$

In the Haar wavelet method the following integrals are used

$$p_{i,1}(x) = \int_0^x h_i(x') dx', \quad p_{i,v+1}(x) = \int_0^x p_{i,v}(x') dx', \quad v = 1, 2, 3, \dots \quad (36)$$

Some of calculated  $p_{i,\nu}(x), \nu = 1, 2, \dots$  are given below

$$p_{i,1}(x) = \begin{cases} x - \xi_1 & x \in [\xi_1, \xi_2), \\ \xi_3 - x & x \in [\xi_2, \xi_3), \\ 0 & otherwise \end{cases} \quad (37)$$

$$p_{i,2}(x) = \begin{cases} 0 & x \in [0, \xi_1), \\ 0.5(x - \xi_1)^2 & x \in [\xi_1, \xi_2), \\ 1/4m^2 - 1/2(\xi_3 - x)^2 & x \in [\xi_2, \xi_3), \\ 1/4m^2 & x \in [\xi_3, 1] \end{cases} \quad (38)$$

$$p_{i,3}(x) = \begin{cases} 0 & x \in [0, \xi_1), \\ 1/6(x - \xi_1)^2 & x \in [\xi_1, \xi_2), \\ 1/4m^2(x - \xi_2) - 1/6(\xi_3 - x)^2 & x \in [\xi_2, \xi_3), \\ 1/4m^2(x - \xi_2) & x \in [\xi_3, 1] \end{cases} \quad (39)$$

$$p_{i,4}(x) = \begin{cases} 0 & x \in [0, \xi_1), \\ 1/24(x - \xi_1)^2 & x \in [\xi_1, \xi_2), \\ 1/8m^2(x - \xi_2) - 1/24(\xi_3 - x)^2 & x \in [\xi_2, \xi_3), \\ 1/8m^2(x - \xi_2)^2 + 1/192m^4 & x \in [\xi_3, 1] \end{cases} \quad (40)$$

In this way, we can find out the higher order  $p_{i,\nu}(x), \nu = 5, 6 \dots$

Any function  $f(x)$  which is square integrable in the interval in  $(0, 1)$  can be expressed as an infinite sum of Haar wavelet as

$$f(x) = \sum_{i=1}^{\infty} a_i h_i(x). \quad (41)$$

The above series terminated at finite terms if  $f(x)$  is piecewise constant or can be approximated as piecewise constant during each subinterval. The best way to understand wavelet is through a multi resolution analysis. Given a function  $f(x) \in L^2(R)$  a multiresolution analysis of  $L^2(R)$  produce a sequence of subspaces  $V_j, V_{j+1} \dots$  such that the projection of  $f$  onto these spaces gives finer and finer approximation of the function  $f$  as  $j \rightarrow \infty$ .

A lot of work has been done on the numerical solutions of differential equations, integral equations and integro-differential equations by using of Haar wavelet in literature. The authors [21, 28, 29, 31, 34-38, 43-45, 47] have developed numerical methods for differential equations, integral equations and integro-differential equations by using of Haar wavelet. The numerical methods are explained in the following sub-sections.

**4.2.1 Haar wavelets based numerical method for solving ordinary differential equations(ODEs)**

By Haar wavelet method we can solve first order, second order and higher order differential equation with constant or variable coefficients. We construct a simple collocation method with the Haar basis function for the numerical solution of linear and non linear boundary value problem and initial value problem arising in mathematical modeling and different engineering application. Haar wavelet methods are used in different ways for numerical solution of ODEs.

Chen and Hsiao [21, 28] realized for the Haar wavelets by expanding the highest derivative appearing in the differential equation into the Haar series and approximation is integrated while the boundary conditions are incorporated by using integration constants as given in [27]. This approach can be called integral method and the main idea of this technique

is to convert a differential equation into an algebraic one. To describe this method, we consider a general second order ordinary differential equation defined as

$$y''(x) = f(x, y, y'), \quad x \in [0, 1], \tag{42}$$

with Dirichlet boundary conditions (the boundary conditions may be Neumann or mixed)

$$y(0) = \alpha, \quad y(1) = \beta, \tag{43}$$

Assume higher order derivative [21, 28]

$$y''(x) = \sum_{i=1}^{2M} a_i h_i(x). \tag{44}$$

Equation (44) is integrated twice from 0 to  $x$  or from  $x$  to 1 depending upon the boundary conditions. Hence, the solution  $y(x)$  with its derivative  $y'(x)$  and  $y''(x)$  are expressed in the term of Haar function and their integrals. We consider the collocation points

