ISSN 1749-3889 (print), 1749-3897 (online)
**International Journal of Nonlinear Science**
Vol.13(2012) No.3,pp.308-316

# A Modified Homotopy Perturbation Method for Solving Linear and Nonlinear Integral Equations

N. Aghazadeh*, S. Mohammadi
Department of Applied Mathematics, Azarbaijan University of Tarbiat Moallem, Tabriz 53751 71379, Iran
*(Received 23 June 2011 , accepted 15 January 2012)*

**Abstract:** In this paper, we present a modification to so-called homotopy perturbation method for solving linear and non-linear integral equations. This method gives an approximate analytic solution to the equations (usually the exact solution of the equations). Some numerical examples presented to show the accuracy and efficiency of the method.

**Keywords:** homotopy perturbation method; integral equation; Fredholm; Volterra

## 1 Introduction

Although perturbation techniques are widely applied to analyze nonlinear problems in science and engineering, they are however so strongly dependent on small parameters appeared in equations under consideration that they are restricted only to weakly nonlinear problems. For strongly nonlinear problems which don't contain any small parameters, perturbation techniques are invalid. So, it seems necessary and worthwhile developing at new kind of analytic technique independent of small parameters.

Liao proposed a new analytic technique in his Ph.D. dissertation [1], namely the Homotopy Analysis Method (HAM). Based on homotopy of topology, the validity of the HAM is independent of whether or not there exist small parameters in considered equations. Therefore, the HAM can overcome the foregoing restrictions and limitations of perturbation techniques so that it provides us with a powerful tool to analyze strongly nonlinear problems. [2]

In [2] some basic ideas about the HAM was described. In [3] some developments of the HAM was presented. Also some lemmas and theorems was proved. In [4] a reliable approach for convergence of the HAM was discussed. In [10]–[14] [46]–[48] [51]–[53] the HAM was applied on some equations. Also, some modifications and improvements was discussed by authors (e.g. see [15]–[17]).

In [19, 20] the homotopy perturbation technique was presented. In [21]–[41] [49]–[50][54]–[55] the homotopy perturbation technique was applied on different equations by some authors and with some modifications (e.g. linear and nonlinear forth-order boundary value problems, functional integral equations, nonlinear problems, system of nonlinear Fredholm integral equations, forth-order integro-differential equations, eighth-order boundary value problems, nonlinear oscillators, partial differential equations, quadratic Riccati differential equation, Volterra integral equations, two-dimensional Fredholm integral equations, Stokes equations and nonlinear ill-posed operator equations). Also the homotopy perturbation method and the HAM was compared by some authors (e.g. [5]–[9]).

We now review [39] to show how HPM applied to the following integral equations. Consider the following integral equation:

$$\gamma(x) = f(x) + \int_a^b k(x,t)\gamma(t)dt, \quad c \leqslant x \leqslant d. \tag{1}$$

Let

$$L(u) = u(x) - f(x) - \int_a^b k(x,t)u(t)dt = 0, \tag{2}$$

*Corresponding author. *E-mail address*: aghazadeh@azaruniv.edu


IJNS.2012.06.15/610

with solution $u(x) = \gamma(x)$, we can define a homotopy $H(u,p)$ by

$$H(u,0) = F(u), \quad H(u,1) = L(u), \tag{3}$$

where $F(u)$ is a functional operator with solution $u_0$.We choose a convex homotopy

$$H(u,p) = (1-p)F(u) + pL(u) = 0, \tag{4}$$

and continuously trace an implicitly defined curve from a starting point $H(u_0)$ to a solution $H(\gamma, 1)$. In fact HPM uses the homotopy parameter $p$ as an expanding parameter [43, 44] to obtain

$$u = u_0 + pu_1 + p^2 u_2 + \cdots, \tag{5}$$

when $p \to 1$, (5) corresponds to (4) and gives an approximation to the solution of (2) as:

$$\gamma = \lim_{p\to 1} u = u_0 + u_1 + u_2 + \cdots, \tag{6}$$

The series (6) converges in most cases, and the rate of convergence depends on L(u). Taking $F(u) = u(x) - f(x)$, and substituting (5) in (4) and equating the terms with identical power of $p$, we obtain

