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Fractal Interpolation Functions on the Stability of Vertical Scale Factor

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Abstract: Fractal interpolation is a new interpolation method, the corresponding vertical scale factor determines the shape of fractal interpolation functions and features. Therefore, choosing the appropriate vertical scale factor, can fit the real rough curve precisely. This paper primarily studies the change of the fractal interpolation curve when giving a small perturbation on vertical scale factor, the fractal interpolation curve produces change. When the vertical scale factor has a small change, in order to satisfy the continuity condition, its iteration function must have the corresponding change, so, give the corresponding iterated function system. Next, this paper discusses the changes of fractal interpolation functions when the vertical scale factor perturbate and give the error estimate.

Keywords: fractal interpolation function; iterated function system; vertical scale factor; continuity conditions; error estimates

1 Introduction

The theory of fractal interpolation has become a powerful and useful tool in applied science and engineering since Barnsley [1] introduced the concept of the fractal interpolation function (FIF). We can fit out the non-smooth data and the non-smooth curve in practical application more vividly by studying the fractal interpolation function. These conclusions are important in the theory and practical application. As we know ,a FIF is essentially the attractor of an iterated function system (IFS).There are also a lot of curve in nature, such as coastline, the top fluctuant curve of the forest, mountains' outlines, the shapes of clouds are specific examples. The simulation of the rough surface is an important application of the fractal interpolation function in recent years. Fractal interpolation is a new interpolation method, the corresponding vertical scale factor determines the shape of fractal interpolation functions and features. Therefore, choosing the appropriate vertical scaling factor, can fit the real rough curve precisely. There are corresponding researches on how the vertical scale factors affect the bounds of the affine fractal interpolation function [3], the bounds of the attractor of the iterated function system with double vertical scale factor [4], the perturbation of the corresponding FIFs when the interpolation data have a small perturbation [5] and discussing the influence of the fractal interpolation function because of the change of the vertical scale factor by numerical experiments [6] etc.

In this paper we first introduce the fractal interpolation theory, then we give a small perturbation on vertical scale factor, we must consider in what circumstances the perturbed iterated function system meet the fractal interpolation continuous conditions so that the perturbed iterated function system can determine a fractal interpolation function. Finally we briefly discuss the error analytical expressions and the estimates of the upper bound in this situation.

2 The theory of fractal interpolation

Let a set of data points $T = \{(x_i, y_i) : i = 0, 1, \cdots, N\}$be given in R^2 with $x_0 < x_1 < \cdots < x_N, N > 1$ is an arbitrary positive integer and $y_0, y_1, \cdots, y_N$ are arbitrary real numbers. Set $I = [x_0, x_N]$, $K = I \times R, I_i = [x_{i-1}, x_i], i = 1, 2, \cdots, N$.

Define mappings : $L_i : I \to I_i, F_i : K \times R, i = 1, 2, \cdots, N$ as follows:

$$\begin{cases} L_i(x) = a_i x + e_i \\ F_i(x, y) = d_i y + \psi_i(x) \end{cases} \tag{1}$$

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where $a_i = (x_i - x_{i-1}) / (x_N - x_0), e_i = (x_N x_{i-1} - x_0 x_i) / (x_N - x_0)$, the d_i obey $|d_i| < 1$, are called vertical scaling factors, and the $\psi_i(x)$ are continuous functions defined on I satisfying the following conditions:

$$F_i(x_0, y_0) = y_{i-1}, F_i(x_N, y_N) = y_i, i = 1, 2, \cdots, N. \tag{2}$$

Let $w_i(x, y) = (L_i(x), F_i(x, y)), i = 1, 2, \cdots, N$; then

$$\{K; w_i(x, y) : i = 1, 2, \cdots N\} \tag{3}$$

constitutes an IFS. According to the IFS theory , such an IFS has a unique attractor G, which is the graph of a continuous function $f : I \times R$ passing through the interpolation points T.
The function f described above is referred to as a FIF generated the IFS (2.3), which satisfies the following fixed point equation:

$$f(x) = d_i f\left(L_i^{-1}(x)\right) + \psi_i\left(L_i^{-1}(x)\right), x \in I_i \tag{4}$$

3 The fractal interpolation functions with perturbation

On the basis of the IFS (2.3), we now change the vertical scaling factors, and other conditions remain unchanged, then we can construct a new IFS as follows:

$$\{K; ((L_i(x), T_i(x, y))) : i = 1, 2, \cdots N\} \tag{5}$$

where $T_i(x, y) = (d_i + \delta_i) y + \psi_i(x)$, and δ_i is the adding variation of d_i, satisfying the condition $0 < |d_i + \delta_i| < 1$.