$$x_j = \frac{j - 0.5}{2M}, \quad j = 1, 2, 3 \dots 2M, \tag{45}$$

Integrate equation (44) twice from 0 to  $x$ , we obtain

$$y'(x) = y'(0) + \sum_{i=1}^{2M} a_i p_{i,1}(x) \tag{46}$$

$$y(x) = \alpha + x y'(0) + \sum_{i=1}^{2M} a_i p_{i,2}(x) \tag{47}$$

The value of unknown term  $y'(0)$  is calculated by integrating (46) from 0 to 1, we have

$$y'(0) = \beta - \alpha - \sum_{i=1}^{2M} a_i C_{i,1}(x), \tag{48}$$

where  $C_{i,1}(x) = \int_0^1 p_{i,1}(x) dx$ .

Substituting, equation (48) into equations (46) and (47), we get

$$y'(x) = \beta - \alpha + \sum_{i=1}^{2M} a_i (p_{i,1}(x) - C_{i,1}), \tag{49}$$

$$y(x) = \alpha + \beta x - \alpha x + \sum_{i=1}^{2M} a_i (p_{i,2}(x) - x C_{i,1}). \tag{50}$$

Substituting these values of  $y(x)$ ,  $y'(x)$ ,  $y''(x)$  in the given differential equation (42) we obtain the system of equations. Solving the system we get the quantity  $y(0)$  and the Haar coefficients  $a_i$ 's. The system may be linear or it may be non linear. We solve the system by suitable numerical methods.

**Remark 2** *The above procedure is for second order ordinary differential equation. We can extend this procedure for  $n$ th order ordinary differential equation by assuming  $y^{(n)}(x) = \sum_{i=1}^{2M} a_i h_i(x)$  and solve it like above procedure.*

The Chen and Hsiao method (CHM) was successfully applied for solving singular, bilinear and stiff systems [29-31]. In this method, the choice of solution steps is essential, if the step is very small the coefficient matrix may be nearly singular and its inversion brings to instability of the solution. Nonlinearity of the system also complicates the solutions, since big systems of nonlinear equations must be solved. Cattani [32] observed that computational complexity can be reduced if the interval of integration is divided into some segments [32], and developed a method, known as a method of segmentation by reducing Haar transform. The number of collocation points in each segment is considerably smaller as in the case of CHM; therefore we can hope that such solutions are more stable.

In this method, we first convert the higher order equation in the system of first order differential equations. The procedure is as follows

Consider a general second order ordinary differential equation (ODE) defined as

$$y''(t) = f(t, y, y'), \quad t \in [0, T] \quad (51)$$

Convert the second order ODE (51) into the system of first-order equations

$$\frac{dy}{dt} = u, \quad \frac{du}{dt} = f(t, y, u) \quad (52)$$

Instead of solving equation (51), we solve the system of first-order equations (52).

Let us divide the time interval into  $N$  segments; the length of the  $n$ -th segment is denoted by  $d^{(n)}$ . If  $t^{(n)}$  is the coordinate of the  $n$ -th dividing point, then

$$t^{(n+1)} = t^{(n)} + d^{(n)}, \quad n = 1, 2, \dots, N \quad (53)$$

with  $t^{(1)} = 0, \quad t^{(N+1)} = 1$ .

Consider the  $n$ -th segment. It is convenient to introduce the local time

$$\tau = \frac{t - t^{(n)}}{d^{(n)}} \quad (54)$$

and choose  $2M$  collocation points

$$\tau_j = \frac{1}{2M} \left( j - \frac{1}{2} \right), \quad j = 1, 2, \dots, 2M \quad (55)$$

The system (52) in local time obtains the form (dots denote differentiation with respect to  $\tau$ ):

$$\dot{y} = d^{(n)}u, \quad \dot{u} = f\left(t^{(n)} + d^{(n)}\tau, y, u\right) \quad (56)$$

In following it is expedient to consider the variables  $u$ ,  $v$  as row vectors with the components  $y_j = y(\tau_j), \quad u_j = u(\tau_j), \quad j = 1, 2, \dots, 2M$ .