$$\begin{aligned} p^0 &: \quad u_0 - f(x) = 0 \Rightarrow u_0 = f(x), \\ p^1 &: \quad u_1 - \int_a^b k(x,t)u_0(t)dt = 0, \\ &\quad\;\; u_1 = \int_a^b k(x,t)u_0(t)dt, \\ &\vdots \end{aligned}$$

and in general we have

$$u_0(x) = f(x), \tag{7}$$

$$u_{n+1}(x) = \int_a^b k(x,t)u_n(t)dt, \quad n = 1, 2, \ldots. \tag{8}$$

# 2 The new modified HPM

In [39] modification applied to HPM for linear integral equations with degenerate kernels, which used a number $m$. In this paper, $m$ is a function (or a number), and our modification can be applied to linear or non-linear integral equations, and there is no limitation on kernel type.

## 2.1 Application to linear second kind Fredholm integral equation

In this section, we apply the modified perturbation method to (1). To this end, we define a new convex homotopy perturbation as

$$H(u,p,m) = (1-p)F(u) + pL(u) + p(1-p)mK^*f = 0, \tag{9}$$

where $F(u) = u(x) - f(x)$, $L(u) = u(x) - f(x) - \int_a^b k(x,t)u(t)dt = 0$ and $K^*f = \int_a^b k(x,t)f(t)dt$, hence we can write

$$(1-p)(u-f) + p\left[u - f - \int_a^b k(x,t)u(t)dt\right] + p(1-p)mK^*f = 0, \tag{10}$$

or

$$u - f - p\int_a^b k(x,t)u(t)dt + p(1-p)mK^*f = 0, \tag{11}$$

Substituting (5) into (11) and equating the coefficients of like terms with the identical powers of $p$ , we obtain

$$\begin{aligned}
p^0: &\quad u_0 - f(x) = 0 \Rightarrow u_0 = f(x),\\
p^1: &\quad u_1 - \int_a^b k(x,t)u_0(t)dt + mK^*f = 0,\\
&\quad u_1 = (1-m)K^*f,\\
p^2: &\quad u_2 - \int_a^b k(x,t)u_1(t)dt - mK^*f = 0,\\
&\quad u_2 = (1-m)K^*K^*f + mK^*f,\\
p^3: &\quad u_3 - \int_a^b k(x,t)u_2(t)dt = 0,\\
&\quad u_3 = \int_a^b k(x,t)u_2(t)dt,\\
&\vdots\\
p^{n+1}: &\quad u_{n+1} = \int_a^b k(x,t)u_n(t)dt, \quad n = 2,3,\ldots,
\end{aligned}$$

now we find $m$ such that $u_2 = 0$. Since if $u_2 = 0$ then $u_3 = u_4 = \cdots = 0$, and the exact solution will be obtained as $u(x) = u_0(x) + u_1(x)$, hence for all values of $x$ we should have

$$(1-m)K^*K^*f + mK^*f = 0,$$

or

$$m(x) = \frac{K^*K^*f}{K^*K^*f - K^*f}.$$

Note that the method can be applied as for Volterra integral equations, in same manner.

## 2.2 Application to non-linear Fredholm integral equations

Consider the following non-linear Fredholm integral equation

$$u(x) = f(x) + \int_a^b k(x,t)T(u(t))dt, \quad a \leqslant x \leqslant b \tag{12}$$

where the function $k$ is given and $T$ is a given nonlinear operator, and $u$ the solution to be determined. We assume that (12) has the unique solution. We define a convex homotopy perturbation as

$$H(u,p,m) = (1-p)F(u) + pL(u) + p(1-p)mK^*T(f) = 0, \tag{13}$$

where

$$F(u) = u(x) - f(x) \quad and \quad L(u) = u(x) - f(x) - \int_a^b k(x,t)T(u(t))dt = 0, \tag{14}$$

hence, we can write

$$(1-p)(u-f) + p\left[u - f - \int_a^b k(x,t)u(t)dt\right] + p(1-p)mK^*T(f) = 0, \tag{15}$$

or

$$u - f - p\int_a^b k(x,t)u(t)dt + p(1-p)mK^*T(f) = 0, \tag{16}$$

Substituting (5) into (16) results into

$$u_0 + pu_1 + p^2u_2 + \cdots - f(x) - p\int_a^b k(x,t)T(u_0 + pu_1 + p^2u_2 + \cdots)dt + p(1-p)mK^*T(f) = 0 \tag{17}$$

