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Nonpolynomial Septic Splines Approach to the Solution of Fourth-order Two Point Boundary Value Problems

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Abstract: In this paper, we develop nonpolynomial septic spline functions for obtaining smooth approximations to the numerical solution of fourth-order two point boundary value problems occurring in a plate deflection theory. Direct methods of second, fourth and sixth order are developed. It is shown that the present method gives approximations, which are better than other existing methods. Three examples are given to demonstrate the practical usefulness and efficiency of our methods.

Keywords: Nonpolynomial septic splines; Fourth-order boundary value problems; Spline function approximation; Plate deflection theory

1 Introduction

We consider the problem of bending a rectangular clamped beam of length l resting on elastic foundation. The vertical deflection w of the plate satisfies the system

$$\left.\begin{array}{c}[L+(\frac{K}{D})]w = D^{(-1)}q(x),\;\; L\equiv d^4/dx^4,\\ w(0)=w(l)=w'(0)=w'(l)=0,\end{array}\right\} \tag{1.1}$$

where D is the flexural rigidity of the beam, k is the spring constant of elastic foundation and the load $q(x)$ acts vertically downwards per unit length of the beam. The details of the mechanical interpretation are given in [13]. Mathematically, the system (1.1) belongs to special class of boundary value problems of the form:

$$\left.\begin{array}{c}y^{(4)}+f(x)y=g(x),\; f(x)\geq 0,\; x\in[a,b],\\ y(a)=A_0,\; y(b)=B_0,\\ y'(a)=A_1,\; y'(b)=B_1,\end{array}\right\} \tag{1.2}$$

where $A_i,\ B_i\ (i=0,1)$ are finite real arbitrary constants. The functions $f(x)$ and $g(x)$ are continuous defined in the interval $x\in[a,b]$. Two point boundary value problems of type (1.2) arise in the plate deflection theory such as the problem of the bending of a uniformly loaded rectangular plate supported over the entire surface by an elastic foundation and rigidly supported along the edges [17]. The analytical solution of system (1.2) for special choices of $f(x)$ and $g(x)$ are easily obtained, but for arbitrary choices the analytical solution can not be determined. Facing with this difficulty we resort a numerical method to find its solution. Usmani [19] has proved that the BVP (1.2) possesses a unique solution provided

$$\inf_{a\leq x\leq b} f(x) = -\eta \geq -\frac{\sigma}{(b-a)^4}, \tag{1.3}$$

where σ=500.5639....

Details of theorems which list the conditions for the existence and uniqueness of solution of such problems are thoroughly discussed in a book by Agarwal [1]. For a brief introduction on the subject by using spline functions for the

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treatment of ordinary differential equations the reader is referred to Ahlberg et al. [2].

Methods of order two and four for the boundary value problem (1.2) based on quintic and sextic spline functions were developed by Usmani [20]. Rashidinia and Aziz [10] derived the quintic spline method for the solution of linear fourth-order boundary value problem (1.2). Later, Rashidinia and Golbabaee [11] discussed convergence of numerical solution of fourth-order boundary value problem.

Chawla and Subramanian [4] given a high accuracy quintic spline solution of the fourth-order two point boundary value problem. Katti [7] discussed five diagonal sixth-order methods for two point boundary value problems involving fourth-order differential equation. Quintic spline collocation method for general fourth-order two point boundary value problems studied by Russell and Shampine [14].

Siddiqi and Akram [15] generated a difference scheme via quintic spline function while Siraj-ul-Islam et al. [16] used second, fourth and sixth-order methods for solving fourth-order boundary value problems (1.2) based on nonpolynomial quintic splines. Recently, Ramadan et al. [8] used a nonpolynomial septic spline function for the numerical solution of sixth-order two point boundary value problems.

Several authors [5, 6, 9, 12, 18, 21, 22] have solved boundary value problem (1.2) prescribed with boundary conditions at the second derivative, such as

$$y(a) = A_0,\ y(b) = B_0,\ y''(a) = A_2,\ y''(b) = B_2.$$

In the present paper, nonpolynomial septic spline functions are applied, which have a polynomial and trigonometric part to develop a family of numerical methods of order two, four and six for obtaining smooth approximations to the solution of fourth-order two point boundary value problems. The new methods perform better than other collocation, finite-difference and spline methods and thus represent an improvement over existing methods. This approach has the advantage that it provides an approximation not only for $y(x)$ at the nodal points but also $y^{(\mu)}(x),\ \mu = 1, 2, \cdots, 5$ at every point in the range of integration.

The nonpolynomial septic spline function proposed in this paper has the form:

$$T_7 = span(1,\ x,\ x^2,\ x^3,\ x^4,\ x^5,\ \sin kx,\ \cos kx),$$

where k is the frequency of the trigonometric part of the spline functions which can be real or pure imaginary. It can be used to raise the accuracy of the method.

The outline of this paper is organised into six sections. In section 2, we have developed the consistency relation in terms of the value of spline and its fourth derivative by using derivative continuities at knots. In section 3, end conditions required to complete the definition of spline are derived. In section 4, we have obtained a class of methods of order two, four and six. In section 5, nonpolynomial septic spline solution approximating the analytic solution of BVP (1.2) is determined, using the consistency relation involving the fourth-order derivatives and the values of the spline along with the end conditions. In section 6, three examples for linear and nonlinear boundary value problems are considered to illustrate the accuracy and performance of the method developed in the paper.