Chen and Hsiao [21, 28] sought the solution is in the following form

$$\begin{aligned} \dot{y} &= aH, & y &= aPH + y_n E \\ \dot{u} &= bH, & u &= bPH + u_n E \end{aligned} \quad (57)$$

where  $H(i, j) = (h_i(t_j)), \quad j = 1, 2, \dots, 2M$  which has the dimension  $2M \times 2M$  and the operational matrix of integration  $P$ , which is a  $2M$  square matrix, is defined by the equation  $(PH)_{ij} = \int_0^{t_j} h_i(t) dt$ . Here  $a$  and  $b$  are  $2M$ -dimensional row vectors,  $E$  a unit vector of the same dimension; symbols  $y_n, u_n$  denote the values of  $u$  and  $v$  at the boundary  $t_n$ .

Substitution of (57) into equation (56) gives

$$\begin{aligned} aH &= d^{(n)}(bPH + u_n E) \\ b\dot{H} &= d^{(n)}f\left(t^{(n)} + d^{(n)}\tau, aPH + y_n E, bPH + u_n E\right) \end{aligned} \quad (58)$$

From these matrix equations the vectors  $a$  and  $b$  are calculated by using some standard numerical method; after that  $u$  and  $v$  can be evaluated from (57).

#### 4.2.2 Haar wavelets based numerical method for solving time dependent partial differential equations (PDEs)

For solving the PDEs the two-dimensional wavelet transform could be applied, as it was proposed e.g. by Newland [33], but more convenient seems to be the following algorithm.

Consider a parabolic equation

$$u_t = f(u, u_x, u_{xx}), \quad x \in [a, b], \quad t \in [t_{\min}, t_{\max}] \quad (59)$$

with initial and boundary conditions

$$u(x, 0) = f_1(x)$$

$$u(a, t) = f_2(x), \quad u(b, t) = f_3(x). \tag{60}$$

Since the Haar wavelets are defined for  $x \in [0, 1]$  we must first normalize equation (59) in regard to  $x$ . Let in the following dots and primes denote differentiation with respect to  $t$  and  $x$ , respectively. Equation (60) can now be rewritten in the form

$$u = f(u, u', u''), \quad x \in [0, 1], \quad t \in [t_{\min}, t_{\max}] \tag{61}$$

with given initial and boundary conditions.

Next let us divide the interval  $[t_{\min}, t_{\max}]$  into  $N$  equal parts of length  $\Delta t = \frac{(t_{\max}-t_{\min})}{N} \lim_{\delta x \rightarrow 0}$  and denote  $t_s = (s - 1) \Delta t, \quad s = 1, 2 \dots N$ . For the subinterval  $t \in [t_s, t_{s+1}]$  the Haar wavelet solution is sought in the form

$$\dot{u}''(x, t) = \sum_{i=1}^{2M} a_s(i) h_i(x), \tag{62}$$

where the row vector  $a_s$  is constant in the subinterval  $t \in [t_s, t_{s+1}]$ .

Integrating Eq. (62) with respect to  $t$  in the limits  $[t_s, t]$  and twice with respect to  $x$  in the limits  $[0, x]$ , we obtain

$$u''(x, t) = (t - t_s) \sum_{i=1}^{2M} a_s(i) h_i(x) + u''(x, t_s), \tag{63}$$

$$u'(x, t) = (t - t_s) \sum_{i=1}^{2M} a_s(i) p_{i,1}(x) + u'(x, t_s) - u'(0, t_s) + u'(0, t), \tag{64}$$

$$u(x, t) = (t - t_s) \sum_{i=1}^{2M} a_s(i) p_{i,2}(x) + u(x, t_s) - u(0, t_s) + x [u'(0, t) - u'(0, t_s)] + u(0, t), \tag{65}$$

$$\dot{u}(x, t) = \sum_{i=1}^{2M} a_s(i) p_{i,2}(x) + \dot{u}(0, t) + x \dot{u}'(0, t). \tag{66}$$

Applying the initial and boundary conditions in the above equations and then put the reduced equations in equation (61). Then, discretized the resulting equation by assuming  $x \rightarrow x_j$  and  $t \rightarrow t_s$ . Finally, we obtained linear or nonlinear algebraic system of equations which can be solved by some numerical method.

### 4.2.3 Haar wavelets based numerical method for solving integral equations

The integral equations have been divided into three categories Fredholm, Volterra and singular integral equations. Such equations occur widely in diverse areas of applied mathematics and physics. They offer a powerful technique for solving a variety of practical problems. In this sub-section, we will explain the Haar wavelet method for these integral equations.

