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Control Chaos in Fractional System by Using a Feasible Scheme of Phase Compression

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Abstract:A feasible scheme of phase space compression is presented to control chaos in the fractional system, and the fractional order Lorenz system is investigated as an example. The output of chaotic system are sampled and compared with the external standard values, the upper and the bottom thresholds. The chaotic system develops autonomously when the output variables are within the scopes between the upper and bottom thresholds, otherwise, the output variables are switched to the corresponding thresholds, thus an intermittent feedback is imposed on the chaotic system. It is found that the chaotic state could be stabilized and certain periodical orbits are approached with appropriate thresholds being used.

Keywords: chaos; fractional order system; phase space compression

1 Instroduction

Fractional calculus is a more than 300 years old topic. In the past, it was considered that this technique is only a mathematical concept [1]. But, in recent years, there has been a rapid grow in the number of applications where fractional calculus has been used. It might be due to the fact that effective methods for differentiation and integration of non-integer order equations have been introduced [2]. Although the fractional calculus has a long history, its applications to physics and engineering have just started in the recent decades [3-5].

The fractional order chaotic system in natural science and technology field has been extensively demonstrated and is very common [6-10]. The fractional order system and its potential application become promising and attractive due to the development of the fractional order calculus. Typically, chaotic systems remain chaotic when their equations become fractional [11-13]. For example, it has been shown that the fractional order Chua's circuit with an appropriate cubic nonlinearity and with an order as low as 2.7 can produce a chaotic attractor [14].

In practice, it is often desired that chaos be avoided and/or that the system performance be improved or changed in some way. In the latest decade, controlling chaos has been given much attention by scientists and technologists because of their potential application [15-22]. Up to date, many control methods have been used to control chaotic system by stabilizing unstable periodic orbits or fixed points. Generally speaking, there are two kinds of method for controlling chaos, feedback control and nonfeedback control, and each one has both advantages and disadvantages. Signal feedback injections can successfully control chaos, and these approaches can be well studied. Nonfeedback control does not need any prior knowledge of the system or explicit changes of system parameters, it is easy to implement, and may be particularly convenient for experimentalists. When the system is under control, the controlling input does not vanish and the controlled target state may or may not be an unstable periodic orbit of the chaotic attractor.

In this paper, an important nonfeedback method is presented by using phase space compression, which compresses the phase space orbit of the chaotic attractor and eliminate chaos[23,24]. To our knowledge, the phase space compression approach has been used in integer order chaotic system and achieved successful control. However, the relevant investigation into the systems described by fractional order differential equations, which are more complicated and precise for many

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realistic situation, is still dull. Furthermore, there are essential differences between ordinary differential equation systems and fractional order differential systems. Most properties and conclusions of ordinary differential equation systems cannot be extended to that of the fractional order differential systems. Therefore, the fractional order systems have been paid more attention. Recently, many investigations were devoted to the chaotic dynamics and chaotic control of fractional order systems. In Ref.[25], Cauchy's integral formula via the modified Riemann-Liouville derivative for analytic functions of fractional order. In Ref.[26], fractional complex transform is proposed to convert fractional differential equations into ordinary differential equations.

The rest of the paper is organized as follows: Section 2 provides the fractional order model. In section 3, the numerical algorithm for the fractional order system is briefly introduced. In section 4, phase space compression of the fractional order system is numerically studied. Finally, concluding comments are given in section 5.

2 Fractional order chaotic system

The model we used in this paper is a fractional order Lorenz model [27]:

$$\begin{cases} \frac{d^{q_1}x}{dt^{q_1}} = \sigma(y-x), \\ \frac{d^{q_2}y}{dt^{q_2}} = rx - xz - y, \\ \frac{d^{q_3}z}{dt^{q_3}} = xy - bz, \end{cases} \tag{1}$$

where the state variables are $(x, y, x) \in R^3$ and σ, r and b are real parameters, q_i is the fractional order, $0 < q_i \leq 1, (i = 1, 2, 3)$. The fractional-order system (1) is called a commensurate fractional order system if $q_1 = q_2 = q_3 = q$, otherwise we call the system (1) a incommensurate fractional order system. For the classical values $q_1 = q_2 = q_3 = 1, \sigma = 10, r = 45.92$ and $b = 4$, the system presents the famous butterfly-shaped chaotic attractor [28]. It is demonstrated that chaos does exist in the fractional order Lorenz system with order less than 3 [27]. For example, when $q_1 = q_2 = q_3 = 0.99, \sigma = 10, r = 45.92, b = 4$ and $q_1 = 0.98, q_2 = 0.99, q_3 = 0.991, \sigma = 10, r = 45.92, b = 4$, chaotic attractors are found and the portraits are shown in figure 1 and 2, respectively.