2 Derivation of the method

In order to develop the spline approximation to the fourth-order boundary value problem (1.2), the interval $[a, b]$ is divided into n equal subintervals. For this we introduce the set of grid points $x_i = a + ih, i = 0, 1, ..., n$ so that

$$x_0 = a,\ x_n = b \text{ and } h = \frac{(b-a)}{n}. \tag{2.1}$$

Let $y(x)$ be the exact solution of the system (1.2) and y_i be an approximation to $y(x_i)$, obtained by the segment $S_i(x)$ of the mixed spline function passing through the points (x_i, y_i) and (x_{i+1}, y_{i+1}). For each i^{th} segment, the nonpolynomial septic spline function $S_i(x)$ in subinterval $[x_i, x_{i+1}], i = 0, 1, \cdots, n-1$ has the form:

$$\begin{aligned} S_i(x) \quad = \quad & a_i + b_i(x-x_i) + c_i(x-x_i)^2 + d_i(x-x_i)^3 + e_i(x-x_i)^4 + f_i(x-x_i)^5 \\ & + g_i \sin k(x-x_i) + l_i \cos k(x-x_i), \end{aligned} \tag{2.2}$$

where $a_i,\ b_i,\ c_i,\ d_i,\ e_i,\ f_i,\ g_i,\ l_i$ are arbitrary real and finite constants and k is a free parameter. If $k \to 0$, then the nonpolynomial septic spline $S_i(x)$ reduces to ordinary septic spline [3].

To determine the eight coefficients of integration of equation (2.2) in terms of $y_i,\ y_{i+1},\ M_i,\ M_{i+1},$ $F_i,\ F_{i+1},\ Q_i,\ Q_{i+1}$, we define the interpolatory conditions at x_i and x_{i+1}

$$\left.\begin{aligned} S_i(x_i) &= y_i,\ \ S_i(x_{i+1}) = y_{i+1},\\ S_i''(x_i) &= M_i,\ \ S_i''(x_{i+1}) = M_{i+1},\\ S_i^{(4)}(x_i) &= F_i,\ \ S_i^{(4)}(x_{i+1}) = F_{i+1},\\ S_i^{(6)}(x_i) &= Q_i,\ \ S_i^{(6)}(x_{i+1}) = Q_{i+1}. \end{aligned}\right\} \tag{2.3}$$

We obtain via a long but straightforward calculation

$$\begin{aligned} a_i &= y_i + \frac{Q_i}{k^6},\\ b_i &= \frac{1}{h}(y_{i+1}-y_i) - \frac{h}{6}(M_{i+1}+2M_i) + \frac{h^3}{360}(7F_{i+1}+8F_i) + \frac{h^5}{\theta^6}(Q_{i+1}-Q_i)\\ &\quad + \frac{h^5}{6\theta^4}(Q_{i+1}+2Q_i) + \frac{h^5}{360\theta^2}(7Q_{i+1}+8Q_i),\\ c_i &= \frac{M_ik^4 - Q_i}{2k^4},\\ d_i &= \frac{1}{6h}(M_{i+1}-M_i) - \frac{h}{36}(F_{i+1}+2F_i) - \frac{h^3}{36\theta^2}(Q_{i+1}+2Q_i) - \frac{h^3}{6\theta^4}(Q_{i+1}-Q_i),\\ e_i &= \frac{F_ik^2+Q_i}{24k^2},\\ f_i &= \frac{1}{120h}(F_{i+1}-F_i) + \frac{h}{120\theta^2}(Q_{i+1}-Q_i),\\ g_i &= \frac{Q_i\cos\theta - Q_{i+1}}{k^6\sin\theta},\\ l_i &= -\frac{Q_i}{k^6},\ \ \theta = kh\ \ and\ \ i = 0(1)n-1. \end{aligned}$$

Using the continuity conditions of the first, third and fifth derivatives, that is $S_{i-1}^{(\mu)}(x_i) = S_i^{(\mu)}(x_i),\ \mu$=1, 3 and 5, we get the following consistency relations [8]

$$\begin{aligned} M_{i+1}+4M_i+M_{i-1} &= \tfrac{6}{h^2}(y_{i+1}-2y_i+y_{i-1}) + \tfrac{h^2}{60}(7F_{i+1}+16F_i+F_{i-1})\\ &\quad +6h^4(\alpha_2Q_{i+1}+2\beta_2Q_i+\alpha_2Q_{i-1}),\ \ i=1(1)n-1, \end{aligned} \tag{2.4}$$

$$\begin{aligned} M_{i+1}-2M_i+M_{i-1} &= \tfrac{h^2}{6}(F_{i+1}+4F_i+F_{i-1})\\ &\quad +h^4(\alpha_1Q_{i+1}+2\beta_1Q_i+\alpha_1Q_{i-1}),\ \ i=1(1)n-1, \end{aligned} \tag{2.5}$$

$$F_{i+1}-2F_i+F_{i-1} = h^2(\alpha Q_{i+1}+2\beta Q_i+\alpha Q_{i-1}),\ \ i=1(1)n-1, \tag{2.6}$$

where

$\alpha_2 = \frac{7}{360\omega^2} + \frac{1}{\omega^2}\left\{\frac{1}{6\omega^2} - \frac{1}{\omega^4}(\omega\csc\omega - 1)\right\},\quad \beta_2 = \frac{8}{360\omega^2} + \frac{1}{\omega^2}\left\{\frac{1}{3\omega^2} - \frac{1}{\omega^4}(1-\omega\cot\omega)\right\},$

$\alpha = \frac{1}{\omega^2}\Big(\omega\csc\omega - 1\Big),\qquad \beta = \frac{1}{\omega^2}\Big(1-\omega\cot\omega\Big),$