Example 1 *Let f is a fractal interpolation function passing through the interpolation points (0,0),(1/4,1),(1,1/2), and* $d_1 = 1/2$, $d_2 = 1/4$*, its corresponding fractal interpolation function systems are as follows:*

$$\begin{cases} L_1(x) = \frac{1}{4}x, F_1(x, y) = \frac{1}{2}y + \frac{3}{4}x \\ L_2(x) = \frac{3}{4}x + \frac{1}{4}, F_2(x, y) = \frac{1}{4}y - \frac{5}{8}x + 1 \end{cases}$$

At this time, this IFS meets the continuity condition $F_2(x_0, y_0) = F_1(x_N, y_N)$.
Now we change the value of the vertical scaling factors with the interpolation points unchanged: $d_1 = \frac{11}{20}, d_2 = -\frac{7}{12}$ *(the corresponding* $\delta_1 = \frac{1}{20}, \delta_2 = \frac{1}{24}$*),then we can obtain:*

$$\begin{cases} L_1(x) = \frac{1}{4}x, F_1'(x, y) = \frac{11}{20}y + \frac{3}{4}x \\ L_2(x) = \frac{3}{4}x + \frac{1}{4}, F_2'(x, y) = -\frac{7}{12}y - \frac{5}{8}x + 1 \end{cases}$$

where $F_2'(x_0, y_0) \neq F_1'(x_N, y_N)$*, so the changed IFS can't satisfy the fractal interpolation continuous conditions.*

We give Theorem 3.1, describe in what circumstances the unique attractor of the perturbed iterated function system is just the graph of the continuous function.

Theorem 1 *Let* $M_i(x, y) = (d_i + \delta_i) y + \psi_i(x) + \lambda_i$*, where the* λ_i *is constant satisfying* $0 < |d_i + \delta_i| < 1$ *and the* $\psi_i(x)$ *are continuous functions defined on I .Suppose* $\lambda_1 = 0$*, then when* λ_n *satisfies* $\lambda_n = \left(\sum_{j=1}^{n-1} \delta_j\right) y_N - \left(\sum_{j=2}^{n} \delta_j\right) y_0 + \lambda_1$*, the new IFS:*

$$\{K; ((L_i(x), M_i(x, y))) : i = 1, 2, \cdots N\} \tag{6}$$

satisfies the fractal interpolation continuous conditions , and determines a FIF, which is denoted as $f_\delta(x)$ *.*

Proof. From the fractal interpolation continuous condition $M_i(x_N, y_N) = M_{i+1}(x_0, y_0)$, we can have:

$$\left(d_{i+1}+\delta_{i+1}\right)y_0+\psi_{i+1}\left(x_0\right)+\lambda_{i+1}=\left(d_i+\delta_i\right)y_N+\psi_i\left(x_N\right)+\lambda_i,$$

then $\delta_{i+1}y_0+\lambda_{i+1}=\delta_i y_N+\lambda_i$, that is, $\lambda_{i+1}=\delta_i y_N-\delta_{i+1}y_0+\lambda_i$,
$\lambda_{i+2}=\delta_{i+1}y_N-\delta_{i+2}y_0+\lambda_{i+1}=\left(\delta_{i+1}+\delta_i\right)y_N-\left(\delta_{i+2}+\delta_{i+1}\right)y_0+\lambda_i$,
$\lambda_{i+3}=\left(\delta_{i+2}+\delta_{i+1}+\delta_i\right)y_N-\left(\delta_{i+3}+\delta_{i+2}+\delta_{i+1}\right)y_0+\lambda_i$,
repeating this process, we can obtain

$$\lambda_{i+n}=\left(\sum_{j=1}^{n-1}\delta_{i+j}\right)y_N-\left(\sum_{j=2}^{n}\delta_{i+j}\right)y_0+\lambda_i,$$

where we let $\lambda_1=0$, then $\lambda_n=\left(\sum\limits_{j=1}^{n-1}\delta_j\right)y_N-\left(\sum\limits_{j=2}^{n}\delta_j\right)y_0+\lambda_1$.