**(i) Fredholm integral equations:** Consider a linear Fredholm integral equation of the form

$$u^*(x^*) - \int_a^b K^*(x^*, t^*) u^*(t^*) dt^* = f^*(x^*), \quad x^* \in [a, b] \tag{67}$$

where the kernel  $K^*$  and the right-hand side function  $f^*$  are prescribed. Since the Haar wavelets are defined only for  $x \in [0, 1]$ , so to change the interval into  $[0, 1]$  by using the following transformations

$$t^* = (b - a) t + a, \quad x^* = (b - a) x + a \tag{68}$$

Apply the transformations (68) on the Eq. (67), we get

$$u(x) - \int_0^1 K(x, t) u(t) dt = f(x), \quad x \in [0, 1] \tag{69}$$

where  $u(t) = u^*(t^*), \quad u(x) = u^*(x^*), \quad f(x) = f^*(x^*), \quad K(x, t) = (b - a) K^*(x^*, t^*)$

Next, we discretize the functions  $h_i(x)$  by dividing the interval  $x \in [0, 1]$  into  $2M$  parts of equal length  $\Delta x = \frac{1}{2M}$  and introduce the collocation points as follows

$$x_j = \frac{(j - 0.5)}{2M}, \quad j = 1, 2, \dots, 2M \tag{70}$$

Consider the solution of the equation (69) in the linear combination of Haar functions as

$$u(x) = \sum_{i=0}^{2M} a_i h_i(x), \tag{71}$$

where  $a_i$  are unknown wavelet coefficients. In the discrete form the equation can be written

$$u(x_j) = \sum_{i=0}^{2M} a_i h_i(x_j) = \sum_{i=0}^{2M} a_i H_{ij}, \tag{72}$$

where  $H_{ij} = h_i(x_j)$  is the coefficients matrix.

Substitute (71) in the integral Eq. (69), we have

$$\sum_{i=0}^{2M} a_i h_i(x) - \sum_{i=0}^{2M} a_i G_i(x) = f(x), \tag{73}$$

where

$$G_i(x) = \int_0^1 K(x,t) h_i(t) dt. \tag{74}$$

In order to find out the wavelet coefficients  $a_i$ , we have to solve the equation (73). There are two ways to evaluate the wavelet coefficients  $a_i$  as follows

**Collocation method:** Satisfying (73) only at the collocation points (70) we get a system of linear equations

$$\sum_{i=0}^{2M} a_i [h_i(x_j) - G_i(x_j)] = f(x_j), \quad j = 1, 2, \dots, 2M \tag{75}$$

The matrix form of this system is

$$a(H - G) = F, \tag{76}$$

where  $a$  and  $F$  are  $2M$  dimensional row vectors and  $H = (h_i(x_j))$ ,  $G = (G_i(x_j))$  are  $2M \times 2M$  matrices.

**Galerkin method:** For realizing this approach each term of (73) is multiplied by  $h_j(x)$  and the result is integrated over  $x \in [0, 1]$ . Due to the orthogonality condition we obtain

$$\frac{a_j}{m_1} - \sum_{i=1}^{2M} a_i \Gamma_{ij} = \int_0^1 f(x) h_j(x) dx \tag{77}$$

Here,  $j = m_1 + k_1 + 1$ ,  $m_1 = 2^{j_1}$ ,  $j_1 = 0, 1, \dots, J$ ,  $k_1 = 0, 1, \dots, m_1 - 1$ ,

$$\Gamma_{ij} = \int_0^1 G_i(x) h_j(x) dx \tag{78}$$

From the above two methods we calculate the wavelet coefficients  $a_i$ , then put wavelet coefficients in (71) we get solution of integral equation.

**(ii) Volterra integral equation:** Consider Volterra integral equation of the form

$$u(x) - \int_0^x K(x,t) u(t) dt = f(x), \quad x \in [0, 1] \tag{79}$$

The discrete form of (79) is

$$u(x_j) - \int_0^{x_j} K(x_j,t) u(t) dt = f(x_j), \quad j = 1, 2 \dots 2M \tag{80}$$

where  $x_j$  are collocation points defined in (70).