$\alpha_1 = \frac{1}{\omega^2}\Big(\frac{1}{6}-\alpha\Big),\qquad \beta_1 = \frac{1}{\omega^2}\Big(\frac{1}{3}-\beta\Big).$

Using equations (2.4)-(2.6), we obtain the following consistency relation in terms of fourth derivative of spline F_i and y_i:

$$\begin{aligned} &6\alpha(y_{i-3}+y_{i+3}) + 12(\beta-2\alpha)(y_{i-2}+y_{i+2}) + 6(7\alpha-8\beta)(y_{i-1}+y_{i+1}) + 12(-4\alpha+6\beta)y_i\\ &= h^4(\lambda\,F_{i-3}+\mu\,F_{i-2}+\tau\,F_{i-1}+\nu\,F_i+\tau\,F_{i+1}+\mu\,F_{i+2}+\lambda\,F_{i+3}),\quad i=3,4,\cdots,n-3, \end{aligned} \tag{2.7}$$

where

$F_i = -f_iy_i + g_i$ with $f_i = f(x_i),\ g_i = g(x_i);\ \ i = 0(1)n$

and

$$\left.\begin{aligned} \lambda &= \tfrac{1}{20}\Big(\alpha - 120(\alpha_2 - \tfrac{\alpha_1}{6})\Big), \\ \mu &= \tfrac{1}{20}\Big(2(13\alpha+\beta) + 40(\alpha_1+\beta_1) + 240(2\alpha_2-\beta_2)\Big), \\ \tau &= \tfrac{1}{20}\Big((67\alpha+52\beta) - 20(5\alpha_1 - 4\beta_1) - 120(7\alpha_2 - 8\beta_2)\Big), \\ \nu &= \tfrac{1}{20}\Big(4(13\alpha+33\beta) + 80(\alpha_1 - 3\beta_1) + 480(2\alpha_2 - 3\beta_2)\Big). \end{aligned}\right\} \tag{2.8}$$

As $\omega \to 0$ then $(\alpha,\beta,\alpha_1,\beta_1,\alpha_2,\beta_2) \to (\frac{1}{6},\frac{1}{3},-\frac{7}{360},-\frac{8}{360},-\frac{31}{15120},-\frac{2}{945})$ and $(\lambda,\mu,\tau,\nu) \to (\frac{1}{840},\frac{120}{840},\frac{1191}{840},\frac{2416}{840})$, then the spline defined by (2.7) reduces into ordinary septic spline and the spline relations reduce to corresponding ordinary septic spline relations (See Ref.[3]).
The relation (2.7) gives $(n-5)$ linear algebraic equations in $(n-1)$ unknowns $(y_i,\ i=1(1)n-1)$, therefore we need four more equations, two at each end of the range of integration to have complete solution of $y_i\ 's$ appearing in equation (2.7), are derived in the following section.

3 Development of boundary equations

For the discretization of the boundary condition, we define

$$\begin{aligned} &\text{(i)} && \sum_{k=0}^{4} a_k y_k + c_1 h y_0' + h^4 \sum_{k=0}^{5} v_k y_k^{(4)} + t_1 = 0, && i=1, \\ &\text{(ii)} && \sum_{k=1}^{5} b_k y_k + c_2 h y_0' + h^4 \sum_{k=1}^{6} w_k y_k^{(4)} + t_2 = 0, && i=2, \\ &\text{(iii)} && \sum_{k=n-5}^{n-1} b_k y_k - c_2 h y_n' + h^4 \sum_{k=n-6}^{n-1} w_k y_k^{(4)} + t_{n-2} = 0, && i=n-2, \\ &\text{(iv)} && \sum_{k=n-4}^{n} a_k y_k - c_1 h y_n' + h^4 \sum_{k=n-5}^{n} v_k y_k^{(4)} + t_{n-1} = 0, && i=n-1, \end{aligned} \tag{3.1}$$

where $c_1,\ c_2,\ a_k\ , b_k,\ v_k$ and w_k are arbitrary parameters to be determined.
To obtain the local truncation error associated with the scheme (2.7), we first rewrite it in the form:

$$\begin{aligned} & y_{i-3} - 9y_{i-1} + 16y_i - 9y_{i+1} + y_{i+3} \\ &\quad = h^4(\lambda\, y_{i-3}^{(4)} + \mu\, y_{i-2}^{(4)} + \tau\, y_{i-1}^{(4)} + \nu\, y_i^{(4)} + \tau\, y_{i+1}^{(4)} + \mu\, y_{i+2}^{(4)} + \lambda\, y_{i+3}^{(4)}) + t_i, \quad i=3,4,\cdots,n-3. \end{aligned} \tag{3.2}$$

Using the Taylor's series expansion, the terms $y_{i-3}^{(4)}$, $y_{i-2}^{(4)}$, etc. are expanded around the point x_i and the expression for $t_i,\ i=3,4,\cdots,n-3$ is obtained:

$$t_i = C_4 h^4 y_i^{(4)} + C_6 h^6 y_i^{(6)} + C_8 h^8 y_i^{(8)} + C_{10} h^{10} y_i^{(10)} + C_{12} h^{12} y_i^{(12)} + O(h^{14}), \tag{3.3}$$

where

$$\left.\begin{aligned} C_4 &= (6 - 2\lambda - 2\mu - 2\tau - \nu), \\ C_6 &= (2 - 9\lambda - 4\mu - \tau), \\ C_8 &= \left(\tfrac{13104}{8!} - \tfrac{(81\lambda+16\mu+\tau)}{12}\right), \\ C_{10} &= \left(\tfrac{118080}{10!} - \tfrac{(729\lambda+64\mu+\tau)}{360}\right), \\ C_{12} &= \left(\tfrac{1062864}{12!} - \tfrac{(6561\lambda+256\mu+\tau)}{20160}\right). \end{aligned}\right\} \tag{3.4}$$

Thus for different choices of parameters $\lambda,\ \mu,\ \tau,\ \nu$ in scheme (2.7), methods of different order are obtained.

