**ISSN** 1749-3889 (print), 1749-3897 (online)
**International Journal of Nonlinear Science**
Vol.13(2012) No.3,pp.353-356

# The Central Variation of Fractal Interpolation Surfaces Derived from Fractal Interpolation Functions

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*(Received 3 January 2012, accepted 27 March 2012)*

**Abstract:** A construction method of fractal interpolation surface on a rectangular domain with arbitrary interpolation nodes is introduced. The variation properties of the bivariate functions corresponding to this type of fractal interpolation surfaces are discussed.

**Keywords:** affine fractal interpolation function; variation; fractal interpolation surface

## 1 Introduction

Barnsley[1,2] proposed the concept of the fractal interpolation function, then a new and more advantageous interpolation method is provided for simulating the rough and irregular curves. Xie et al.[3,4] proposed a mathematical model of the fractal interpolation surfaces on a rectangular domain they derived the box-counting dimension formula of the fractal interpolation surfaces. Dalla[5] gave a method of constructions of bivariate fractal interpolation functions. Dalla' results improve and correct a construction quoted in the paper[3]. Feng[6] studied the continuity condition for the fractal interpolation surface on rectangular domain, and discussed the variation properties of the corresponding bivariate continuous functions. They also obtained the box-counting dimension of the fractal interpolation surface. In these papers, in order to ensure the fractal interpolation surface continuous, the interpolation data on each edge is assumed to be collinear, or they must meet the conditions which are difficult to verify. Malysz[7] used some reflections to construct the iterated function system with same contraction factors, and then proposed a new construction method of fractal interpolation surface for arbitrary interpolation data. Based on the construction of recurrent fractal interpolation functions, Bouboulis and Dalla[8] presented a new construction of fractal interpolation surfaces for arbitrary interpolation data on the grids. They also proved some properties of the fractal interpolation surfaces, and provided a lower bound of their box-counting dimension. Bouboulis and Dalla[8] gave a method for construction of a class of fractal interpolation surfaces on the rectangle domain derived from affine fractal interpolation functions. In this article, we discuses some properties of this class of fractal interpolation surfaces, especially about their variations.

## 2 Fractal interpolation surfaces derived from fractal interpolation functions

Let $I = [a, b]$. For a number $m \in N^+$ and $m \geq 2$, let $\{(x_i, z_i) : i = 0, 1, \cdots, m\}$ be a data set on $I \times R$, where $a = x_0 < x_1 < \cdots < x_m = b$. For a given real array $\{s_i\}_{i=1}^{m}$, where $s = \max\{|s_i| : i = 1, 2, \cdots, m\} < 1$, called vertical scaling factors, we define the affine mappings from $I \times R$ to $I \times R$:

$$\omega_i \begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} L_i(x) \\ F_i(x, z) \end{pmatrix} = \begin{pmatrix} a_i & 0 \\ c_i & s_i \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} b_i \\ d_i \end{pmatrix}, \quad (1)$$

where $a_i = (x_i - x_{i-1})/(x_m - x_0)$, $b_i = x_{i-1} - x_0 (x_i - x_{i-1})/(x_m - x_0)$,
$c_i = (z_i - z_{i-1})/(x_m - x_0) - s_i(z_m - z_0)/(x_m - x_0)$,

$$d_i = (x_m z_{i-1} - x_0 z_i)/(x_m - x_0) - s_i(x_m z_0 - x_0 z_m)/(x_m - x_0),$$

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IJNS.2012.06.15/614

for $i = 1, 2, \cdots, m$. Then an iterated function system associated with the interpolation knots $\{(x_i, z_i) : i = 0, 1, \cdots, m\}$

$$\{I \times R; \omega_i, i = 1, 2, \cdots, m\} \tag{2}$$

is constructed. According to the paper [1,2]we have the following theorems.

**Theorem 1** *If the vertical scaling factors $|s_i| < 1$, $i = 1, 2, \cdots, m$, then the iterated function system (2) has a unique invariant set $K \subset R^2$. it is a graph of a continuous function $f$ on I, which interpolates the data set $\{(x_i, z_i) : i = 0, 1, \cdots, m\}$. That is, $K = \{(x, z) : z = f(x), x \in I\}$, where $f(x_i) = z_i$, for $i = 0, 1, 2, \cdots, m$. If $\sum_{i=1}^{m} |s_i| > 1$ and the interpolation data do not all lie on a single straight line, then the box-counting dimension of $K$ is the unique real solution $D$ of*

