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Homotopy Invariance of Perturbation of D_∞ -differential Module

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Abstract: In this article we use the concepts of perturbation of differential module introduced by Gugenheim, Lambe, and Stasheff [2–4] and D_∞-differential module introduced by Lapin [5] to define the perturbation of D_∞-differential module. We give some examples and study the homotopy invariance property of these modules.

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1 Introduction

Gugenheim, Lambe, and Stasheff [2–4] introduced the notion of perturbation of differential module and established the homotopy invariance property of perturbation of differential module. In [5] Lapin introduced the concept of D_∞-differential module (in short D_∞-module), which is a homotopy invariance , quantum analogue of the concept of a differential module. He found the connection between perturbations differential module and D_∞-differential module . The interest thing, in perturbation of D_∞-modules , is mainly due to the development of task management techniques of differential in spectral sequences. In[5] it is shown that the category of spectral sequence over field equivalent to category of D_∞-differential module over field. The classic results of Serre[7] on the multiplicative structure of of cohomology of spectral sequence of a filtration is generalized by Lapin [6] on E_∞-algebras. In the present work we give the notion of perturbations D_∞-module and show that it has a homotopy invariant property. The main result is theorem 3 related to the homotopy invariant of perturbation of D_∞-module over field.

1-Homotopy Invariance of perturbation of D_∞ -module

We begin by recalling some basic definitions and facts essential in the sequel. The useful references for this material are [2–4] ,[5] and[1]. Let K be an arbitrary commutative ring with identity. All the modules and module maps considered are K-modules and K-linear maps of modules, If not stated otherwise.

Definition 1 *The differential module (X, d) is a module X equipped with a map of modules $d : X \to X$, called the differential of the module X, such that $d^2 = dd = 0$.*

A map of differential modules $f : (X, d) \to (Y, d)$ is a mapping of modules $f : X \to Y$ such that $df = fd$.

Definition 2 *A differential homotopy between maps $f, g : (X, d) \to (Y, d)$ of differential modules is a map of modules $h : X \to Y$ such that $dh + hd = f - g$.*

Using the notion of differential homotopy between mappings of differential modules, we can define the notion of homotopy equivalence of differential modules.

Definition 3 *Consider the modules (X, d) and (X, Y) . The triple $(\eta : X \leftrightarrows Y : \xi, h)$ is called strong differential retract of differential modules, such that the maps $\eta : X \to Y, \xi : Y \to X$ are differential module morphisms satisfy $\eta\xi = 1_Y$ and h is a homotopy between morphisms $\xi\eta$ and 1_X. If the following identities $\eta h = 0$, $h\xi = 0$, $hh = 0$ hold, the triple $(\eta : X \leftrightarrows Y : \xi, h)$ is called SDR-case of differential modules. The good example of differential SDR-case is*

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the homology $H(X)$ of a differential module X over a field. Let $H(X) = Ker\, d \,/\, Im\, d$ be the homology module of an arbitrary differential module (X, d) over a field K. If we regard $H(X)$ as a differential module with zero differential, then by using the fixed direct sum decomposition $Ker\, d = H(X) \oplus Im\, d$ we get the SDR-case $(\eta : X \leftrightarrows Y : \xi, h)$ for differential modules which is called homology SDR-case for differential modules.

Definition 4 *A differential perturbation of module (X, d) is a map of modules $t : X \to X$ such that the pair $(X, d + t)$ is a differential module, i.e., the mapping $d + t : X \to X$ satisfies the condition $(d + t)^2 = 0$. It is easy to see that any differential perturbation $t : X \to X$ of a differential module (X, d) satisfies the condition $dt + td = -t^2$. For given module (X, d) there is a new map $D : X \to X$ such that $t = D - d : X \to X$.*

Definition 5 *A D_∞-differential module is a pair (X, d^i) where X is an arbitrary module and is d^i a family of homeomorphisms $\{d^i : X \to X,\, i \in Z, i \geq 0\}$ such that the following relation holds for each non-negative Integer $k \geq 0$:*

$$\sum_{i+j=0} d^i d^j = 0 \tag{1}$$

From relation (1), if $k = 0$ we have $d^0 d^0 = 0$, this means that the pair is a differential module if $k = 0$ we have , this means that the pair (X, d^i) is a differential module. For $k = 1$ we have $d^1 d^0 + d^0 d^1 = 0$, that is the maps d^0 and $d^1$are anticommuting maps. Hence the composite map is an endomorphism $d^1 d^1 : X \to X$ of the differential module (X, d^0) .