In order to find the numerical solution of (79) we proceed like the Fredholm integral equation. On this equation we can only apply collocation method as discussed above. After applying the collocation method, we obtained the system

$$\sum_{i=1}^{2M} a_i [h_i(x_j) - G_i(x_j)] = f(x_j), \quad j = 1, 2 \dots 2M \tag{81}$$

Here,  $G_i(x_j)$  is given by

$$G_i(x_j) = \int_0^{x_j} K(x_j, t) h_i(t) dt \tag{82}$$

By computing these integrals the following cases should be distinguished:

- (a)  $G_{ij} = 0$  if  $x_j < \xi_1$ ,
- (b)  $G_{ij} = \int_{\xi_1}^{x_j} K(x_j, t) dt$  if  $\xi_1 \leq x_j \leq \xi_2$ ,
- (c)  $G_{ij} = \int_{\xi_1}^{\xi_2} K(x_j, t) dt - \int_{\xi_2}^{x_j} K(x_j, t) dt$  if  $\xi_2 \leq x_j \leq \xi_3$ ,
- (d)

$$G_{ij} = \int_{\xi_1}^{\xi_2} K(x_j, t) dt - \int_{\xi_2}^{\xi_3} K(x_j, t) dt \text{ if } \xi_3 \leq x_j \leq 1, \tag{83}$$

Where the quantities  $\xi_1, \xi_2, \xi_3$  are defined with the formulas (34).

(iv) **Weakly singular integral equation:** Consider weakly singular integral equation of the form

$$u(x) = \int_0^x \frac{K(x, t)}{(x-t)^\alpha} u(t) dt + f(x), \quad 0 < \alpha < 1, \quad 0 \leq t \leq x \leq 1, \tag{84}$$

This is a weakly singular equation. Since it is also a Volterra equation, then we can apply the same method applied for Volterra integral equations.

#### 4.2.4 Haar wavelets based numerical method for solving integral-differential equations

Let us consider an integral-differential equation of the type

$$u'(x) + p(x) u(x) = \int_0^1 K(x, t) [\alpha u(t) + \beta u'(t)] dt + f(x), \tag{85}$$

where  $\alpha, \beta$  are constants and  $p(x), f(x)$  are prescribed functions. To this equation belongs the initial condition  $u(0) = \gamma$ .

According to the method suggested by Chen and Hsiao [21, 28] we do not develop the Haar series for the function  $u(x)$ , instead we develop its derivative  $u'(x)$  for the numerical solution of integral-differential equation (85) as follows:

$$u'(x) = \sum_{i=0}^{2M} a_i h_i(x) \tag{86}$$

Integrate (86) with respect  $x$  over the domain  $[0, x]$ , we have

$$u(x) = \sum_{i=0}^{2M} a_i p_{i,1}(x) + u(0), \tag{87}$$

where  $p_{i,1}(x) = \int_0^x h_i(y) dy$  defined in (37).

After substitution of (86) and (87) into (85), we have

$$\sum_{i=0}^{2M} a_i [h_i(x) + p(x) p_{i,1}(x) - \alpha R_i(x) - \beta G_i(x)] = -u(0) p(x) + \alpha u(0) Q(x) + f(x), \tag{88}$$

where

$$G_i(x) = \int_0^1 K(x, t) h_i(t) dt \quad R_i(x) = \int_0^1 K(x, t) p_{i,1}(t) dt$$

$$Q(x) = \int_0^1 K(x, t) dt. \tag{89}$$

Now, apply Collocation method on the equation (88), we have matrix form

$$a(H + V - \alpha R - \beta G) = -\gamma p + \alpha \gamma Q + F, \tag{90}$$

where  $H = (h_i(x_j))$ ,  $G = (G_i(x_j))$ ,  $Q_j = (Q(x_j))$ ,  $R = (R_{ij}) = (R_i(x_j))$  and  $V = (V_{ij}) = p(x_j) p_{i,1}(x_j)$ ,

and  $p = p(x_j)$ ,  $F = f(x_j)$  and  $Q = Q(x_j)$  are understood as  $2M$ -vectors.

The system (90) can be solved by some standard numerical method such as Gauss-elimination method.