$$\sum_{i=1}^{m} |s_i| \cdot a_i^{D-1} = 1 \tag{3}$$

Otherwise the box-counting dimension of $K$ is 1. This $f$ is called affine fractal interpolation function. $f$ is the affine fractal interpolation fractal function generated by the iterated function system (2) if and only if for $i = 1, 2, \cdots, m$, the $f$ satisfied the equations

$$f(L_i(x)) = F_i(x, f(x)), x \in I. \tag{4}$$

Now let$G = [a, b] \times [c, d]$ be a rectangular domain in $R^2$, $\{(x_i, y_j, z_{i,j}) : i = 0, 1, \cdots, m; j = 0, 1, \cdots, n\}$ be a data set in $R^3$, where $a = x_0 < x_1 < \cdots < x_m = b$ and $c = y_0 < y_1 < \cdots < y_n = d$. Let $u_i(y)$ $(i = 0, 1, \cdots, m)$, be $m+1$continuous functions on $J = [c, d]$, satisfying the interpolation conditions $u_i(y_j) = z_{i,j}$, for $j = 0, 1, \cdots, m$ .

For any $y \in [a, b]$, we can get an affine fractal interpolation function $g_y(x)$, which interpolates the data set

$$\{(x_i, u_i(y)) : i = 0, 1, \cdots, m\}$$

, i.e. $g_y(x_i) = u_i(y)$ for $i = 0, 1, 2, \cdots, m$. Let $f(x, y) = g_y(x)$, for $(x, y) \in [a, b] \times [c, d]$. Then $f(x, y)$is a bivariate function on $G$, and satisfies the interpolation condition $f(x_i, y_j) = z_{i,j}$, for $i = 0, 1, 2, \cdots, m$,$j = 0, 1, 2, \cdots, n$. It has been proved in [8] that the function $f$ is continuous on $G$. The graph of the function $z = f(x, y)$, $(x, y) \in G$, is called fractal interpolation surface derived from interpolation functions. For this kind of the fractal interpolation surfaces, we have the following result.

**Theorem 2** *Let $u_i$ , $i = 0, 1, \cdots, m$ be all continuous functions on $[c, d]$, if there exits $y_0 \in [c, d]$ such that the data set $\{(x_i, u_i(y)) : i = 0, 1, \cdots, m\}$ is not collinear, then there exists interval $[\alpha, \beta] \subset [c, d]$ ,such that the data set $\{(x_i, u_i(y)) : i = 0, 1, \cdots, m\}$is not collinear for any $y \in [\alpha, \beta]$.*

**Proof.** Let $\{(x_i, u_i(y_0)) : i = 0, 1, \cdots, m\}$ is not collinear for $y_0 \in [c, d]$. Then there exists an $i_0 \in \{1, 2, \cdots, m-1\}$, such that

$$u_{i_0}(y_0) - \left[u_0(y_0) + \frac{x_{i_0} - x_0}{x_m - x_0}(u_m(y_0) - u_0(y_0))\right] \neq 0$$

Because function $F(y) = u_{i_0}(y) - \left[u_0(y) + \frac{x_{i_0} - x_0}{x_m - x_0}(u_m(y) - u_0(y))\right]$ is a continuous function on $[c, d]$ and $F(y_0) \neq 0$, there exists a interval $[\alpha, \beta]$, such that $F(y) \neq 0$ for$y \in [\alpha, \beta]$ Therefore, the data set $\{(x_i, u_i(y)) : i = 0, 1, \cdots, m\}$ is not collinear for any $y \in [\alpha, \beta]$. ■

# 3 The central variation of the fractal interpolation surface

Let $I = [a, b]$, $f$ be a continuous function on I. For a real non-negative number $\gamma$ and any$x \in I$, denote$I[x, \gamma] = I \cap [x - \gamma, x + \gamma]$, then the value

$$O^I_{f;\gamma}(x) = \sup_{x' \in I[x,\gamma]} |f(x') - f(x)|$$

is $\gamma-$called central oscillation of function $f$ at the point $x$ about I. It is also denoted by $O_{f;\gamma}(x)$ for short. Because $f$ is a continuous function on the closed domain G, it is obvious that $O_{f;\gamma}(x)$ is a continuous function of $x$ on I. Then $O_{f;\gamma}(x)$ is Riemann integrable. The Riemann integral of $O_{f;\gamma}(x)$ on I,$\int_I O_{f;\gamma}(x)dx$, is called $\gamma-$ central variation of function $f$ on the domain I, denoted by $V_{f;\gamma}(I)$.

In order to discuss the properties of central variation of the fractal interpolation surface derived from interpolation functions, we give a lemma first.