For $k = 2$ we have $d^0 d^2 + d^2 d^0 = 0 - d^1 d^1$. This means that the map $d^2 : X \to X$ is a homotopy between the zero map and the map $d^1 d^1 : (X, d^0) \to (X, d^0)$ of differential modules. Hence $d^1 : X \to X$ is a differential.

Examples of D_∞-differential modules:

1- Let X be an arbitrary module X with two anticommutating differentials $d^{'}, d^{''} : X \to X$. The pair (X, d^i) is D_∞-module such that $d^0 = d^{'}$, $d^1 = d^{''}$, $d^i = 0,\, i \geq 0$.

2- The differential module with filtration : For given module (X, d) over field consider the filtration $\{X^n\},\, n \geq 0$, let Y^n is submodule of X^n such that $X^n = Y^n \oplus X^{n-1}$ such that $d(X^n) \subseteq X^n$ Define the map $d^i_n : Y^n \to Y^{n-1},\, n \geq 0$,where $d : Y^n \to X^n = Y^n \oplus ... \oplus Y^{n-i} \oplus ... \oplus Y^0$.Clearly the family

$$\{d^i = \oplus_{n \geq 0} \oplus d^i_n : X^n \to X^n,\, i \geq 0\}$$

satisfies the relation (1) and hence its D_∞-differential module homeomorphisms.

Definition 6 *The morphism of D_∞-modules $f : (X, d^i) \to (Y, d^i)$ is a family of homomorphisms $f^i : X \to Y,\, i \geq 0$, such that for each Integer $k \geq 0$ the following identity holds*

$$\sum_{i+j=0} f^i d^j = \sum_{i+j=0} d^j f^j \tag{2}$$

In equation (2), For $k = 0$ we have $f^0 d^1 - d^1 f^0 : (X, d^0) \to (Y, d^0)$, which gives the well-known map $f^0 : (X, d^0) \to (Y, d^0)$. For $k = 1$ we have $f^0 d^1 - d^1 f^0 = d^0 f^1 - f^1 d^0$, this mean that the map $f^1 : X \to X$ is a homotopy between the zero map and the map $f^0 d^1 - d^1 f^0 : (X, d^0) \to (Y, d^0)$. On other words, the map f^0, with respect to d^1 , is a map of differential modules within a homotopy.

Definition 7 *The composition of homomorphisms of D_∞-modules $f = \{f^i\} : X \to Y$ and $g = \{g^i\} : Y \to Z$ is defined by the formula :*

$$(gf)^i = \sum_{i+j=0} g^n f^m : X \to Z,\, i \geq 0$$

Note that

- The direct calculation show that $g\,f = \{(gf)^i\}$ is D_∞-modules morphism.
- The composition operator is associative.
- The identity homorphism $1_X = \{1^i_X\}$ of D_∞-module is a family $1_X = \{1^i_X : X \to X\}$, where $1_X = 0,\, i > 0$ and 1^0_X is the identity map. In this way we have defined the category of D_∞-modules.

Definition 8 *A family of homeomorphisms $h = \{h^i : X \to Y, i \in Z,\ i \geq 0\}$ is called a homotopy between morphisms of D_∞-modules if the following relation holds for each integer $k \geq 0$:*

$$\sum_{i+j=k} d^i h^j + h^i d^j = f^{\,k} - g^{\,k} \tag{3}$$

For $k = 0$ we have $d^0 h^0 + h^0 d^0 = f^{\,0} - g^0$. Hence the map of modules $h^0 : X \to Y$ is a differential homotopy between the maps of differential modules $f^{\,0}, g^0 : (X, d^0) \to (Y, d^0)$.

A direct checking shows that the relation between morphisms of D_∞-modules, in terms of the concept of a homotopy, is an equivalence relation. The special important notions of homotopy equivalent are the strong deformation retraction of and SDR-case of D_∞-modules. We detail these concepts by generalizing the definition 1.3.