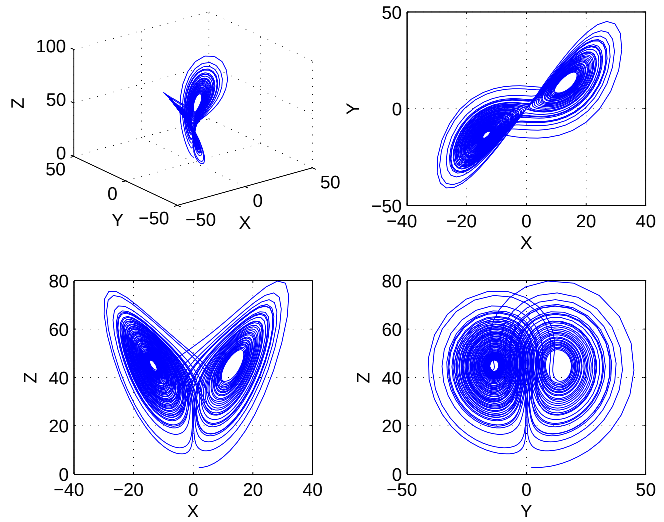

Figure 1: Phase diagrams of commensurate fractional Lorenz system.

3 Fractional derivative and numerical algorithm

There are two approximation methods for solving fractional differential equations. The first one is an improved version of the Adams-Bashforth-Moulton algorithm, and the rest one is the frequency domain approximation. The Caputo derivative definition involves a time-domain computation, in which nonhomogenous initial conditions are needed, and those values are readily determined. In this paper, the Caputo fractional derivative defined in [29] is often described by

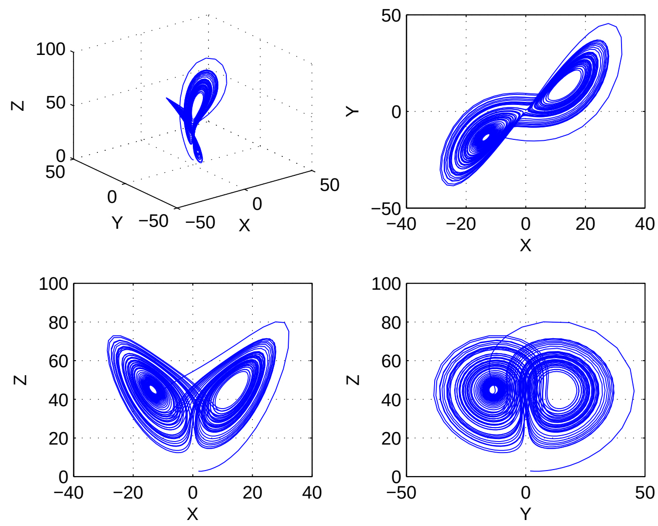

Figure 2: Phase diagrams of incommensurate fractional Lorenz system.

$$D^q f(t) = J^{n-q} f^{(n)}(t), q > 0,$$

when n is the first integer that is not less than q, J^α is the α-order Riemann-Liouville integral operator which defined by

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau,$$

where Γ is the Gamma function, $0 < \alpha \leq 1$.