In (17) we can write $T(u_0 + pu_1 + p^2u_2 + \cdots)$ as follows

$$T(u_0 + pu_1 + p^2u_2 + \cdots) = A_0 + pA_1 + p^2A_2 + \cdots, \tag{18}$$

where $A_k$ are Adomian polynomials which depend upon $u_0, u_1, u_2, \ldots, u_k$. [39, 45] By differentiating both sides of (18) we can write

$$\frac{d^k}{dp^k}T(u_0 + pu_1 + p^2u_2 + \cdots)\mid_{p=0} = \frac{d^k}{dp^k}(A_0 + pA_1 + p^2A_2 + \cdots)\mid_{p=0}. \tag{19}$$

From (19) we have

$$A_k = A_k(u_0, u_1, u_2, \ldots, u_k) = \frac{1}{k!}\frac{d^k}{dp^k}T(u_0 + pu_1 + p^2u_2 + \cdots)\mid_{p=0}, \quad k = 0, 1, \ldots \tag{20}$$

By substituting (18) into (17) we have

$$u_0 + pu_1 + p^2u_2 + \cdots - f(x) - p\int_a^b k(x,t)(A_0 + pA_1 + p^2A_2 + \cdots)dt + p(1-p)mK^*T(f) = 0. \tag{21}$$

Equating the terms with identical powers of $p$, we have

$$\begin{aligned}
p^0: &\quad u_0 - f(x) = 0 \Rightarrow u_0 = f(x),\\
p^1: &\quad u_1 - \int_a^b k(x,t)A_0(t)dt + mK^*T(f) = 0,\\
&\quad u_1 = (1-m)K^*T(f),\\
p^2: &\quad u_2 - \int_a^b k(x,t)A_1(t)dt - mK^*T(f) = 0,\\
&\quad u_2 = K^*((1-m)K^*T(f)T'(f)) + mK^*T(f),\\
p^3: &\quad u_3 - \int_a^b k(x,t)A_2(t)dt = 0,\\
&\quad u_3 = \int_a^b k(x,t)A_2(t)dt,\\
&\vdots\\
p^{n+1}: &\quad u_{n+1} = \int_a^b k(x,t)A_n(t)dt, \quad n = 3, 4\ldots,
\end{aligned}$$

now we find $m$ such that $u_2 = 0$. Since if $u_2 = 0$ then $u_3 = u_4 = \cdots = 0$, and the exact solution will be obtained as $u(x) = u_0(x) + u_1(x)$, hence for all values of $x$ we should have

$$(1-m)K^*(K^*T(f)T'(f)) + mK^*T(f) = 0,$$

or

$$m(x) = \frac{K^*(K^*T(f)T'(f))}{K^*(K^*T(f)T'(f)) - K^*T(f)}.$$

Note that the method can be applied as for Volterra integral equations, in same manner.

## 3 Numerical Examples

**Example 1** *Consider the equation*

$$u(x) = (1-2\pi)\cos x + \sin x + \int_0^\pi 4\cos x \cos t u(t)dt, \tag{22}$$

*with exact solution $u(x) = \sin x + \cos x$. Using the method, we have*

$$\begin{aligned}
&u_0 = f(x),\\
&u_1 = (1-m)K^*f, \quad where \quad m(x) = \frac{K^*K^*f}{K^*K^*f - K^*f}
\end{aligned}$$

*and*

$$K^*f = \int_a^b k(x,t)f(t)dt$$
$$u(x) = u_0 + u_1.$$

*In this example* $K^*f = 2(1-2\pi)\pi\cos x$ *and* $K^*K^*f = 4(1-2\pi)\pi^2\cos x$ *hence*

$$m(x) = \frac{2\pi}{-1+2\pi},$$
$$u_0 = (1-2\pi)\cos x + \sin x,$$
$$u_1 = 2\pi\cos x$$