In the above sub-section, we have explained the procedure of numerical methods based on Haar wavelet for the numerical solutions of differential, integral and intego-differential equations. On the other hand, this approach has also some limitations. The conventional form of the Haar wavelet approach is applicable for the range of the argument  $x \in [0, 1]$ ; besides it is assumed that this interval is distributed into subintervals of equal length. If we want to raise the exactness of the results, we must increase the number of the grid points. In the course of the solution we have to invert some matrices, but by increasing the number of calculation points these matrices become nearly singular and therefore the inverse matrices cannot be evaluated with necessary accuracy [38]. There are also many problems where the uniform Haar wavelet method is not suitable (e.g. differential equations under local excitations, boundary layer problems, weakly singular equations, problems for which the region of the variation of the argument is infinite). In [20], the method of segmentation for solving such problems, but in this case the solution becomes more complicated. One possibility to find a way out of these difficulties is to make use of the non-uniform Haar method for which the length of the subintervals is unequal. This idea was proposed in [39] and applied for analyzing the function approximation problems. The non-uniform Haar wavelet has described in the following sub-section.

### 4.3 Non-uniform haar wavelet

Haar wavelets are characterized by two numbers: the dilatation parameter  $j = 0, 1 \dots J$  ( $J$  is maximal level of resolution) and the translation parameter  $k = 0, 1 \dots m - 1$ , where  $m = 2^j$ . The number of the wavelet is identified as  $i = m + k + 1$ . The maximal value is  $i = 2M$  where  $M = 2^J$ . Consider the interval  $x \in [a, b]$ . We shall partition this interval into  $2M$  subintervals. We shall define the non-uniform Haar wavelet as

$$h_i(x) = \begin{cases} 1 & \text{for } x \in [\xi_1(i), \xi_2(i)], \\ -c_i & \text{for } x \in [\xi_2(i), \xi_3(i)], \\ 0 & \text{elsewhere,} \end{cases} \quad i = 2, 3, \dots, 2M, \tag{91}$$

where

$$\begin{aligned} \xi_1(i) &= x(2k\mu), & \xi_2(i) &= x((2k+1)\mu), \\ \xi_3(i) &= x((2k+2)\mu), & \mu &= M/m. \end{aligned} \tag{92}$$

The coefficients  $c_i$  are calculated from the requirement

$$\int_a^b h_i(x) dx = 0. \tag{93}$$

The values of  $c_i$  are calculated by the formula

$$c_i = \frac{(\xi_2(i) - \xi_1(i))}{(\xi_3(i) - \xi_2(i))}. \tag{94}$$

The integrals which are introduced in the Haar wavelet method, the same integrals are also used in the non-uniform Haar wavelet method. The integrals are as follows

$$p_{i,1}(x) = \int_a^x h_i(z) dz \quad p_{i,\nu+1}(x) = \int_a^x p_{i,\nu}(z) dz, \quad \nu = 1, 2, \dots \tag{95}$$

Some of calculated  $p_{i,\nu}(x)$ ,  $\nu = 1, 2, \dots$  are given below

$$p_{i,1}(x) = \begin{cases} x - \xi_1 & \text{for } x \in [\xi_1, \xi_2], \\ c_i(\xi_3 - x) & \text{for } x \in [\xi_2, \xi_3], \\ 0 & \text{elsewhere,} \end{cases} \quad i = 2, 3, \dots, 2M, \tag{96}$$

$$p_{i,2}(x) = \begin{cases} \frac{1}{2}(x - \xi_1)^2 & \text{for } x \in [\xi_1, \xi_2], \\ K - \frac{c_i}{2}(\xi_3 - x)^2 & \text{for } x \in [\xi_2, \xi_3], \\ K & \text{for } x \in [\xi_2, b] \\ 0 & \text{elsewhere,} \end{cases} \quad i = 2, 3, \dots, 2M, \tag{97}$$

where

$$K = \frac{1}{2}(\xi_2 - \xi_1)(\xi_3 - \xi_1) \tag{98}$$



The Haar wavelet has some limitations to solve the differential equations under local excitations, boundary layer problems, singularly perturbed problems, problems for which the region of the variation of the argument is infinite. By the non-uniform Haar wavelet method these limitations can be removed. The procedure of numerical methods based on the non-uniform Haar wavelet is same as the Haar numerical wavelet method for differential, integral and intego-differential equations. The non-uniform Haar wavelet method was successively used by the authors [40, 41] for differential and integral equations.

## 5 Application of wavelet transform

Some fields of applications are geophysics, astrophysics, quality control, biology and aural signals in medicine, imagery in all its aspects and medical imagery in particular, coding of video signal, modeling of traffic in communication networks like the internet analysis of atmospheric or wind tunnel turbulence. The other fields of applications are computer science and mathematics. Today wavelets are not only workspace in computer imaging and animation; they are also used by the FBI to encode its database of million fingerprints. In future, scientist may put wavelet analysis for diagnosing breast cancer, looking for heart abnormalities or predicting the weather. The following are the some important application of wavelet transformation in science and engineering. In some application we will give mathematical model of engineering problems tackled by Haar wavelet and Daubechies wavelet transform. The detail description for solution of the problems using these wavelets can be found in the references cited again the problems.