By exploiting the Adams-Bashforth-Moulton scheme[30], the fractional order system (1) can be discretized as followings:

$$x_{n+1} = x_0 + \frac{h^{q_1}}{\Gamma(q_1+2)} \sigma(y^\rho_{n+1} - x^\rho_{n+1}) + \frac{h^{q_1}}{\Gamma(q_1+2)} \sum_{j=0}^{n} \beta_{1,j,n+1} \sigma(y_j - z_j),$$

$$y_{n+1} = y_0 + \frac{h^{q_2}}{\Gamma(q_2+2)} (r x^\rho_{n+1} + x^\rho_{n+1} z^\rho_{n+1} - y^\rho_{n+1}) + \frac{h^{q_2}}{\Gamma(q_2+2)} \sum_{j=0}^{n} \beta_{2,j,n+1} (r x_j + x_j z_j - y_j),$$

$$z_{n+1} = z_0 + \frac{h^{q_3}}{\Gamma(q_3+2)} (x^\rho_{n+1} y^\rho_{n+1} - b z^\rho_{n+1}) + \frac{h^{q_3}}{\Gamma(q_3+2)} \sum_{j=0}^{n} \beta_{3,j,n+1} (x_j y_j - b z_j),$$

$$x^\rho_{n+1} = x_0 + \frac{1}{\Gamma(q_1)} \sum_{j=0}^{n} \gamma_{1,j,n+1} \sigma(y_j - z_j),$$

$$y^\rho_{n+1} = y_0 + \frac{1}{\Gamma(q_2)} \sum_{j=0}^{n} \gamma_{2,j,n+1} (r x_j + x_j z_j - y_j),$$

$$z^\rho_{n+1} = z_0 + \frac{1}{\Gamma(q_3)} \sum_{j=0}^{n} \gamma_{3,j,n+1} (x_j y_j - b z_j),$$

$$\beta_{i,j,n+1} = \begin{cases} n^{q_i+1} - (n-q_i)(n+1)^{q_i} & j = 0, \\ (n-j+2)^{q_i+1} + (n-j)^{q_i+1} - 2(n-j+1)^{q_i+1} & 1 \leq j \leq n, \\ 1 & j = n+1, \end{cases}$$

$$\gamma_{i,j,n+1} = \frac{h^{q_i}}{q_i} ((n-j+1)^{q_i} - (n-j)^{q_i}), 0 \leq j \leq n, i = 1, 2, 3.$$

4 Phase space compression

Phase space compression is interesting because of being unnecessary to detect the system parameters and easy to be operated, therefore it has some conveniences. In this section, we consider the fractional order Lorenz system with phase space compression and take $\sigma = 10, r = 45.92, b = 4$. We interfere the system variable by confining system variable in $[x_{min}, x_{max}], [y_{min}, y_{max}], [z_{min}, z_{max}]$, i.e., when the control operation starts, let

$$x = \begin{cases} x_{max} & x \geq x_{max}, \\ x & x_{min} < x < x_{max}, \\ x_{min} & x \leq x_{min}, \end{cases}$$

$$y = \begin{cases} y_{max} & y \geq y_{max}, \\ y & y_{min} < y < y_{max}, \\ y_{min} & y \leq y_{min}, \end{cases}$$

$$z = \begin{cases} z_{max} & z \geq z_{max}, \\ z & z_{min} < z < z_{max}, \\ z_{min} & z \leq z_{min}, \end{cases}$$

where $x_{min}, x_{max}, y_{min}, y_{max}$ and z_{min}, z_{max} are the maximum and minimum suppression limitations for the system variables x, y and z.

4.1 Phase space compression for commensurate fractional Lorenz system

To explore controlling results with fractional order Lorenz system, we set $q_1 = q_2 = q_3 = 0.99$. Assume that the system is in the chaotic region, i.e., system parameters are chosen from chaotic area. Figure 1 plots the trajectory in the phase space, in which the system parameters are exemplified as $q_1 = q_2 = q_3 = 0.99, \sigma = 10, r = 45.92, b = 4$. The chaotic motion is obvious. Now we use phase space compression to control chaos in the fractional order Lorenz system and give the control parameter ranges that lead to orbits of any desired period or fixed points. To illustrate this, we compress system variables x, y, z into $[-15, 13.45], [-15, 13.45], [20, 60]$ when $t \geq 20s$, respectively. The fixed point is obtained, the trajectory of system is illustrate in figure 3. Then we increase the compression limitation in figure 4 with $[-20, 25], [-35, 35], [10, 70]$, the period-1 orbit is induced to substitute chaos. In figure 5, we analyze the fractional order Lorenz system with compression limitation $[-23, 26], [-25, 35], [10, 70]$ and obtain a stable period-10 orbit.