*so we have* $u(x) = \cos x + \sin x$*, which is the exact solution.*

**Example 2** *Consider the following Fredholm integral equation*

$$u(x) = e^{2x+\frac{1}{3}} - \frac{1}{3}\int_0^1 e^{2x+\frac{5}{3}t}u(t)dt,$$

*with exact solution* $u(x) = e^{2x}$*.*

*In this example* $K^*f = -(-1+e^{\frac{1}{3}})e^{\frac{1}{3}+2x}$ *and* $K^*K^*f = (-1+e^{\frac{1}{3}})^2e^{\frac{1}{3}+2x}$ *hence*

$$m(x) = 1 - \frac{1}{e^{\frac{1}{3}}}$$

$$u_0 = e^{2x+\frac{1}{3}},$$
$$u_1 = -(-1+e^{\frac{1}{3}})e^{2x}$$

*so we have* $u(x) = e^{2x}$*, which is the exact solution.*

**Example 3** *Consider the following Volterra integral equation*

$$u(x) = 12x + x^2 - 2x^3 - \frac{x^6}{30} - 11\sin x + 2\int_0^x (x-t)^3u(t)dt,$$

*with exact solution* $u(x) = x^2 + \sin x$*.*

*In this example*

$$K^*f = 2\left(66x - 11x^3 + \frac{3x^5}{5} + \frac{x^6}{60} - \frac{x^7}{70} - \frac{x^{10}}{25200} - 66\sin x\right)$$

*and*

$$K^*K^*f = 1584x - 264x^3 + \frac{66x^5}{5} - \frac{11x^7}{35} + \frac{x^9}{210} + \frac{x^{10}}{12600} - \frac{x^{11}}{23100} - \frac{x^{14}}{25225200} - 1584\sin x$$

*hence*

$$m(x) = \frac{x(-39956716800 + 6659452800x^2 - 332972640x^4 + 7927920x^6 - 120120x^8 - 2002x^9 + 1092x^{10} + x^{13}) + 39956716800\sin x}{x(-36626990400 + 6104498400x^2 - 302702400x^4 + 840840x^5 + 7207200x^6 - 120120x^8 - 4004x^9 + 1092x^{10} + x^{13}) + 36626990400\sin x}$$
$$u_0 = 12x + x^2 - 2x^3 - \frac{x^6}{30} - 11\sin x,$$
$$u_1 = \frac{143(x(-1663200 + 277200x^2 - 15120x^4 - 420x^5 + 360x^6 + x^9) + 1663200\sin x)^2}{900(x(-36626990400 + 6104498400x^2 - 302702400x^4 + 840840x^5 + 7207200x^6 - 120120x^8 - 4004x^9 + 1092x^{10} + x^{13}) + 36626990400\sin x)}$$
$$u(x) = 12x + x^2 - 2x^3 - \frac{x^6}{30} - 11\sin x + \frac{143(x(-1663200 + 277200x^2 - 15120x^4 - 420x^5 + 360x^6 + x^9) + 1663200\sin x)^2}{900(x(-36626990400 + 6104498400x^2 - 302702400x^4 + 840840x^5 + 7207200x^6 - 120120x^8 - 4004x^9 + 1092x^{10} + x^{13}) + 36626990400\sin x)}$$

**Example 4** *Consider the following Volterra integral equation*

$$u(x) = x - \int_0^x \sinh(x-t)u(t)dt,$$

*with exact solution* $u(x) = x - \frac{x^3}{6}$.