**Lemma 3** *(1) If $g(x)$ is a differential function on the interval $I$, then* $0 \le V_{g;\gamma}(I) \le \sup_{x\in I}|g'(x)| \cdot |I| \cdot \gamma$.
*(2) If $\tau(t) = \lambda t + \varsigma, t \in I$ and $\lambda \ne 0$ and $g(x)$ is a continuous function on* $\tau(I)$*, then* $V_{g\circ\tau;\gamma}(I) = \frac{1}{|\lambda|}V_{g;|\lambda|\gamma}(\tau(I))$.
*(3) Let $g$ be a continuous function on $I = [a,b]$, $a = x_0 < x_1 < \cdots < x_m = b$, and $I_i = [x_{i-1}, x_i]$, then*

$$\sum_{i=1}^{m} V_{g;\gamma}(I_i) \le V_{g;\gamma}(I) \le \sum_{i=1}^{m} V_{g;\gamma}(I_i) + 2(m-1)V_g(I)\gamma$$

*where* $V_g(I) = \sup_{x\in I} g(x) - \inf_{x\in I} g(x)$.

**Proof.** (1) Because

$$0 \le O^I_{g;\gamma}(x) = \sup_{x'\in I[x;\gamma]} |g(x') - g(x)| \le \sup_{x'\in I[x;\gamma]} |g'(\xi)||x'-x| \le \sup_{x\in I}|g'(\xi)| \cdot \gamma$$

integrating it on $I$, (1) is proved.
(2) Let $x = \tau(t)$, for $t \in I$. Because

$$O^I_{g\circ\tau;\gamma}(t) = \sup_{t'\in I[t;\gamma]} |g\circ\tau(t') - g\circ\tau(t)| = \sup_{x'\in L(I)[x;|\lambda|\gamma]} |g(x') - g(x)| = O^{L(I)}_{g;|\lambda|\gamma}(x)$$

then $V_{g\circ\tau;\gamma}(I) = \int_I O^I_{g\circ\tau;\gamma}(t)dt = \frac{1}{|\lambda|}\int_{L(I)} O^{L(I)}_{g;|\lambda|\gamma}(x)\,dx = \frac{1}{|\lambda|}V_{g;|\lambda|\gamma}(\tau(I))$.
(3) For any $i = 1,2,\cdots,m$, $x \in I_i$, we have $I_i[x;\gamma] \subseteq I[x;\gamma]$,then$O^{I_i}_{g;\gamma}(x) \le O^I_{g;\gamma}(x)$. therefore the left inequality is true. Now let we prove the right one.
Let $I^c_1 = [\alpha, x_1-\gamma]$, $I^c_m = [x_{m-1}+\gamma, b]$, $I^c_i = [x_{i-1}+\gamma, x_i-\gamma]$, $i = 2,3,\cdots,m-1$, and $I^s_i = [x_i-\gamma, x_i+\gamma] \cap [a,b]$, $i = 1,2,3,\cdots,m-1$, where assuming $[\alpha,\beta]$ is an empty set if $\alpha > \beta$.Then $\cap^m_{i=1} I^c_i \cap \cap^{m-1}_{i=1} I^c_i = [a,b]$. It is obvious that $O^I_{g;\gamma}(x) = O^{I_i}_{g;\gamma}(x)$ for $x \in I^c_i$, and $O^I_{g;\gamma}(x) \le V_g(I)$ for $x \in I^s_i$. Therefore

$$V_{g;\gamma}(I) \le \sum\nolimits_{i=1}^{m} \int_{I^c_i} O^{I_i}_{g;\gamma}(x)dx + \sum\nolimits_{i=1}^{m-1} \int_{I^s_i} V_g(I)\,dx \le \sum\nolimits_{i=1}^{m} V_{g;\gamma}(I_i) + (m-1)V_g(I)\cdot 2\gamma.$$

The right inequality is also true. ■
Let $u_i$, $i = 0,1,\cdots,m$ be continuous functions on $[c,d]$, and for $y \in [c,d]$, the sections $f(\cdot,y)$ be affine fractal interpolation function with the data set$\{(x_i, u_i(y)) : i = 0,1,\cdots,m\}$ and vertical scaling factors $\{s_i : i = 1,2,\cdots,m\}$. According to the method given in Section 4, we can obtain a fractal interpolation surface $z = f(x,y)$, $(x,y) \in [a,b] \times [c,d]$.