Let (X, d^i) and (Y, d^i) be D_∞-modules. Further, let $\eta = \{\eta^i\} : (X, d^i) \to (Y, d^i) : \{\xi_i\} = \xi$ be morphisms of -modules and let $h = \{h^i\} : X \to X$ be a homotopy between morphisms of -modules such that:

$$\sum_{s+t=i} \eta^s \xi^t = (1_Y)^i,\ \sum_{s+t=i} d^s h^t + h^s d^t = \sum_{s+t=i} \xi^s \eta^t - (1_X),\ i \geq 0.$$

The expression $(\{\eta^i\} : (X, d^i) \rightleftarrows (Y, d^i) : \{\xi^i\} = \{h^i\}$ is called the strong deformation retraction of D_∞-modules. If the following are satisfied:

$$\sum_{i+\ j=k} \eta^i \xi^j = 0,\ \sum_{i+\ j=k} h^i \xi^j = 0,\ \sum_{i+\ j=k} h^i h^j = 0,\ k \geq 0.$$

then the strong deformation retraction of D_∞-modules is called SDR-data for D_∞-modules.

The following theorems 4 and 10 about the invariant structure of D_∞-modules. It's related to the strong deformation retract as a homotopical equivalent type [5].

Theorem 1 *: Let (X, d^i) be an arbitrary D_∞-modules, for the differential module (Y, d) suppose the strong deformation retraction*

$$\{\eta : (X, d^0) \to (Y, d) : \xi, h\}$$

of differential modules, then the family $(d_*^i : Y \to Y)$, that satisfies the formula

$$d_*^0 = d, d_*^i = \eta \left(\sum_{\substack{1 \leq k \leq i \\ i_1 + \cdots + i_k = i \\ i_1 \geq 1, \cdots, i_k \geq 1}} d^{i_1} \left(hd^{i_2}\right) \cdots \left(hd^{i_2}\right) \right) \xi, i \geq 1 \tag{4}$$

is a D_∞-modules Y and the strong deformation retraction $(\{\eta_*^i\} : (X, d^i) \to (Y, d_*^i) : \{\xi_*^i\}, \{h_*^i\})$ of D_∞-modules, wher $\{\xi_*^i : Y \to X\}, \{\eta_*^i : X \to Y\}$ and $\{h_*^i : X \to X\}$ are given by the following identities

$$\xi_*^0 = d, \xi_*^i = h \left(\sum_{\substack{1 \leq k \leq i \\ i_1 + \cdots + i_k = i \\ i_1 \geq 1, \cdots, i_k \geq 1}} d^{i_1} \left(hd^{i_2}\right) \cdots \left(hd^{i_2}\right) \right) \xi, i \geq 1 \tag{5}$$

$$\eta_*^0=\eta,\xi_*^i=\eta\left(\sum_{\substack{1\le k\le i\\ i_1+\cdots+i_k=i\\ i_1\ge 1,\cdots,i_k\ge 1}} d^{i_1}\left(hd^{i_2}\right)\cdots\left(hd^{i_2}\right)\right)\xi,i\ge 1 \tag{6}$$

$$h_*^0=h,h_*^i=h\left(\sum_{\substack{1\le k\le i\\ i_1+\cdots+i_k=i\\ i_1\ge 1,\cdots,i_k\ge 1}} d^{i_1}\left(hd^{i_2}\right)\cdots\left(hd^{i_2}\right)\right)\xi,i\ge 1 \tag{7}$$

If the strong deformation retraction $(\eta : (X, d^0) \rightleftarrows (Y, d) : \xi, h)$ is SDR-case , then $(\{\eta_*^i\} : (X, d^i) \rightleftarrows (Y, d_*^i) : \{\xi_*^i\}, \{h_*^i\})$ is also SDR-case.

Theorem 2 *Suppose that $\{\eta^i\} : (X, d^i) \rightleftarrows (Y, d^i) : \{\xi^i\}, \{h^i\}$ is an arbitrary strong deformation retraction of D_∞-module and $(\{\eta_*^i\} : (X, d^i) \rightleftarrows (Y, d_*^i) : \{\xi_*^i\}, \{h_*^i\})$ is strong deformation retraction of D_∞-modules which is given from the strong deformation retraction $\{\eta^0\} : (X, d^0) \rightleftarrows (Y, d^0) : \{\xi^0\}, \{h^0\}$, then the D_∞ -module morphisms $\eta * \xi = \{\eta_*^i\}\{\xi_*^i\} : (Y, d^i) \to (Y, d_*^i)$,where $(\eta * \xi)^0 = \eta^0 * \xi^0 = \eta^0 * \xi^0 = 1_Y$, is the inverse -modules isomorphisms $g = \{g^i\} : (Y, d^i) \to (Y, d_*^i)$, such that*

$$g^0=1_Y,g^i=\sum_{\substack{1\le k\le i\\ i_1+\cdots+i_k=i\\ i_1\ge 1,\cdots,i_k\ge 1}}(-1)^k\,(\eta*\xi)^{i_1}\cdots(\eta*\xi)^{i_k},i\ge 1$$

In what follows, we are concerned with perturbation of -differential module and its properties.