*Here,* $K^*f = x - \sinh x$ *and* $K^*K^*f = x + \frac{1}{2}x\cosh x - \frac{3\sinh x}{2}$ *hence*

$$m(x) = \frac{2x + x\cosh x - 3\sinh x}{x\cosh x - \sinh x}$$

$$u_0 = x,$$
$$u_1 = -\frac{2(x-\sinh x)^2}{x\cosh x - \sinh x}$$

$$u(x) = \frac{-2x^2 + x^2\cosh x + 3x\sinh x - 2\sinh x^2}{x\cosh x - \sinh x}$$

**Example 5** *Consider the following Volterra integral equation*

$$u(x) = \frac{1}{16}(7\cos x + 9\cos 3x + 4x\sin x) - \int_0^x (x-t)\cos(x-t)u(t)dt,$$

*with exact solution* $u(x) = \frac{1}{3}(2\cos 3x + 1)$.

*Here,* $K^*f = \frac{1}{1536}\left(-3(45+56x^2)\cos x + 135\cos 3x - 4x(15+8x^2)\sin x\right)$ *and*

$$K^*K^*f = \frac{1}{245760}\left(5(-675+24x^2+224x^4)\cos x + 3375\cos 3x + 4x(3345+40x^2+32x^4)\sin x\right)$$

*hence*

$$m(x) = \frac{5(-675+24x^2+224x^4)\cos x + 3375\cos 3x + 4x(3345+40x^2+32x^4)\sin x}{5(3645+5400x^2+224x^4)\cos x - 18225\cos 3x + 4x(5745+1320x^2+32x^4)\sin x}$$

$$u_0 = \frac{1}{16}(7\cos x + 9\cos 3x + 4x\sin x),$$
$$u_1 = -\frac{5(3(45+56x^2)\cos x - 135\cos 3x + 4x(15+8x^2)\sin x)^2}{48(5(3645+5400x^2+224x^4)\cos x - 18225\cos 3x + 4x(5745+1320x^2+32x^4)\sin x)}$$

$$u(x) = \frac{1}{1536}\left(96(7\cos x + 9\cos 3x + 4x\sin x) - \frac{160\left(3(45+56x^2)\cos x - 135\cos 3x + 4x(15+8x^2)\sin x\right)^2}{5(3645+5400x^2+224x^4)\cos x - 18225\cos 3x + 4x(5745+1320x^2+32x^4)\sin x}\right)$$

**Example 6** *Consider the following nonlinear Volterra integral equation*

$$u(x) = 1 + sin^2 x - 3\int_0^x \sin(x-t)u^2(t)dt,$$

*with exact solution* $u(x) = \cos x$. *Using the method, we have*

$$u_0 = f(x),$$
$$u_1 = (1-m)K^*T(f), \quad where \quad m(x) = \frac{K^*(K^*T(f)T'(f))}{K^*(K^*T(f)T'(f)) - K^*T(f)}$$

*and*

$$K^*T(f) = \int_a^b k(x,t)T(f(t))dt$$
$$u(x) = u_0 + u_1.$$

*In this problem* $K^*T(f) = \frac{1}{40}(-285 + 344\cos x - 60\cos 2x + \cos 4x)$ *and*

$$K^*(K^*T(f)T'(f)) = \frac{1}{5600}(-176509\cos x + 7315\cos 2x - 3(-57750 + 1505\cos 3x - 154\cos 4x + \cos 6x + 30100x\sin x))$$

Table 1: The values of absolute error for examples 3–6.

| x | example 3 | example 4 | example 5 | example 6 |
|---|---|---|---|---|
| 0.1 | 1.40e-14 | 2.183e-11 | 2.082e-9 | 5.735e-10 |
| 0.2 | 2.15e-14 | 2.790e-9 | 1.329e-7 | 1.42e-7 |
| 0.3 | 2.78e-13 | 4.759e-8 | 1.509e-6 | 3.423e-6 |
| 0.4 | 8.85e-12 | 3.556e-7 | 8.435e-6 | 3.148e-5 |
| 0.5 | 1.63e-10 | 1.691e-6 | 3.196e-5 | 1.689e-4 |
| 0.6 | 1.76e-9 | 6.036e-6 | 9.464e-5 | 6.397e-4 |
| 0.7 | 1.32e-8 | 1.768e-5 | 2.363e-4 | 1.895e-3 |
| 0.8 | 7.54e-8 | 4.480e-5 | 5.202e-4 | 4.668e-3 |
| 0.9 | 3.52e-7 | 1.016e-4 | 1.040e-3 | 9.964e-3 |
| 1.0 | 1.40e-6 | 2.111e-4 | 1.927e-3 | 1.896e-2 |