### 5.1 Estimation of depth profile of soil temperature using Haar wavelet transform [42]

Let  $u$  be the mean of measurement of the heat stored,  $K$  means thermal diffusivity, where  $K = \frac{c_1}{c_v}$ ;  $c_v$  is the volumetric heat capacity and  $c_1$  is the thermal conductivity. Where  $K$  is useful to measure of how fast the temperature of soil layer changes. Then Heat Diffusion equation in soil is governing by

$$\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial z^2} \quad (99)$$

To find the solution of the diffusion equation by Haar wavelet transform they assume that

$$\frac{\partial u}{\partial t} = a_0 e^{(-x\sqrt{1/2K})^p} H(t) \quad (100)$$

Then applying the integration matrix  $P$  [ 20,38 ] they obtained

$$u(x, t) = a^t PH(t). \quad (101)$$

Using (100-101) in (99) they obtain the solution in the form of

$$u(x, t) = a_0 e^{(-x\frac{1}{\sqrt{K}})} PH(t).$$

Thus Haar wavelet is used for physical problem solution.

#### 5.1.1 Fingerprint verification

Fingerprint verification is one of the most reliable personal identification methods and its contribution in forensic and civilian application is quite important. However, manual fingerprint verification is so boring, time-consuming and expensive in that it is incapable of meeting today's increasing performance requirements. Hence automatic fingerprint identification system is widely needed. In Singapore, a new security system was introduced in Hitachi Tower (a 37-storey office building) in 2003, and the 1500 employees get access to the building by scanning fingers. The scanner uses infrared rays to trace the hemoglobin in blood in order to capture the vein patterns in the finger; these patterns determine the person uniquely. After comparing with the scanned data in an electronic archive, it is decided whether the person can get in or not, see [10, 11].

### 5.1.2 Storing fingerprint electronically using wavelet

We now discuss how the FBI (Federal Bureau of Investigation) in USA use wavelets as a tool to store fingerprints electronically. For many years, the FBI stored their fingerprints in paper format in a highly secured building in Washington; it occupied an area which had the same size as football field. If one wants to compare a fingerprint in San Francisco with the stored fingerprints was done manually, so it was quite slow process. For these reasons FBI started to search for way to store the fingerprints electronically; this would facilitate transmission of the information and the search in the archive. We can consider a fingerprint as a small picture, so a natural idea is to split each square-inch into, say,  $256 \times 256$  pixels, to which we associate a grey-tone on a scale from for example 0 (completely white) to 256 (completely black). This way we have kept the essential information in the form of a sequence of pairs of numbers, namely, the pairs consisting of a numbering of the pixels and the associated grey-tones. This sequence can easily be stored and transmitted electronically, that is, it can if necessary be sent rapidly to San-Francisco and be compared with a given fresh fingerprints. This procedure will represent each fingerprint by a sequence of numbers which use 10 Mb, i.e. information corresponding to about 10 standard diskettes. The FBI has more than 30 million set of fingerprint (each consisting of 10 fingers) and receive each 30000 fingerprints every day. There we are speaking about the tremendous set of data, and it is necessary to do some compression in order to be handle them. There we, use wavelet transform in data compression to handle it.

### 5.1.3 Denoising noisy data

In different fields, from planetary science to molecular spectroscopy, scientists are faced with the problem of recovering a true signal incomplete, indirect or noisy data. Can wavelets help to solve this problem? The answer is certainly yes, through a technique called, *wavelet shrink age and thresholding*, that David Donoho of Stanford University has work on for a number of years [12]. The technique work in the following way. When you decompose a data set using wavelets you use filters and other tools that produce details some of the resulting wavelet coefficient correspond to detail in the data set. If the details are small, they might be omitted without substantially affecting the main feature of the data set. The idea of thresholding, then, is to set all coefficients, to zero that are less than a particular threshold. These coefficients are used in inverse wavelet transformation to reconstruct the data set. This technique is best for handling noisy data set.