Definition 9 *A differential perturbation of D_∞-module (X, d^i) is defined to be a family of modules maps $\{t^i : X \to X,\ i \ge 1,\ i \in Z,\ t^0 = 0\}$ such that , for every integer $k \ge 1$, the following hold:*

$$\sum_{i+j=k} d^i t^j+\sum_{i+j=k} t^i d^j=-\sum_{i+j=k} t^i t^j \tag{8}$$

From relation clearly for any differential perturbation $\{t^i : X \to X,\ i \ge 1,\ i \in Z,\ t^0 = 0\}$ of a D_∞ −differential module (X, d^i) , there is a new D_∞ differential $\{D^i = d^i + t^i,\ i \ge 0,\ i \in Z\}$.

Note that for $k = 1$ the relation (8) has the form: $d^0t^1 + t^1d^0 = 0$, that is and are anticommutative i. e.$d^0t^1 = -t^1d^0$. If $k = 2$, then $d^0t^2 + t^2d^0 = 0 - (d^1t^1 + t^1d^1 + (t^1)^2)$.Thus the map $t^1 : X \to X$ homotopic to perturbation map $d^1 : X \to X$. Thus we can define for D_∞-differential module (X, d^i) the perturbation $\{t^i\}$, such that $d^0 = 0, t^1 = t$ and $t^i = 0, i \ge 2$.

An examples of perturbation D_∞-differential module can be established by considering the differential module with filtration. Suppose there are , over an arbitrary field , a differential module with increasing filtration $\{X^n\}, d(X^n) \subseteq X^n, n \ge 0$ also there is a differential perturbation of differential module(X, d) satisfies $t(X^n) \subseteq X^{n-1},\ n \ge 1$.Suppose the submodule Y^n on X^n, such that $X^n = Y^n \oplus X^{n-1}$, then $t_n^i : Y^n \to Y^{n-1},\ i \ge 1$such that $t : Y^n \to X^{n-1} = Y^{n-1} \oplus ... \oplus Y^{n-i} \oplus ... \oplus Y^0$. Clearly the family $t^i : X \to X,\ i \ge 1$, where $t^i = 0,\ t^i = \oplus_{n\ge 0} \oplus t_n^i,\ i \ge 1$ is a perturbation of differential module (X, d^i) , since $dt+td = -t^2$. To study the homotopy invariant of perturbation of D_∞-differential module suppose the strong deformation retraction $(\{\eta^i\} : (X, d^i) \rightleftarrows (Y, d^i) : \{\xi^i\}, \{h^i\})$ of D_∞-differential module, suppose also and the perturbation $\{t^i : X \to X\}$ for D_∞-differential module (X, d^i) . Our aim to establish the perturbation $\{t_*^i : Y \to Y\}$ of D_∞-differential module (Y, d^i) . Clearly $t_*^0 = 0$. Let t_*^1 is given by $t_*^1 = \eta^0t^1\xi^0$, then using the relation $d^0t^1 + t^1d^0 = 0$ we have

$$d^0t_*^1+t_*^1d^0=d^0(\eta^0t^1\xi^0)+(\eta^0t^1\xi^0)d^0=\eta^0d^0t^1\xi^0+\eta^0t^1d^0\xi^0=\eta^0(d^0t^1+t^1d^0)\xi^0=0\quad(i)$$