*hence*

$$m(x) = \frac{176509\cos x - 7315\cos 2x + 3(-57750 + 1505\cos 3x - 154\cos 4x + \cos 6x + 30100x\sin x)}{-193200 + 200589\cos x - 11515\cos 2x + 4515\cos 3x - 392\cos 4x + 3\cos 6x + 90300x\sin x},$$

$$u_0 = 1 + sin^2 x,$$

$$u_1 = \frac{7(-285 + 344\cos x - 60\cos 2x + \cos 4x)^2}{4(-193200 + 200589\cos x - 11515\cos 2x + 4515\cos 3x - 392\cos 4x + 3\cos 6x + 90300x\sin x)},$$

$$u(x) = 1 + \sin x^2 + \frac{7(-285 + 344\cos x - 60\cos 2x + \cos 4x)^2}{4(-193200 + 200589\cos x - 11515\cos 2x + 4515\cos 3x - 392\cos 4x + 3\cos 6x + 90300x\sin x)}.$$

# References


[1] S.J. Liao, The homotopy analysis method and its applications in mechanics, Ph.D. thesis, Shanghai Jiaotong University. 1992.

[2] S.J. Liao, Homotopy analysis method: a new analytic method for nonlinear problems, *Applied Mathematics and Mechanics*, 19(10)(1998):957–962.

[3] S.J. Liao, Notes on the homotopy analysis method: Some definitions and theorems, *Commun Nonlinear Sci Numer Simulat*, 14(2009):983–997.

[4] Z.M. Odibat, A study on the convergence of homotopy analysis method, *Applied Mathematics and Computation*, 217(2010):782–789.

[5] S. Liang and D. Jeffrey, Comparison of homotopy analysis method and homotopy perturbation method through an evolution equation, *Commun Nonlinear Sci Numer Simulat*, 14(2009):4057–4064.

[6] M.S.H. Chowdhury, I. Hashim and O. Abdulaziz, Comparison of homotopy analysis method and homotopy perturbation method for purely nonlinear fin-type problems, *Commun Nonlinear Sci and Numer Simulat*, 14(2009):371–378.

[7] M. Sajid and T. Hayat, Comparison of HAM and HPM methods in nonlinear heat conduction and convection equations, *Nonlinear Analysis: Real World Applications*, 9(2008):2296-2301.

[8] S. Liao, Comparison between the homotopy analysis method and homotopy perturbation method, *Applied Mathematics and Computation*, 169(2005):1186–1194.

[9] S. Liang and D.J. Jeffrey,Comparison of homotopy analysis method and homotopy perturbation method through an evolution equation, *Commun Nonlinear Sci Numer Simulat*, 14(2009):4057-4064.

[10] Z. Niu and C. Wang, A one-step optimal homotopy analysis method for nonlinear differential equations, Commun *Nonlinear Sci Numer Simulat*, 15(2010):2026–2036.

[11] S.J. Liao, An optimal homotopy analysis approach for strongly nonlinear differential equations, *Commun Nonlinear Sci Numer Simulat*, 15(2010):2003–2016.

[12] H. Jafari and M.A. Firoozjaee, Multistage homotopy analysis method for solving nonlinear integral equations, *Applications and Applied Mathematics: An International Journal*, Special Issue (1)(2010):34–45.

[13] Y.H. Qian, W. Zhang , B.W. Lin and S.K. Lai, Analytical approximate periodic solutions for two-of-freedom coupled van der pol-Duffing oscillators by extended homotopy analysis method, *Acta Mech*, DOI 10.1007/s00707-010-0433-3.

[14] A.S. Bataineh, M.S.M. Noorani and I. Hashim, Solving systems of ODEs by homotopy analysis method, *Commun Nonlinear Sci Numer Simulat*, 13(2008):2060–2070.

[15] S.S. Motsa, P. Sibanda, G.T. Marewo and S. Shateyi, A note on improved homotopy analysis method for solving the Jeffery-Hamel flow, *Mathematical Problems in Engineering*, Volume 2010, Article ID 359297, 11 pages.