**Theorem 4** *Let $f(x,y)$, $(x,y) \in D$, be a fractal interpolation surface constructed above. There exist positive real numbers $\beta_1$ and $\beta_2$, such that for any $\gamma \ge 0$ any $y \in [c,d]$,*

$$\sum_{i=1}^{m} |s_i| a_i V_{f(\cdot,y);\frac{\gamma}{a_i}}(I) - \beta_1\gamma \le V_{f(\cdot,y);\gamma}(I) \le \sum_{i=1}^{m} |s_i| a_i V_{f(\cdot,y);\frac{\gamma}{a_i}}(I) + \beta_2\gamma \tag{5}$$

**Proof.** Because $f(\cdot,y)$ is an affine fractal interpolation function with the data
set $\{(x_i, u_i(y)) : i = 0,1,\cdots m\}$ and the vertical scaling factors $\{s_i : i = 1,2,\cdots,m\}$. According to Theorem 2.1, if $x_i \in I_i$, $f(x,y) = s_i f\left(L^{-1}_i(x), y\right) + p_{i,y}\left(L^{-1}_i(x)\right)$, where $p_{i,y}(t) = c_{i,y}t + d_{i,y}$, for $i = 1,2,3,\cdots,m$. Because $L^{-1}_i(x) = (x - b_{i,y})/a_{i,y}$ for $x \in I_i$, with Lemma 3.1 we have

$$\begin{aligned} V_{f(\cdot,y);\gamma}(I) &\ge \sum\nolimits_{i=1}^{m} \left[|s_i| V_{f\left(L^{-1}_i(\cdot),y\right);\gamma}(I_i) - V_{p_{i,y}\circ L^{-1}_i;\gamma}(I_i)\right] \\ &= \sum\nolimits_{i=1}^{m} \left[|s_i| a_i V_{f(\cdot,y);\gamma/a_i} - a_i V_{p_{i,y};\gamma/a_i}(I)\right] \\ &\ge \sum\nolimits_{i=1}^{m} |s_i| a_i V_{f(\cdot,y);\gamma/a_i}(I) - \sum\nolimits_{i=1}^{m} \max\{|c_{i,y}| : i = 1,2,\cdots,m\}|I|\gamma. \end{aligned}$$

On the other hand, with Lemma 3.1, we have

$$\begin{aligned} &V_{f(\cdot,y);\gamma}(I) \le \sum\nolimits_{i=1}^{m} \left[|s_i| V_{f\left(L^{-1}_i(\cdot),y\right);\gamma}(I_i) + V_{p_{i,y}\circ L^{-1}_i;\gamma}(I_i)\right] + 2(m-1)V_{f(\cdot,y)}(I)\gamma \\ &\le \sum\nolimits_{i=1}^{m} |s_i| a_i V_{f(\cdot,y);\gamma/a_i}(I) + \left(\sum\nolimits_{i=1}^{m} \max\{|c_{i,y}| : i = 1,2,\cdots,m\}|I| + 2(m-1)V_{f(\cdot,y)}(I)\right)\gamma \end{aligned}$$

According to Eqn (1), $c_{i,y} = [u_i(y) - u_{i-1}(y) - s_i(u_m(y) - u_0(y))] / (x_m - x_0)$, and $u_i$ are all continuous on $[c, d]$, so there exists $M > 0$, such that$|c_{i,y}| \le M$ for any $y \in [c, d]$ and $i \in \{1, 2, \cdots, m\}$. And because $V_{f(\cdot,y)}(I) \le \sup_{(x,y)\in D} f(x, y) - \inf_{(x,y)\in D} f(x, y) = V_f(D)$ ,then we can let$\beta_1 = mM|I|$ and $\beta_2 = \beta_1 + 2(m-1)V_f(D)$, which are not dependent on $y$ and $\gamma$. The proof of the theorem is complete. ■

**Theorem 5** *Let* $\Lambda = \sum_{i=1}^{m}|s_i| > 1$, $\underline{a} = \min\{a_i : i = 1, 2, \cdots, m\}$*and for any* $y \in [\alpha, \beta] \subseteq [c, d]$ *the data set*$\{(x_i, u_i(y)) : i = 0, 1, \cdots, m\}$*be not collinear. Then there exists positive constant C such that for any* $y \in [\alpha, \beta]$*, and any* $\gamma \in [0, \underline{a}^k]$*, we have*

$$V_{f(\cdot,y);\gamma}(I) \ge C \cdot \Lambda^k \cdot \gamma. \tag{6}$$

**Proof.** Let $l(x, y) = u_0(y) + \frac{x - x_0}{x_m - x_0}(u_m(y) - u_0(y))$. For any$y \in [\alpha, \beta]$, $l(\cdot, y)$ is a linear function on I, satisfying $l(a, y) = f(a, y)$ and $l(b, y) = f(b, y)$. let $D(y) = \sup_{x\in I}|l(x, y) - f(x, y)|$ , then $D(y)$ is positive and continuous on $[\alpha, \beta]$, therefore $D = \inf_{y\in[\alpha,\delta]} D(y) > 0$.