This result of controlling chaos can transform the system from a chaotic state into a periodic state or stationary when the system orbit is limited to the compressed phase space, while the system parameters are not changed. This is because the control works before the next evolution process and phase space compression compels the chaotic attractor to take only the values decided by the control when the orbit exceeds the subspace boundaries. These definite values decide the subsequent process according to the dynamical function, suppress the possible evolution orbits in the original phase space, and form the new orbit distributions in the system. In substance, phase space compression limits free contraction and expansion of the chaotic attractor in phase space, thus changing the system's dynamical character. Hence, by appropriately selecting different phase space compression parameters, one can control chaos to different periodic or other states. In a word, the formation of the chaotic attractor of a nonlinear system needs both appropriate parameter values and a large enough phase space; changing either of these two conditions will vary the system dynamics.

4.2 Phase space compression for incommensurate fractional Lorenz system

For the incommensurate fractional order Lorenz system, we select system parameters $q_1 = 0.98, q_2 = 0.99, q_3 = 0.991, \sigma = 10, r = 45.92, b = 4$. After transient iterations, the evolution orbits of system represent a chaotic attractor in phase space (see figure 2); this chaotic attractor is limited in a bounded phase space V. To start to control chaos in system (1), we select a subspace $W, W \subset V, \emptyset \not\subset W$, and compress the orbits of system (1) into W. We use the discretization scheme in section 3. The simulation results of incommensurate fractional order Lorenz system are showed in figure 6 to figure 9. We choose $W = \{[-10, 13.42], [-10, 13.42], [30, 50]\}$ when $t \geq 20s$, the trajectory of system (1) convergences to a fixed point (see figure 6). When we increase the compression limitation in figure 7 with $[-15, 30], [-25, 45], [20, 75]$, the period-1 orbit is obtained. We further expand the area of subspace W, the period-4 orbit is obtained (see figure 8). Power spectrums of period-4 orbit is plotted in figure 9. From figure 9, the power spectrum of system (1) demonstrates

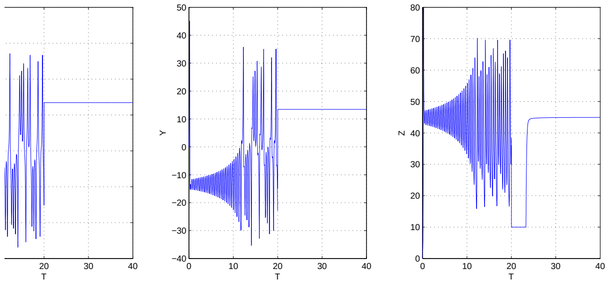

Figure 3: Time evolution of commensurate fractional Lorenz system with [-15,13.45], [-15,13.45],[20,60].

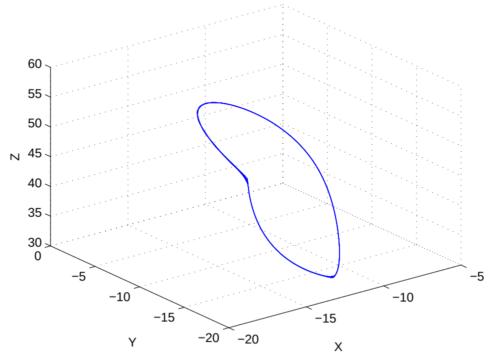

Figure 4: Phase diagrams of commensurate fractional Lorenz system with [-20,25],[-35,35],[10,70].

discrete feature. In general, when the bounded of subspace W close to V, we can obtain the high period orbit. While the area of subspace W is small, the low period orbit is induced.

Finally, it is important to discuss its feasibility in experiments. The output variables are monitored and sampled to compare with the external standard signal, which the amplitudes are selected as the upper and bottom thresholds. When the output variables are beyond the selected thresholds, the output variables are switched to the corresponding threshold, otherwise, the system develops itself freely without external perturbation, and it could be realized by Heaviside function which can be practiced in experiments by using diodes in circuits.