[16] A.S. Bataineh, M.S.M. Noorani and I. Hashim, Modified homotopy analysis method for solving systems of second-order BVPs, *Commun in Nonlinear Scie and Numer Simulat*, 14(2009):430–442.

[17] R.A. Van Gorder and K. Vajravelu, On the selection of auxiliary functions, operators, and convergence control parameters in the application of the homotopy analysis method to nonlinear differential equations: a general approach, *Commun Nonlinear Sci Numer Simulat*, 14(2009):4078–4089.

[18] S. Abbasbandy, Y. Tan and S.J. Liao, Newton-homotopy analysis method for nonlinear equations, *Applied Mathematics and Computation*, 188(2007):1794–1800.

[19] J.H. He, Homotopy perturbation technique, *Comput. Methods Appl. Mech. Engrg.* 178(1999):257–262.

[20] J.H. He, An elementary introduction to the homotopy perturbation method, *Computers and Mathematics with Applications*, 57(2009):410–412.

[21] E. Roohi, F.R. Marzabadi, and Y. Farjami, Application of the homotopy perturbation method to linear and nonlinear fourth-order boundary value problems, *Phys. Scr.*, 77(055004)(2008):1–5.

[22] S. Abbasbandy, Application of Hes homotopy perturbation method to functional integral equations, *Chaos, Solitons and Fractals*, 31(2007):1243–1247.

[23] J.H. He, A coupling method of a homotopy technique and a perturbation technique for non-linear problems, *International Journal of Non-Linear Mechanics*, 35(2000):37–43.

[24] R.K. Saeed, Homotopy perturbation method for solving system of nonlinear Fredholm integral equations of the second kind, *Journal of Applied Sciences Research*, 4(10)(2008):1166–1173.

[25] A. Yildirim, Solution of BVPs for fourth-order integro-differential equations by using homotopy perturbation method, *Computers and Mathematics with Applications*, 56(2008):3175–3180.

[26] J. Saberi-Nadjafi and A. Ghorbani, Hes homotopy perturbation method: an effective tool for solving nonlinear integral and integro-differential equations, *Computers and Mathematics with Applications*, 58(2009):2379–2390.

[27] A. Golbabai and M. Javidi, Application of homotopy perturbation method for solving eighth-order boundary value problems, *Applied Mathematics and Computation*, 191(2007):334–346.

[28] E. Babolian, J. Saeidian and A. Azizi, Application of homotopy perturbation method to some nonlinear problems, *Applied Mathematical Sciences*, 3(45)(2009):2215–2226.

[29] J.H. He, The homotopy perturbation method for nonlinear oscillators with discontinuities, *Applied Mathematics and Computation*, 151(2004):287-292.

[30] H. Jafari, M. Alipour and H. Tajadodi, Convergence of homotopy perturbation method for solving integral equations, *Thai Journal of Mathematics*, 8(3)(2010):511–520.

[31] O. Abdulaziz, M.S.H. Chowdhury, I. Hashim and S. Momani, Direct solution of second-order BVPs by homotopy-perturbation method, *Sains Malaysiana*, 38(5)(2009):717–721.

[32] Z. Odibat and S. Momani, Modified homotopy perturbation method: application to quadratic Riccati differential equation of fractional order, *Chaos, Solitons and Fractals*, 36(2008):167–174.

[33] J. Biazar and M. Eslami, Modified HPM for solving systems of Volterra integral equations of the second kind, *Journal of King Saud University (Science)*, 23(2011):35–39.

[34] M. Javidi, Modified homotopy perturbation method for solving system of linear Fredholm integral equations, *Mathematical and Computer Modelling*, 50(2009): 159–165.

[35] M. Javidi and A. Golbabai, Modified homotopy perturbation method for solving non-linear Fredholm integral equations, *Chaos, Solitons and Fractals*, 40(2009):1408–1412.

[36] A. Tari, Modified homotopy perturbation method for solving two-dimensional Fredholm integral equations, *International Journal of Computational and Applied Mathematics*, 5(5)(2010):585–593.