Finally, we can give the conclusion: when λ_n satisfies $\lambda_n=\left(\sum\limits_{j=1}^{n-1}\delta_j\right)y_N-\left(\sum\limits_{j=2}^{n}\delta_j\right)y_0+\lambda_1$, the new IFS: $\{K;\left(\left(L_i\left(x\right),M_i\left(x,y\right)\right)\right)\ :\ i=1,2,\cdots N\}$ satisfies the fractal interpolation continuous conditions , and the unique attractor of the perturbed iterated function system is just the graph of the continuous function $f_\delta\left(x\right)$. ■

Example 2 *In the Example3.1 above, we can have $\lambda_2=\delta_1 y_N-\delta_2 y_0=\frac{1}{20}\times\frac{1}{2}=0$.
if we let $\lambda_1=0$, thus, $F_2'\left(x,y\right)=-\frac{7}{12}y-\frac{5}{8}x+\frac{41}{40}$, $F_2'\left(x_0,y_0\right)=\frac{41}{40}$, $F_1'\left(x_N,y_N\right)=\frac{11}{40}+\frac{3}{4}=\frac{41}{40}$,
At last, we can obtain $F_2'\left(x_0,y_0\right)=F_1'\left(x_N,y_N\right)$, this is to say, the IFS (3.1) satisfies the fractal interpolation continuous conditions when adding λ_i.*

4 Error estimation

We will investigate the internal relations between f(x) and $f_\delta\left(x\right)$. For this purpose, we first present a useful lemma.
For any $x\in I$, let $L_{i_1 i_2\cdots i_n}\left(x\right)=L_{i_n}\circ L_{i_{n-1}}\circ\cdots\circ L_{i_1}\left(x\right)$, set $\prod\limits_{j=1}^{0}a_{i_j}=1$ and $\prod\limits_{j=1}^{0}d_{i_j}=1$.From (2.1) and (2.4), applying the successive iteration method and mathematical induction ,we are able to show the following lemma.

Lemma 2 *Let $f_\delta\left(x\right)$ be the fractal interpolation function generated with (3.2) and $\lambda_{i_1}=0$. Then for any $j=1,2,\cdots,n$, we have*

$$L_{i_1 i_2\cdots i_n}\left(x\right)=\left(\prod_{j=1}^{n}a_{i_j}\right)x+\sum_{k=1}^{n}\left(\prod_{j=1}^{k-1}a_{i_j}\right)e_{i_k},$$

$$f_\delta\left(L_{i_1 i_2\cdots i_n}\left(x\right)\right)=\left(\prod_{j=1}^{n}\left(d_{i_j}+\delta_{i_j}\right)\right)f_\delta\left(x\right)+\left(\prod_{j=1}^{n}\left(d_{i_j}+\delta_{i_j}\right)\right)\psi_{i_n}\left(x\right)+$$

$$\sum_{k=1}^{n-1}\left(\prod_{j=1}^{k-1}\left(d_{i_j}+\delta_{i_j}\right)\right)\psi_{i_k}\left(\overline{x}\right)+\sum_{k=1}^{n}\left(\prod_{j=1}^{k-1}\left(d_{i_j}+\delta_{i_j}\right)\right)\lambda_{i_k},$$

where

$$\overline{x}=\left(\prod_{j=1}^{n-k}a_{i_{k+j}}\right)x+\sum_{l=1}^{n-k}\left(\prod_{j=1}^{n-k}a_{i_{k+j}}\right)e_{i_{k+l}}.$$

Proof. From the successive iteration method and mathematical induction, we give simple proof as follows:

$$L_{i_1 i_2\cdots i_n}\left(x\right)=L_{i_n}\circ L_{i_{n-1}}\circ\cdots\circ L_{i_1}\left(x\right)=L_{i_n}\circ L_{i_{n-1}}\circ\cdots\circ L_{i_2}\left(L_{i_1}\left(x\right)\right)$$

$$= L_{i_n} \circ L_{i_{n-1}} \circ \cdots \circ L_{i_2}\left(a_{i_1}x + e_{i_1}\right) = L_{i_n} \circ L_{i_{n-1}} \circ \cdots \circ L_{i_3}\left(a_{i_2}\left(a_{i_1}x + e_{i_1}\right) + e_{i_2}\right)$$

$$= \cdots = \left(\prod_{j=1}^{n} a_{i_j}\right)x + \sum_{k=1}^{n}\left(\prod_{j=1}^{k-1} a_{i_j}\right)e_{i_k}$$

$$f_\delta\left(L_{i_1 i_2 \cdots i_n}(x)\right) = \left(\prod_{j=1}^{n}\left(d_{i_j} + \delta_{i_j}\right)\right) f_\delta(x) + \left(\prod_{j=1}^{n-1}\left(d_{i_j} + \delta_{i_j}\right)\right)\left[\psi_{i_n}(x) + \lambda_{i_n}\right] + \sum_{k=1}^{n-1}\left(\prod_{j=1}^{k-1}\left(d_{i_j} + \delta_{i_j}\right)\right)\left[\psi_{i_k}(\overline{x}) + \lambda_{i_k}\right]$$