4 Numerical methods of different order

4.1 Second-order methods

In order to obtain the second-order method we find that
$(a_0, a_1, a_2, a_3, a_4, v_0, v_1, v_2, v_3, v_4, v_5) = (6, -\frac{134}{11}, \frac{111}{11}, -\frac{54}{11}, 1, \frac{19}{22}, -\frac{13}{11}, 0, 0, 0, 0)$,
$(b_1, b_2, b_3, b_4, b_5, w_1, w_2, w_3, w_4, w_5, w_6) = (5, -\frac{166}{13}, \frac{162}{13}, -\frac{74}{13}, 1, \frac{77}{65}, -\frac{17}{65}, 0, 0, 0, 0)$,
$(b_{n-5}, b_{n-4}, b_{n-3}, b_{n-2}, b_{n-1}, w_{n-6}, w_{n-5}, w_{n-4}, w_{n-3}, w_{n-2}, w_{n-1})$
$= (1, -\frac{74}{13}, \frac{162}{13}, -\frac{166}{13}, 5, 0, 0, 0, 0, -\frac{17}{65}, \frac{77}{65})$,
$(a_{n-4}, a_{n-3}, a_{n-2}, a_{n-1}, a_n, v_{n-5}, v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1}, v_n)$
$= (1, -\frac{54}{11}, \frac{111}{11}, -\frac{134}{11}, 6, 0, 0, 0, 0, -\frac{13}{11}, \frac{19}{22})$,
$(c_1, c_2) = (\frac{30}{11}, \frac{12}{13})$ and the local truncation error is

$$t_i = \begin{cases} \frac{133}{132} h^6 y_i^{(6)} + O(h^7), & i = 1, n-1, \\ \frac{57}{65} h^6 y_i^{(6)} + O(h^7), & i = 2, n-2. \end{cases} \tag{4.1}$$

Case 1: For $(\lambda,\ \mu,\ \tau,\ \nu) = (0, \frac{1}{120}, \frac{124}{120}, \frac{470}{120})$ the truncation error is given by

$$t_i = \frac{14}{15} h^6 y_i^{(6)} + O(h^8), \qquad i = 3(1)n-3. \tag{4.2}$$

Case 2: For $(\lambda,\ \mu,\ \tau,\ \nu) = (0, 0, 0, 6)$ the truncation error is given by

$$t_i = 2h^6 y_i^{(6)} + O(h^8), \qquad i = 3(1)n-3. \tag{4.3}$$

4.2 Fourth-order methods

In order to obtain the fourth-order method we find that
$(a_0, a_1, a_2, a_3, a_4, v_0, v_1, v_2, v_3, v_4, v_5) = (6, -\frac{134}{11}, \frac{111}{11}, -\frac{54}{11}, 1, \frac{2}{77}, \frac{299}{924}, -\frac{115}{231}, -\frac{157}{924}, 0, 0)$,
$(b_1, b_2, b_3, b_4, b_5, w_1, w_2, w_3, w_4, w_5, w_6) = (5, -\frac{166}{13}, \frac{162}{13}, -\frac{74}{13}, 1, \frac{1319}{2730}, \frac{879}{910}, -\frac{319}{910}, -\frac{479}{2370}, 0, 0)$,
$(b_{n-5}, b_{n-4}, b_{n-3}, b_{n-2}, b_{n-1}, w_{n-6}, w_{n-5}, w_{n-4}, w_{n-3}, w_{n-2}, w_{n-1})$
$= (1, -\frac{74}{13}, \frac{162}{13}, -\frac{166}{13}, 5, 0, 0, -\frac{479}{2370}, -\frac{319}{910}, \frac{879}{910}, \frac{1319}{2730})$,
$(a_{n-4}, a_{n-3}, a_{n-2}, a_{n-1}, a_n, v_{n-5}, v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1}, v_n)$
$= (1, -\frac{54}{11}, \frac{111}{11}, -\frac{134}{11}, 6, 0, 0, -\frac{157}{924}, -\frac{115}{231}, \frac{299}{924}, \frac{2}{77})$,
$(c_1, c_2) = (\frac{30}{11}, \frac{12}{13})$ and the local truncation error is

$$t_i = \begin{cases} -\frac{21}{13778} h^8 y_i^{(8)} + O(h^9), & i = 1, n-1, \\ -\frac{123}{421} h^8 y_i^{(8)} + O(h^9), & i = 2, n-2. \end{cases} \tag{4.4}$$

Case 1: For $(\lambda,\ \mu,\ \tau,\ \nu) = (\frac{1}{840}, \frac{120}{840}, \frac{1191}{840}, \frac{2416}{840})$ the truncation error is given by

$$t_i = \frac{1}{120} h^8 y_i^{(8)} + O(h^{10}), \qquad i = 3(1)n-3. \tag{4.5}$$

Case 2: For $(\lambda,\ \mu,\ \tau,\ \nu) = (0, 0, 2, 2)$ the truncation error is given by

$$t_i = \frac{19}{120} h^8 y_i^{(8)} + O(h^{10}), \qquad i = 3(1)n-3. \tag{4.6}$$