[37] X. Fenga and Y. He, Modified homotopy perturbation method for solving the Stokes equations, *Computers and Mathematics with Applications*, 61(2011):2262–2266.

[38] Z.M. Odibat, A new modification of the homotopy perturbation method for linear and nonlinear operators, *Applied Mathematics and Computation*, 189(2007):746-753.

[39] A. Golbabai and B. Keramati, Modified homotopy perturbation method for solving Fredholm integral equations, *Chaos, Solitons and Fractals*, 37(2008):1528–1537.

[40] L. Cao and B. Han, Convergence analysis of the homotopy perturbation method for solving nonlinear ill-posed operator equations, *Computers Mathematics with Applications*, 61(2011):2058–2061.

[41] Jian-Lin Li,Adomian's decomposition method and homotopy perturbation method in solving nonlinear equations, *Journal of Computational and Applied Mathematics*, 228(2009):168–173.

[42] J. Biazar and H. Ghazvini, Convergence of the homotopy perturbation method for partial differential equations, *Nonlinear Analysis: Real World Applications*, 10(2009):2633–2640.

[43] AH. Nayef, Problems in perturbation. New york: John Wiley. 1985.

[44] C. Hillermeier, Generalized homotopy approach to multiobjective optimization, *Int J Optim Theory Appl*, 110(3)(2001):557–583.

[45] Ismail HNA, Raslan K and Rabboh AAA, Adomian decomposition method for Burger's Huxley and Burger's Fisher equations, *Applied Mathematics and Computation*, 159(2004):291–301.

[46] H. Saberi nik, S. Effati, R. Bouzhabadi and M. Golchaman. Solution of The Smoluchowski's Equation by Homotopy Analysis Method. *International Journal of Nonlinear Science*, 11(3)(2011): 330-337.

[47] M. Saeidy, M. Matinfar, J.Vahidi. Analytical Solution of BVPs for Fourth-order Integro-differential Equations by Using Homotopy Analysis Method. *International Journal of Nonlinear Science*, 9(4)(2010): 414-421.

[48] G. A. Afrouzi, J. Vahidi, M. Saeidy. Numerical Solutions of Generalized Drinfeld-Sokolov Equations Using the Homotopy Analysis Method. *International Journal of Nonlinear Science*, 9(2) (2010): 165-170.

[49] J. Biazar, M. Eslami. Exact Solutions for Non-linear Volterra-Fredholm Integro-Differential Equations by He's Homotopy Perturbation Method. *International Journal of Nonlinear Science*, 9(3)(2010): 285-289.

[50] Marwan Alquran, Mahmoud Mohammad. Approximate Solutions to System of Nonlinear Partial Differential Equations Using Homotopy Perturbation Method. *International Journal of Nonlinear Science*, 12(4)(2011): 485-497.

[51] H. Saberi nik, S. Effati, R. Bouzhabadi and M. Golchaman. Solution of The Smoluchowski's Equation by Homotopy Analysis Method. *International Journal of Nonlinear Science*, 11(3)(2011): 330-337.

[52] M. Saeidy, M. Matinfar, J.Vahidi. Analytical Solution of BVPs for Fourth-order Integro-differential Equations by Using Homotopy Analysis Method. *International Journal of Nonlinear Science*, 9(4)(2010): 414-421.

[53] G. A. Afrouzi, J. Vahidi, M. Saeidy. Numerical Solutions of Generalized Drinfeld-Sokolov Equations Using the Homotopy Analysis Method. *International Journal of Nonlinear Science*, 9(2) (2010): 165-170.

[54] J. Biazar, M. Eslami. Exact Solutions for Non-linear Volterra-Fredholm Integro-Differential Equations by He's Homotopy Perturbation Method. *International Journal of Nonlinear Science*, 9(3)(2010): 285-289.

[55] Marwan Alquran, Mahmoud Mohammad. Approximate Solutions to System of Nonlinear Partial Differential Equations Using Homotopy Perturbation Method. *International Journal of Nonlinear Science*, 12(4)(2011): 485-497.