$$= \left(\prod_{j=1}^{n}\left(d_{i_j} + \delta_{i_j}\right)\right) f_\delta(x) + \left(\prod_{j=1}^{n-1}\left(d_{i_j} + \delta_{i_j}\right)\right)\psi_{i_n}(x) + \left(\prod_{j=1}^{n-1}\left(d_{i_j} + \delta_{i_j}\right)\right)\lambda_{i_n} +$$

$$\sum_{k=1}^{n-1}\left(\prod_{j=1}^{k-1}\left(d_{i_j} + \delta_{i_j}\right)\right)\psi_{i_k}(\overline{x}) + \sum_{k=1}^{n-1}\left(\prod_{j=1}^{k-1}\left(d_{i_j} + \delta_{i_j}\right)\right)\lambda_{i_k}$$

$$= \left(\prod_{j=1}^{n}\left(d_{i_j} + \delta_{i_j}\right)\right) f_\delta(x) + \left(\prod_{j=1}^{n}\left(d_{i_j} + \delta_{i_j}\right)\right)\psi_{i_n}(x) +$$

$$\sum_{k=1}^{n-1}\left(\prod_{j=1}^{k-1}\left(d_{i_j} + \delta_{i_j}\right)\right)\psi_{i_k}(\overline{x}) + \sum_{k=1}^{n}\left(\prod_{j=1}^{k-1}\left(d_{i_j} + \delta_{i_j}\right)\right)\lambda_{i_k}$$

■

From the lemma 4.1, we can discuss the error estimate between f(x) and $f_\delta(x)$.

Theorem 3 *Let f(x) and $f_\delta(x)$be the FIFs generated with the IFS (2.3) and (3.2), respectively. For any given $x \in I$,let $i_j \in \{1, 2, \cdots, N\}$,$j = 1, 2, \cdots, n$, be the sequence such that x satisfies*

$$x = \sum_{k=1}^{\infty}\left(\prod_{j=1}^{k-1} a_{i_j}\right)e_{i_k}.$$

Then

$$f_\delta(x) - f(x) = \sum_{k=1}^{\infty}\left(\prod_{j=1}^{k-1}\left(d_{i_j} + \delta_{i_j}\right) - \prod_{j=1}^{k-1} d_{i_j}\right)\psi_{i_k}\left(\sum_{l=1}^{\infty}\left(\prod_{j=1}^{l-1} a_{i_{k+j}}\right)e_{i_{k+l}}\right) +$$

$$\sum_{k=1}^{\infty}\left(\prod_{j=1}^{k-1}\left(d_{i_j} + \delta_{i_j}\right)\right)\lambda_{i_k}.$$

Proof. The proof is similar to that of theorem 1 in the reference [7]. ■

Corollary 4 *Let f(x) and$f_\delta(x)$be the FIFs generated with the IFS (2.3) and (3.2), respectively. Let $d = \max\limits_{1\le i\le N}\{|d_i|\} < 1$ and $M = \max\limits_{1\le i\le N}\{\|\psi_i\|\}$, where $\|\psi_i\| = \max\limits_{x\in I}|\psi_i(x)|$, $\lambda = \max\limits_{1\le i\le N}\{\lambda_i\}$, suppose $0 < \delta = \max\limits_{1\le i\le N}\{|\delta_i|\}$ and $d + \delta < 1$; then*

$$|f_\delta(x) - f(x)| \le \frac{\delta\left[M + N\left(|y_0| + |y_N|\right)\right]}{(1 - d - \delta)(1 - d)}.$$

Proof. From $\lambda_n = \left(\sum_{j=1}^{n-1}\delta_j\right)y_N - \left(\sum_{j=2}^{n}\delta_j\right)y_0 + \lambda_1$, we can have

$$|\lambda_n| < |(n-1)\delta \cdot y_N + (n-1)\delta \cdot y_0| < N\delta\left(|y_N| + |y_0|\right).$$

From the reference[7], we can prove this corollary. ■

5 Conclusion

This paper have discussed the vertical scaling factors on the influence of the fractal interpolation function. We have analysed the expression for the perturbation errors and upper bounds of the two iterated function systems when the vertical scaling factors have small changes theoretically, clearly reflected that the fractal interpolation function changes successively with the change of the vertical scaling factors. In this paper, the interpolation points of the perturbed iterated function systems have been changed, so, in the future research, we can study how the image shape of the fractal interpolation functions will change when the vertical scaling factors perturb but the interpolation points of the perturbed iterated function systems are unchanged .

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