4.3 Sixth-order methods

In order to obtain the sixth-order method we find that
$(a_0, a_1, a_2, a_3, a_4, v_0, v_1, v_2, v_3, v_4, v_5) = (6, -\frac{134}{11}, \frac{111}{11}, -\frac{54}{11}, 1, \frac{323}{12079}, \frac{1511}{4703}, -\frac{731}{1473}, -\frac{270}{1603}, -\frac{85}{37549}, \frac{1}{1320})$,
$(b_1, b_2, b_3, b_4, b_5, w_1, w_2, w_3, w_4, w_5, w_6) = (5, -\frac{166}{13}, \frac{162}{13}, -\frac{74}{13}, 1, \frac{453}{911}, \frac{523}{579}, -\frac{930}{3859}, -\frac{339}{1259}, \frac{101}{2588}, -\frac{53}{8507})$,
$(b_{n-5}, b_{n-4}, b_{n-3}, b_{n-2}, b_{n-1}, w_{n-6}, w_{n-5}, w_{n-4}, w_{n-3}, w_{n-2}, w_{n-1})$
$= (1, -\frac{74}{13}, \frac{162}{13}, -\frac{166}{13}, 5, -\frac{53}{8507}, \frac{101}{2588}, -\frac{339}{1259}, -\frac{930}{3859}, \frac{523}{579}, \frac{453}{911})$,
$(a_{n-4}, a_{n-3}, a_{n-2}, a_{n-1}, a_n, v_{n-5}, v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1}, v_n)$
$= (1, -\frac{54}{11}, \frac{111}{11}, -\frac{134}{11}, 6, \frac{1}{1320}, -\frac{85}{37549}, -\frac{270}{1603}, -\frac{731}{1473}, \frac{1511}{4703}, \frac{323}{12079})$,

$(c_1, c_2) = (\frac{30}{11}, \frac{12}{13})$ and the local truncation error is

$$t_i = \begin{cases} \frac{171568}{37} h^{10} y_i^{(10)} + O(h^{11}), & i = 1, n-1, \\ -\frac{86589}{4} h^{10} y_i^{(10)} + O(h^{11}), & i = 2, n-2. \end{cases} \quad (4.7)$$

Case 1: For $(\lambda,\ \mu,\ \tau,\ \nu) = (-\frac{41}{450}, \frac{141}{200}, 0, \frac{859}{180})$ the truncation error is given by

$$t_i = \frac{366}{3991} h^{10} y_i^{(10)} + O(h^{12}), \qquad i = 3(1)n-3. \quad (4.8)$$

Case 2: For $(\lambda,\ \mu,\ \tau,\ \nu) = (\frac{59}{400}, -\frac{109}{150}, \frac{859}{240}, 0)$ the truncation error is given by

$$t_i = -\frac{2149}{758} h^{10} y_i^{(10)} + O(h^{12}), \qquad i = 3(1)n-3. \quad (4.9)$$

4.4 Eighth-order method

For $(\lambda,\ \mu,\ \tau,\ \nu) = (\frac{1}{1680}, \frac{13}{84}, \frac{868}{631}, \frac{617}{210})$, the local truncation error is given by

$$t_i = -\frac{3}{362882} h^{12} y_i^{(12)} + O(h^{14}), \qquad i = 3(1)n-3. \quad (4.10)$$

Boundary equations of eighth-order method are not constructed, however these equations can be obtained by the procedure adapted in previous cases.

5 Nonpolynomial spline solution

The nonpolynomial septic spline solution of system (1.2) is based on the system of linear equations given by (2.7). Let $Y = (y_i),\ (\bar{Y}) = (\bar{y}_i),\ V = (v_i),\ T = (t_i),\ E = (e_i) = Y - \bar{Y},\ i = 1, 2, \cdots, n-1$ be an exact column vectors, where $Y,\ \bar{Y},\ T,\ E$ are exact, approximate, local truncation error, discretization error respectively. We can write the standard matrix equation for the method developed in the following form:

$$\left.\begin{array}{ll} \text{(i)} & MY = V + T, \\ \text{(ii)} & M\bar{Y} = V, \\ \text{(iii)} & ME = T, \end{array}\right\} \quad (5.1)$$

where M is a matrix of order $(n-1)$ with

$$M = (M_0 + h^4 BF) \quad (5.2)$$

and M_0, B, F=diag$(f_i),\ i = 1, 2, \cdots, n-1$ are $(n-1) \times (n-1)$ matrices.

Remark 1 *In case of the nonlinear two point boundary value problem $y^{(4)} = g(x, y)$, Eq.(ii) in (5.1) takes the form:*

$$M_0 \bar{Y}^{(i+1)} + h^4 B g(x, \bar{Y}^{(i)}) = W,\ i = 0, 1, 2, 3... \quad (5.3)$$

The seven-band matrix M_0 has the form:

$$M_0 = \begin{bmatrix} -\frac{134}{11} & \frac{111}{11} & -\frac{54}{11} & 1 & & & & & & \\ 5 & -\frac{166}{13} & \frac{162}{13} & -\frac{74}{13} & 1 & & & & & \\ D_2 & D_3 & D_4 & D_3 & D_2 & D_1 & & & & \\ D_1 & D_2 & D_3 & D_4 & D_3 & D_2 & D_1 & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & & D_1 & D_2 & D_3 & D_4 & D_3 & D_2 & D_1 \\ & & & & D_1 & D_2 & D_3 & D_4 & D_3 & D_2 \\ & & & & & 1 & -\frac{74}{13} & \frac{162}{13} & -\frac{166}{13} & 5 \\ & & & & & & 1 & -\frac{54}{11} & \frac{111}{11} & -\frac{134}{11} \end{bmatrix}, \quad (5.4)$$