For $y \in [\alpha, \beta]$ and $i_1 \in \{1, 2, \cdots, m\}$,let $l_{i_1}(x, y) = s_{i_1} l\left(L_{i_1}^{-1}(x), y\right) + c_{i_1,y} L_{i_1}^{-1}(x) + d_{i_1,y}$. Is a linear function on $I_{i_1} = L_{i_1}(I)$ and $f(L_{i_1}(x_0), y) = l_{i_i}(L_{i_1}(x_0), y)$, $f(L_{i_1}(x_m), y) = l_{i_i}(L_{i_1}(x_m), y)$ . With Theorem 2.1, $D_{i_1}(y) = \sup_{x\in Li_1(I)}|l_{i_1}(x, y) - f(x, y)|$
$= \sup_{x\in L_{i_1}(I)}|s_{i_1}|\left|l\left(L_{i_1}^{-1}(x), y\right) - f\left(L_{i_1}^{-1}(x), y\right)\right| = \sup_{\xi\in I}|s_{i_1}||l(\xi, y) - f(\xi, y)|$
$= |s_{i_1}| \cdot D(y) \ge D \cdot |s_{i_1}|$ .

Recursively defining $l_{i_k,i_{k-1},\cdots,i_1}(x, y) = s_{i_k} l_{i_{k-1},i_{i-2},\cdots,i_1}\left(L_{i_k}^{-1}(x), y\right) + c_{i_k,y} L_{i_k}^{-1}(x) + d_{i_k,y}$, for $i_k \in \{1, 2, \cdots, m\}$, $k = 1, 2, \cdots$ . With mathematical induction, we can prove that $l_{i_k,i_{k-1},\cdots,i_1}(x, y)$ is a linear function on $L_{i_k,i_{k-1},\cdots,i_1}(I) = L_{i_k} \circ L_{i_{k-1}} \circ \cdots \circ L_{i_1}(I)$ , $f\left(L_{i_k,i_{k-1},\cdots,i_1}(x_0, y)\right)$
$= l_{i_k,i_{k-1},\cdots,i_1}\left(L_{i_k,i_{k-1},\cdots,i_1}(x_0), y\right)$ , $f\left(L_{i_k,i_{k-1},\cdots,i_1}(x_m), y\right) = l_{i_k,i_{k-1},\cdots,i_1}\left(L_{i_k,i_{k-1},\cdots,i_1}(x_m), y\right)$ ,and

$$D_{i_k,i_{k-1},\cdots,i_1}(y) = \sup_{x\in L_{i_k,,i_{k-1},\cdots,,i_1}(I)}\left|l_{i_k,i_{k-1},\cdots,i_1}(x, y) - f(x, y)\right| = \left|s_{i_k,i_{k-1},\cdots,i_1}\right| D(y) \ge D \cdot \left|s_{i_k,i_{k-1},\cdots,i_1}\right|$$

For a continuous function $g(t)$on$[t_1, t_2]$, if $0 \le \gamma \le t_2 - t_1$, then $V_{g;\gamma}([t_1, t_2]) \ge V_g([t_1, t_2]) \cdot \gamma$,where $V_g([t_1, t_2]) = \sup_{t\in[t_1,t_2]} g(t) - \inf_{t\in[t_1,t_2]} g(t)$. It is obvious that $V_{f(\cdot,y)}\left(L_{i_k,i_{k-1},\cdots,i_1}(I)\right) \ge D_{i_k,i_{k-1},\cdots,i_1}(y) \ge D \cdot \left|s_{i_k} s_{i_{k-1}} \cdots s_{i_1}\right|$.

Therefore when $\gamma \in [0, \underline{a}^k]$, with Lemma 3.1(3),

$$V_{f(\cdot,y);\gamma}(I) \ge \sum_{i_k,i_{k-1},\cdots,i_1=1}^{m} V_{f(\cdot,y);\gamma}\left(L_{i_k,i_{k-1}\cdots i_1}(I)\right)$$

$$\ge \sum_{i_k,i_{k-1},\cdots,i_1=1}^{m} D \cdot \left|s_{i_k} s_{i_{k-1}} \cdots s_{i_1}\right| \cdot \gamma = D \cdot \Lambda^k \cdot \gamma$$

The proof of Theorem 3.2 is complete. ■

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