Define t_*^2 by the relation $t_*^2 = \eta^0 t^2 \xi^0 + \eta^1 t^1 \xi^0 + \eta^0 t^2 \xi^1 + \eta^0 t^1 h^0 t^1 \xi^0$.By using the relations (i) we have

$$d^0 t^2 + d^1 t^1 + t^2 d^0 = -t^1 t^1 \qquad (ii)$$

we can easily get, for given t_*^1 and t_*^2 the following relation
$d^0 t_*^2 + d^1 t_*^1 + t_*^1 d^1 + t_*^2 d^0 = -t^1 t_*^1$ *, since*

$$\begin{aligned}
d^1 t_*^1 &= d^1(\eta^0 t^1 \xi^0) = \eta^1 d^0 t^1 \xi^0,\ t_*^1 d^1 = (\eta^0 t^1 \xi^0) d^1 = \eta^0 t^1 d^0 t^1 \xi^1,\\
d^0 t_*^2 &= \eta^0 d^0 t^2 \xi^0 + \eta^0 d^1 t^1 \xi^0 + \eta^0 t^1 d^1 \xi^0 + d^0 \eta^0 t^1 h^0 t^1 \xi^0,\\
t_*^2 d^0 &= \eta^0 t^2 d^0 \xi^0 + \eta^0 t^1 d^0 t^1 \xi^0 + \eta^0 t^1 d^1 \xi^0 + \eta^0 t^1 h^0 t^1 \xi^0 d^0,\\
t_*^1 t_*^1 &= (\eta^0 t^1 \xi^0)(\eta^0 t^1 \xi^0) = \eta^0 t^1 (d^0 h^0 + h^0 d^0 - 1) t^1 \xi^0\\
&= \eta^0 t^1 d^0 h^0 t^1 \xi^0 + \eta^0 t^1 h^0 d^0 t^1 \xi^0 - \eta^0 t^1 t^1 \xi^0\\
&= -d^0 \eta^0 t^1 h^0 t^1 \xi^0 - \eta^0 t^1 h^0 t^1 \xi^0 d^0 -^0 \eta^0 t^1 t^1 \xi^0.
\end{aligned}$$

Consequently By considering the relations (i), (ii) and $d^0 t^3 + d^1 t^2 + d^2 t^1 + t^2 d^1 + d^1 t^2 + t^3 d^0 = -(t^1 t^2 + t^2 t^1)$ we get the map t_*^3 as follows

$$\begin{aligned}
t_*^3 \;=\; & \eta^0 t^3 \xi^0 + \eta^1 t^2 \xi^0 + \eta^2 t^1 \xi^0 + \eta^0 t^1 \xi^2 + \eta^1 t^1 \xi^1 + \eta^0 t^2 h^0 t^1 \xi^0 +\\
& + \eta^0 t^1 h^0 t^2 \xi^0 + \eta^0 t^1 h^1 t^1 \xi^0 + \eta^1 t^1 h^0 t^1 \xi^0 + \eta^0 t^1 h^0 t^1 \xi^1 + \eta^0 t^1 h^0 t^1 h^0 t^1 \xi^0,
\end{aligned}$$

such that $d^0 t_*^3 + d^1 t_*^2 + t_*^2 d^1 + t_*^1 d^2 + t_*^3 d^1 = -(t^1 t_*^2 + t_*^2 t^1)$.

The following assertion gives new perturbation $t_*^i,\ i \geq 0$. of D_∞-differential (strong deformation retraction) by helping of homotopy invariant concept.

Theorem 3 *Suppose the strong deformation retraction $(\{\eta^i\} : (X, d^i) \rightleftarrows (Y, d^i) : \{\xi^i\}, \{h^i\})$ of D_∞-differential module and differential perturbation $\{t^i : X \to X\}$. Then*

1. on the D_∞-differential module (Y, d^i) we can establish the perturbation $\{\tilde{t^i} : Y \to Y\}$as follows :

$$t_*^0 = 0,\ t_*^i = \sum_{\substack{1 \leq k \leq i\\ i_1 + \cdots + i_k + j_1\\ + j_2 + \cdots + j_{k+1} = i}} (h^{j_1} t^{i_1})(h^{j_2} t^{i_2}) \cdots (h^{j_k} t^{i_k}) \xi^{j_{k+1}}, i \geq 1 \qquad (9)$$