where
$D_1 = 6\alpha,\ \ D_2 = 12(\beta - 2\alpha),\ \ D_3 = 6(7\alpha - 8\beta),\ \ D_4 = 12(-4\alpha + 6\beta)$

and the matrix B has the form:

$$B = \begin{bmatrix} -v_1 & -v_2 & -v_3 & -v_4 & -v_5 & & & & & & \\ -w_1 & -w_2 & -w_3 & -w_4 & -w_5 & -w_6 & & & & & \\ \mu & \tau & \nu & \tau & \mu & \lambda & & & & & \\ \lambda & \mu & \tau & \nu & \tau & \mu & \lambda & & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & & \\ & & & \lambda & \mu & \tau & \nu & \tau & \mu & \lambda \\ & & & & \lambda & \mu & \tau & \nu & \tau & \mu \\ & & & & -w_{n-6} & -w_{n-5} & -w_{n-4} & -w_{n-3} & -w_{n-2} & -w_{n-1} \\ & & & & & -v_{n-5} & -v_{n-4} & -v_{n-3} & -v_{n-2} & -v_{n-1} \end{bmatrix}. \quad (5.5)$$

The $(n-1)$ *column vector* V *is defined by*

$$v_i = \begin{cases} -a_0A_0 - c_1hA_1 - h^4(v_0(g_0 - f_0A_0) + v_1g_1 + v_2g_2 + v_3g_3 + v_4g_4 + v_5g_5), & i = 1, \\ -c_2hA_1 - h^4(w_1g_1 + w_2g_2 + w_3g_3 + w_4g_4 + w_5g_5 + w_6g_6), & i = 2, \\ h^4(\lambda(g_0 - f_0A_0) + \mu g_1 + \tau g_2 + \nu g_3 + \tau g_4 + \mu g_5 + \lambda g_6) - 6\alpha A_0, & i = 3, \\ h^4(\lambda g_{i-3} + \mu g_{i-2} + \tau g_{i-1} + \nu g_i + \tau g_{i+1} + \mu g_{i+2} + \lambda g_{i+3}), & 4 \le i \le (n-4), \\ h^4(\lambda g_{n-6} + \mu g_{n-5} + \tau g_{n-4} + \nu g_{n-3} + \mu g_{n-2} + \lambda g_{n-1}) \\ \quad + h^4\lambda(g_n - f_nB_0) - 6\alpha B_0, & i = n-3, \\ c_2hB_1 - h^4(w_{n-6}g_{n-6} + w_{n-5}g_{n-5} + w_{n-4}g_{n-4} + w_{n-3}g_{n-3} \\ \quad + w_{n-2}g_{n-2} + w_{n-1}g_{n-1}), & i = n-2, \\ -a_nB_0 + c_1hB_1 - h^4v_n(g_n - f_nB_0) \\ \quad - h^4(v_{n-5}g_{n-5} + v_{n-4}g_{n-4} + v_{n-3}g_{n-3} + v_{n-2}g_{n-2} + v_{n-1}g_{n-1}), & i = n-1. \end{cases} \quad (5.6)$$

and the form of vector W *we have*

$$w_i = \begin{cases} -a_0A_0 - c_1hA_1 - h^4v_0g_0, & i = 1, \\ -c_2hA_1, & i = 2, \\ -A_0 + h^4(\lambda g_0), & i = 3, \\ 0, & 4 \le i \le n-4, \\ -B_0 + h^4(\lambda g_n), & i = n-3, \\ c_2hB_1, & i = n-2 \\ -a_nB_0 + c_1hB_1 - h^4v_ng_n, & i = n-1. \end{cases} \quad (5.7)$$

We have the following cases:
(i) For second-order methods, the truncation error is $\| T \| = O(h^6)$.
It follows that $\| E \| = O(h^2)$.
(ii) For fourth-order methods, the truncation error is $\| T \| = O(h^8)$.
It follows that $\| E \| = O(h^4)$.
(iii) For sixth-order methods, the truncation error is $\| T \| = O(h^{10})$.
It follows that $\| E \| = O(h^6)$.
(iv) For eighth-order method, the truncation error is $\| T \| = O(h^{12})$.
It follows that $\| E \| = O(h^8)$.

6 Numerical results and discussions

We now consider three numerical problems illustrating the performance of our methods and compared the results with the existing methods. All computations are performed by using MATLAB 7.

Example 1 *Consider the boundary value problem, which is discussed in [10]:*

$$\left.\begin{array}{c} y^{(4)} - xy = -(11+9x+x^2-x^3)e^x, \quad x \in [-1,1], \\ y(-1)=0,\ y(1)=0, \\ y'(-1)=\frac{2}{e},\ y'(1)=-2e. \end{array}\right\} \tag{6.1}$$

The analytical solution of the above problem is

$$y(x) = (1-x^2)e^x. \tag{6.2}$$

Example 2 *Consider the boundary value problem, which is discussed in [15]:*

$$\left.\begin{array}{c} y^{(4)} - y = -4(2x\cos(x)+3\sin(x)), \quad x \in [-1,1], \\ y(-1)=y(1)=0, \\ y'(-1)=\ y'(1)=2\sin(1). \end{array}\right\} \tag{6.3}$$

The analytical solution of the above problem is

$$y(x) = (x^2-1)\sin(x). \tag{6.4}$$

Example 3 *Consider the nonlinear boundary value problem:*

$$\left.\begin{array}{c} y^{(4)} - 6\exp[-4y(x)] = -12(1+x)^{-4}, \quad x \in [0,1], \\ y(0)=0,\ y(1)=\log 2, \\ y'(0)=1,\ y'(1)=0.5. \end{array}\right\} \tag{6.5}$$

The analytical solution of the above problem is

$$y(x) = \log(1+x). \tag{6.6}$$

Table 1: Observed maximum absolute errors, Example 1.