### 5.1.4 Musical tones

Victor Wickerhauser has suggested that wavelet packets could be useful in sound synthesis. Wickerhauser feels that sound synthesis is a natural use of wavelets. For example, you want to approximate the sound of musical instrument, a sample of the notes produce by the instrument could be decomposed into wavelet packets coefficients. Reproducing the notes would then require reloading those coefficients in to a wavelet packet generator and playing back the result. The intensity variation of how the sound starts and ends- could be controlled separately, or by using longer wave packets.

### 5.1.5 Computer graphics

The construction and manipulation of curves and surface is core area of computer graphics. Basic operation such as interactive editing, variational modeling and compact representation of geometry, provide many opportunities to take advantage of unique feature of wavelet transform. Similar observations apply to the area of animation.

### 5.1.6 ECG

The ECG is nothing but the recording of the heart's electrical activity. The earlier method of ECG signal analysis is based on the time domain method. So the frequency representation methods are required. So Fast Fourier Transform (FFT) is applied. But the unavoidable limitation of the FFT is that the technique fails to provide the information regarding the exact location of frequency domain in time. As the frequency content of the ECG varying in time, for this Short Term Fourier Transformation (STFT) is available on that time. But the major drawback of the STTF is time frequency precision is not optimal. Hence we choose a more suitable technique to overcome from this drawback. For this, wavelet transform is more suitable, simple and valuable. At present Db 6 is use for ECG.

### 5.1.7 Breast cancer [13]

Mammography is one of the principal kinds of the medical images which are considered as the most efficient for the detection of breast cancer at its first step. Microclassification and clustered microclassification are known to be first sign of development of an eventual cancer. They appear as small and bright regions with irregular shape in breast.

Their diversity in their shape, their orientation, their size, and localization in a dense mammogram are the cause of the major difficulty for their classification. The aim of our scheme is the development of a method for the detection and classification of all type micro classification. The wavelet transform and multiresolution is powerful tool for non stationary signal analysis. Due to multiresolution analysis, different resolution levels which are sensitive to different frequency band by choosing an appropriate wavelet with a right resolution level, we can effectively detect the micro classifications in digital mammogram. By wavelet transform, different resolution levels are explored for detecting the micro classification. Discrete wavelet transform is used to provide a method for enhancing and controlling the detection of all type of small scale object, microclassification, and separating them from large background structures. Daubechies wavelet with 4<sup>th</sup> level of decomposition achieves the best detective result.

### 5.1.8 Face recognition

Face recognition has recently obtained significant attention. It plays an important role in many application areas, such as human machine interaction, authentication and surveillance. However a large range variation of human face, due to pose, illumination, and expression, result in a extremely complex distribution and deteriorate the recognition performance. In addition, the problem of machine recognition of human faces continue to attract researchers from discipline such as image processing, pattern recognition, neural network, computer vision, computer graphics and psychology. Wavelet transform has been used successively in image processing. Its ability to capture localized time-frequency information of image motivates its use for its feature extraction. The decomposition of the data into different frequency range allows us to isolate the frequency component introduced by intrinsic deformations due to expression or extrinsic factors (like illumination) into certain subbands. Wavelet-based methods prune away these variable subbands, and focus on the subbands that contain the most relevant information to better represent the data. For more details read [14]. In general daubechies wavelet is use for face recognition.

## 6 Conclusion

This paper gives a comprehensive overview of both the fundamentals of wavelet transform and related tools, and of the most active recent developments towards applications. It offers a state-of-the-art in several active areas of research where numerical methods for solving differential equations, integral equation and integro differential equation have proved particularly effective. Finally, it has been pointed out that the Haar and Daubechies wavelet transform method used presently in virtually every conceivable area of engineering and science that can make use of models of nature characterized by differential and integral equations. In this paper, we introduc the important wavelet transform (Haar wavelet transform and Daubechies wavelet transform) used in the solution of the differential and integral equations, which are important for the development of new research in the field of numerical analysis and beneficial for new researchers. This paper shows different generalization and application of Haar function and Haar and Daubechies transform. The benefits of Haar wavelet approach are sparse matrices of representation, fast transformation and possibility of implementation of fast algorithms. Most of references used in this paper are of great practical importance. The main goal of this paper was to demonstrate that the Haar wavelet and Daubechies wavelet methods are powerful tool for solving different type problems arise in engineering and sciences. Finally, it can be observe from this paper that a significant amount of work is correctly being done on differential, integral and integro-differential equations by using these methods.

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