5 Conclusions

The phase space compression is one of the nonfeedback methods in controlling chaos. This scheme is very simple and effective, and there is no need to know the prior condition of detecting system parameters in many occasions. Moreover, this method is easy to be operated in practical applications. In the above analysis and simulations, we have shown that the phase space compression is effective for controlling fractional order Lorenz system. The chaos in same fractional order or different fractional orders is successfully controlled to period orbits or fixed points by appropriately selecting the phase space compression. Using this control rule, we obtain various desired periods orbits or fixed points by different phase space compressions. Numerical simulations also show that phase space compression is effective for controlling chaos. The method avoids complex mathematical calculations in numerical simulations and may require only a multichannel threshold detector or a multichannel amplitude limiter in experiments. Controlling chaos is a very difficult and significant task in real systems such as hydrodynamic systems, laser systems, chemical reactions, biological systems, and so on. We are sure the simple and effective method in this paper will have very important applications in practice.

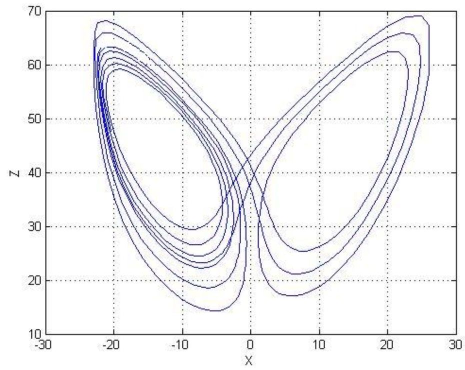

Figure 5: Phase diagrams of commensurate fractional Lorenz system with [-23,26],[-25,35],[10,70].

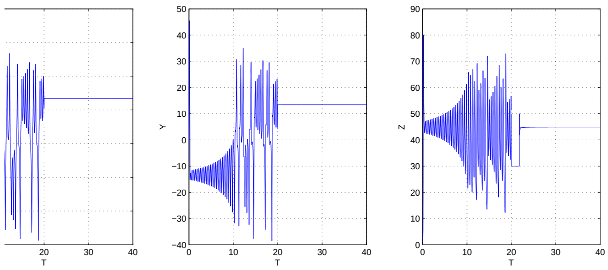

Figure 6: Time evolution of incommensurate fractional Lorenz system with [-10,13.42],[-10,13.42],[30,50].

Acknowledgements

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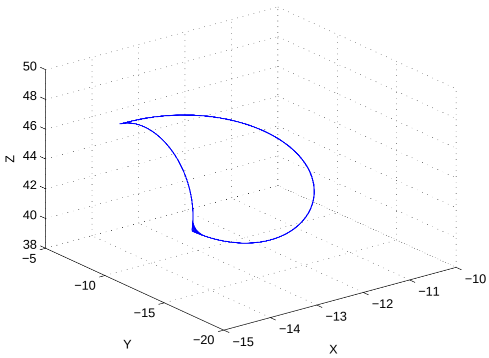

Figure 7: Phase diagrams of incommensurate fractional Lorenz system with [-15,30],[-25,45],[20,75].

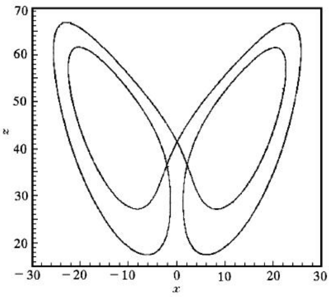

Figure 8: Phase diagrams of incommensurate fractional Lorenz system with [-30,30],[-25,45],[10,70].

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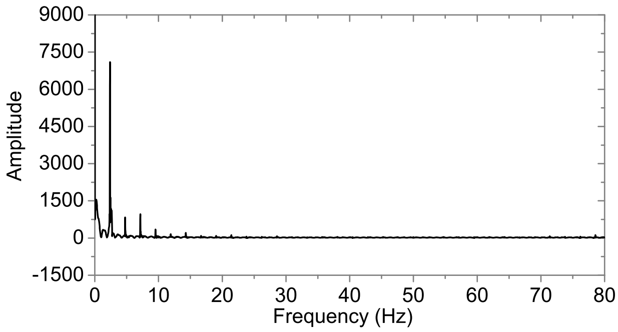

Figure 9: Power spectrum of incommensurate fractional Lorenz system with [-30,30],[-25,45],[10,70].

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