Methods ↓	$\mathbf{h=\frac{1}{4}}$	$\mathbf{h=\frac{1}{8}}$	$\mathbf{h=\frac{1}{16}}$	$\mathbf{h=\frac{1}{32}}$
Second order method				
$(\lambda,\ \mu,\ \tau,\ \nu)=(0,\frac{1}{120},\frac{124}{120},\frac{470}{120})$	4.89×10^{-3}	2.77×10^{-3}	8.19×10^{-4}	2.09×10^{-4}
$(\lambda,\ \mu,\ \tau,\ \nu)=(0,0,0,6)$	1.25×10^{-2}	6.48×10^{-3}	1.77×10^{-3}	4.89×10^{-4}
Rashidinia and Aziz [10]	6.70×10^{-2}	1.45×10^{-2}	3.58×10^{-3}	9.00×10^{-4}
Fourth order method				
$(\lambda,\ \mu,\ \tau,\ \nu)=(\frac{1}{840},\frac{120}{840},\frac{1191}{840},\frac{2416}{840})$	2.09×10^{-5}	9.43×10^{-7}	5.52×10^{-8}	3.41×10^{-9}
$(\lambda,\ \mu,\ \tau,\ \nu)=(0,0,2,2)$	2.38×10^{-4}	1.62×10^{-5}	1.03×10^{-6}	6.47×10^{-8}
Rashidinia and Aziz [10]	1.28×10^{-4}	1.27×10^{-5}	8.25×10^{-7}	2.98×10^{-8}
Sixth order method				
$(\lambda,\ \mu,\ \tau,\ \nu)=(-\frac{41}{450},\frac{141}{200},0,\frac{859}{180})$	1.40×10^{-5}	2.34×10^{-7}	3.48×10^{-9}	2.30×10^{-11}
$(\lambda,\ \mu,\ \tau,\ \nu)=(\frac{59}{400},-\frac{109}{150},\frac{859}{240},0)$	2.10×10^{-5}	3.75×10^{-7}	6.25×10^{-9}	1.29×10^{-10}
Rashidinia and Aziz [10]	−	3.31×10^{-7}	5.15×10^{-9}	8.10×10^{-11}

The observed maximum absolute errors (in absolute values) corresponding to the examples 1-3 for our second, fourth and sixth-order methods are briefly summarized in tables 1-3. These problems have been solved by the presented method with step lengths $h=2^{-m},\ m=2,3,....$ Comparison with other existing methods are also listed in tables 1-3. These tables show that our class of methods are more accurate than the existing methods.

It is verified from the tables 1-3 that on reducing the step-size from h to $h/2$, the maximum observed error $\parallel E \parallel$ is approximately reduces by a factor $1/2^p$, where p is the theoretical order of numerical method, except possibly when the rounding errors are significant.

Table 2: Observed maximum absolute errors, Example 2.

Methods $\downarrow$	$\mathbf{h}=\frac{1}{4}$	$\mathbf{h}=\frac{1}{8}$	$\mathbf{h}=\frac{1}{16}$	$\mathbf{h}=\frac{1}{32}$
Second order method				
$(\lambda,\ \mu,\ \tau,\ \nu)=(0,\frac{1}{120},\frac{124}{120},\frac{470}{120})$	1.87×10^{-3}	1.21×10^{-4}	5.33×10^{-5}	1.42×10^{-5}
$(\lambda,\ \mu,\ \tau,\ \nu)=(0,0,0,6)$	1.38×10^{-3}	3.63×10^{-4}	1.18×10^{-4}	3.05×10^{-5}
Fourth order method				
$(\lambda,\ \mu,\ \tau,\ \nu)=(\frac{1}{840},\frac{120}{840},\frac{1191}{840},\frac{2416}{840})$	1.93×10^{-6}	6.96×10^{-8}	3.56×10^{-9}	2.16×10^{-10}
$(\lambda,\ \mu,\ \tau,\ \nu)=(0,0,2,2)$	1.07×10^{-5}	9.82×10^{-7}	6.47×10^{-8}	4.06×10^{-9}
Sixth order method				
$(\lambda,\ \mu,\ \tau,\ \nu)=(-\frac{41}{450},\frac{141}{200},0,\frac{859}{180})$	5.61×10^{-7}	1.40×10^{-8}	2.78×10^{-10}	1.11×10^{-11}
$(\lambda,\ \mu,\ \tau,\ \nu)=(\frac{59}{400},-\frac{109}{150},\frac{859}{240},0)$	7.63×10^{-7}	2.06×10^{-8}	3.02×10^{-10}	4.11×10^{-12}
Siddiqi and Akram [15]	$-$	1.93×10^{-4}	3.40×10^{-5}	7.83×10^{-6}

Table 3: Observed maximum absolute errors, Example 3.