Definition 10 *We also get the strong deformation retraction retraction $(\{\tilde{\eta}^i\} : (X, d^i + t^i) \rightleftarrows (Y, d^i + t_*^i) : \{\tilde{\xi^i}\}, \{\tilde{h}^i\})$ such that:*

$$\tilde{\xi}^0 = \xi^0, \tilde{\xi}^i = \sum_{\substack{1 \leq k \leq i\\ i_1 + \cdots + i_k + j_1\\ + j_2 + \cdots + j_{k+1} = i}} (h^{j_1} t^{i_1})(h^{j_2} t^{i_2}) \cdots (h^{j_k} t^{i_k}) \xi^{j_{k+1}}, i \geq 1 \qquad (10)$$

$$\tilde{\eta}^0 = \xi^0, \tilde{\eta}^i = \sum_{\substack{1 \leq k \leq i\\ i_1 + \cdots + i_k + j_1\\ + j_2 + \cdots + j_{k+1} = i}} (h^{j_1} t^{i_1})(h^{j_2} t^{i_2}) \cdots (h^{j_k} t^{i_k}) \xi^{j_{k+1}}, i \geq 1 \qquad (11)$$

2. If the strong deformation retraction $(\{\eta^i\} : (X, d^i) \rightleftarrows (Y, d^i) : \{\xi^i\}, \{h^i\})$ is SDR-case of D_∞-differential module, then

$$(\{\tilde{\eta}^i\} : (X, d^i + t^i) \rightleftarrows (Y, d^i + t_*^i) : \{\tilde{\xi^i}\}, \{\tilde{h}^i\}), \qquad (12)$$

is also SDR-case of D_∞-differential module.

Proof. *The strong deformation retraction $(\{\eta_*^i\} : (X, d^i) \rightleftarrows (Y, d_*^i) : \{\xi_*^i\}, \{h_*^i\})$ of D_∞-module which is defined by formula (4)- (7) and is the strong deformation retraction$(\{\eta^0\} : (X, d^0) \rightleftarrows (Y, d^0) : \{\xi^0\}, \{h^0\})$*

Above all, the strong deformation retraction

$$(\{\bar{\eta}^i\} : (X, D^i = d^i + t^i) \rightleftarrows (Y, \bar{D}^i = d^i + \bar{t}^i) : \{\bar{\xi}^i\}, \{\bar{h}^i\}), \tag{13}$$

where $\bar{t}^i = \bar{D}^i - d_*^i$ is strong deformation retraction $(i = 0)$: $(\{\eta^0\} : (X, D^0 = d^0 + t^0) \rightleftarrows (Y, \bar{D}^0 = d^0 :$ $\{\xi^0\}, \{h^0\})$ .By considering the isomorphism $\eta * \xi = \{(\eta * \xi)^i\} : (Y, d^0) \rightleftarrows (Y, d_*^0) : \{g^i\} = g$, from theorem 1.2 and relation (13) we can define the following relation

$$(\{\tilde{\eta}^i\} : (X, D^i = d^i + t^i) \rightleftarrows (Y, \tilde{D}^i = d^i + t_*^i) : \{\tilde{\xi}^i\}, \{\tilde{h}^i\}), \tag{14}$$

As follows :

$$\{t_*^i\} = g\{\bar{t}^i\}\,\eta * \xi,\ \tilde{\eta} = \{\tilde{\eta}^i\} = g\,\tilde{\eta},\ \tilde{\xi} = \{\tilde{\xi}^i\} = \bar{\xi}\,\eta * \xi,\ \tilde{h} = \{\tilde{h}^i\} = \bar{h}, \tag{15}$$

The direct calculation shows that the strong deformation retraction of D_∞-module (14) is unknown i. e. the formula (15) equivalent to formula (9)-(12). ■

For any D_∞-differential module over field consider its SDR-case homology and apply theorem 10 for any perturbation of this module, we get the following assertion

Corollary 4 *Suppose the D_∞-differential module (X, d^i) over field and its perturbation $\{t^i : X \to X\}$, then On the homology of D_∞-differential module $(H(X), d_*^i)$ there is a perturbation $\{t_*^i : H(X) \to H(X)\}$, and also appear the SDR-case D_∞-differential module $(\{\tilde{\eta}^i\} : (X, d^i + t^i) \rightleftarrows (H(X), d_*^i + t_*^i) : \{\tilde{\xi}^i\}, \{\tilde{h}^i\})$, which is given by relations (10)-(12).*

Finally if the -differential module is stable [5], then theorem 3 can be given by helping of the perturbation of D_∞-differential module from [2-4]. If D_∞-differential module is not stable, the perturbation techniques in [2-4], to prove theorem 12, is not applicable.

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