Methods $\downarrow$	$\mathbf{h}=\frac{1}{8}$	$\mathbf{h}=\frac{1}{16}$	$\mathbf{h}=\frac{1}{32}$	$\mathbf{h}=\frac{1}{64}$
Second order method				
$(\lambda,\ \mu,\ \tau,\ \nu)=(0,\frac{1}{120},\frac{124}{120},\frac{470}{120})$	6.23×10^{-5}	1.67×10^{-5}	5.56×10^{-6}	1.45×10^{-6}
$(\lambda,\ \mu,\ \tau,\ \nu)=(0,0,0,6)$	4.97×10^{-5}	4.17×10^{-5}	1.22×10^{-5}	3.12×10^{-6}
Fourth order method				
$(\lambda,\ \mu,\ \tau,\ \nu)=(\frac{1}{840},\frac{120}{840},\frac{1191}{840},\frac{2416}{840})$	4.46×10^{-7}	2.23×10^{-8}	1.26×10^{-9}	7.76×10^{-11}
$(\lambda,\ \mu,\ \tau,\ \nu)=(0,0,2,2)$	4.19×10^{-6}	3.46×10^{-7}	2.29×10^{-8}	1.45×10^{-9}
Sixth order method				
$(\lambda,\ \mu,\ \tau,\ \nu)=(-\frac{41}{450},\frac{141}{200},0,\frac{859}{180})$	1.48×10^{-7}	3.33×10^{-8}	5.62×10^{-10}	9.39×10^{-12}
$(\lambda,\ \mu,\ \tau,\ \nu)=(\frac{59}{400},-\frac{109}{150},\frac{859}{240},0)$	2.22×10^{-7}	5.22×10^{-8}	8.99×10^{-10}	1.41×10^{-11}

Conclusion

In this paper we have presented new methods for solving fourth-order two point boundary value problems using nonpolynomial septic spline. The present methods enable us to approximate the solution at every point of the range of integration. The comparison of the method is also depicted through tables 1-3 which shown our methods have an improvement over existing methods.

References

[1] R.P.Agarwal. Boundary Value Problems for Higher-Order Differential Equations. World Scientific, Singapore, 1986 .

[2] J.H.Ahlberg, E.N.Nilson, J.L.Walsh. The theory of splines and their applications. Academic Press, New York, 1967.

[3] G.Akram, S.S.Siddiqi. End conditions for interpolatory septic splines. *Int. J. Comput. Math.*, 82 (2005): 1525-1540.

[4] M.M.Chawla, R.Subramanian. High accuracy quintic spline solution of fourth-order two-point boundary value problems. *Int. J. Comput. Math.*, 31 (1989): 87 − 94.

[5] Y.Gupta, M.Kumar. Numerical method for solving boundary value problem arising in deflection of beams. *Canadian J. Comput. Math., Nat. Sci., Eng. and Med.*, 2(7) (2011): 166 − 169.

[6] M.K.Jain, S.R.K.Iyengar, J.S.V.Saldanha. Numerical solution of a fourth-order ordinary differential equation. *J. Eng. Math.*, 11(4) (1977): 373 − 380.

[7] C.P.Katti. Five diagonal sixth order methods for two-point boundary value problems involving fourth order differential equation. *Math. Comput.*, 35 (1980): 1177 − 1179.

[8] M.A.Ramadan, I.F.Lashien, W.K.Zahra. A class of methods based on a septic non-polynomial spline function for the solution of sixth order two-point boundary value problems. *Int. J. Comput. Math.*, 85 (2008): 759 − 770.

[9] M.A.Ramadan, I.F.Lashien, W.K.Zahra. Quintic non-polynomial spline solutions for fourth-order two-point boundary value problem. *Comm. Nonlinear Sci. Numer. Simulation*, 14 (2009): 1105 − 1114.

[10] J.Rashidinia, T.Aziz. Quintic spline solution of a fourth-order two-point boundary value problem. *Int. J. Appl. Sci. Comput.*, 3(3) (1997): 191 − 197.

[11] J.Rashidinia, A.Golbabaee. Convergence of numerical solution of a fourth-order boundary value problem. *Appl. Math. Comput.*, 171 (2005): 1296 − 1305.

[12] J.Rashidinia, R.Jalilian. Non-polynomial spline for solution of boundary value problems in plate deflection theory. *Int. J. Comput. Math.*, 84(10) (2007): 1483 − 1494.

[13] E.L.Reiss, A.J.Callegari, D.S.Ahluwalia. Ordinary Differential Equation with Applications. Holt, Rinehart and Winston, New York, 1976.

[14] R.D.Russell, L.F.Shampine. A collocation method for boundary value problems. *Numer. Math.*, 19 (1972): 1 − 28.

[15] S.S.Siddiqi, G.Akram. Quintic spline solutions of fourth order boundary value problems. *Int. J. Numer. Anal. Model.*, 5(1) (2008): 101 − 111.

[16] Siraj-ul-Islam, I.A.Tirmizi, S.Ashraf. A class of methods based on non-polynomial spline functions for the solution of special fourth-order boundary value problems with engineering applications. *Appl. Math. Comput.*, 174 (2006): 1169 − 1180.

[17] S.Timoshenko, S.Woinowsky Krieger. Theory of plates and shells. McGraw-Hill, New York, 1959.

[18] R.A.Usmani, M.J.Marsden. Numerical solution of some ordinary differential equations occuring in a plate deflection theory. *J. Eng. Math.*, 9(1) (1975): 1 − 10.

[19] R.A.Usmani. Discrete variable methods for a boundary value problem with engineering applications. *Math. Comput.*, 32(144) (1978): 1087 − 1096.

[20] R.A.Usmani. Smooth spline approximations for the solution of a boundary value problem with engineering applications. *J. Comput. Appl. Math.*, 6(2) (1980): 93 − 98.

[21] R.A.Usmani, S.A.Warsi. Smooth spline solutions for boundary value problems in plate deflection theory. *Comput. Math. Appl.*, 6 (1980): 205 − 211.

[22] R.A.Usmani. The use of quartic splines in the numerical solution of a fourth-order boundary value problem. *J. Comput. Appl. Math.*, 44 (1992): 187 − 199.