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Parametric Spline Method for Solving Bratu's Problem

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Abstract: In this paper, parametric spline method is introduced for solving Bratu's problem. The convergence analysis of the presented method is discussed. The method is illustrated with two numerical examples and the results show that the method converges rapidly and approximates the exact solution very accurately.

Keywords: Numerical method; Parametric spline; Quintic spline; Convergence analysis; Bratu's problem; Boundary-value problem.

1 Introduction

Consider the Liouville-Bratu-Gelfand equation[1-3]:

$$\begin{cases} \Delta u(t) + \lambda e^{u(t)} = 0, & t \in \Omega, \\ u(t) = 0, & t \in \partial\Omega, \end{cases} \tag{1}$$

where $\lambda > 0$, and Ω is a bounded domain. The Bratu-type model[2-4] in one-dimensional planar coordinates is of the form:

$$u''(t) + \lambda e^{u(t)} = 0, \quad 0 \le t \le 1, \tag{2}$$

$$u(0) = u(1) = 0, \tag{3}$$

and is used to model a combustion problem in a numerical slab. The Bratu-type models appear in a number of applications such as the fuel ignition of the thermal combustion theory and in the Chandrasekhar model of the expansion of the universe. It stimulates a thermal reaction process in a rigid material where the process depends on the balance between chemically generated heat and heat transfer by conduction, see [5-9] and the references there in.

Several numerical methods for approximating the solution of Bratu's problem are known. Laplace transform decomposition numerical algorithm is used for solving Bratu's problem [10]. The Perturbation-iteration algorithm [11], applied to Bratu-type equations.Mohsen et.al. [12] introduced new smoother to enhance multigrid-based methods for Bratu problem. One-point pseudospectral collocation method has been used for the solution of the one-dimensional Bratu equation [13]. The main purpose of the present paper is to use parametric quintic spline method [14-16] for the numerical solution of nonlinear boundary value problem (2). The method consists of reducing the problem to a set of nonlinear algebraic equations.

The outline of the paper is as follows. First, in Section 2 we introduce parametric quintic spline method and describe the basic formulation of spline approximation required for our subsequent development. Section 3 outlines convergence analysis of the parametric spline method for solution of Bratu's problem. Finally numerical examples are given in section 4 to illustrate the efficiency of the presented method.

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2 Description of the method

Consider the partition Δ of $[a,b] \subset R$. Let $S_k(\Delta)$ denote the set of piecewise polynomials of degree k on subinterval $I_i = [t_{i-1}, t_i]$ of partition Δ. In this work, we consider parametric quintic spline method for finding approximate solution of variational problems.

Consider the grid points t_i on the interval $[a,b]$ as follows:

$$\begin{aligned}
&a = t_0 < t_1 < t_2 < \ldots < t_{n-1} < t_n = b,\\
&t_i = t_0 + ih, \quad i = 0,1,2,\ldots,n,\\
&h = \frac{b-a}{n},
\end{aligned}$$

where n is a positive integer. Let $S_\Delta(t,\tau)$ be quintic spline function of class $C^4[a,b]$ that interpolates $u(t)$ at the grid points $\{t_i\}_{i=0}^n$. Also, $S_\Delta(t,\tau)$ depends on a parameter $\tau > 0$ that is called a parametric spline function also, $S_\Delta(t,\tau)$ reduces to a ordinary quintic spline as $\tau \to 0$. By considering parametric quintic spline $S_\Delta(t,\tau) = S_\Delta(t)$, the spline function $S_\Delta(t)$ satisfies in the following equation:

$$S_\Delta^{(4)}(t) + \tau^2 S_\Delta^{(2)}(t) = \big(S_\Delta^{(4)}(t_i) + \tau^2 S_\Delta^{(2)}(t_i)\big)\big[\frac{t-t_{i-1}}{h}\big] + \big(S_\Delta^{(4)}(t_{i-1}) + \tau^2 S_\Delta^{(2)}(t_{i-1})\big)\big[\frac{t_i - t}{h}\big], \tag{4}$$

where $t \in [t_{i-1}, t_i]$, $S_\Delta(t_i) = u(t_i)$, and $h = t_i - t_{i-1}$. The Eq.(4) is a inhomogeneous ordinary differential equation. We solve the Eq.(4) and obtain the constants of integration by using interpolation conditions at the endpoints of the interval $[t_{i-1}, t_i]$, then we get:

$$\begin{aligned}
S_\Delta(t) &= \big(\frac{t-t_{i-1}}{h}\big)u_i + \big(\frac{t_i-t}{h}\big)u_{i-1} + \big(\frac{h^2}{3!}\big)\Big[M_i\Big(\big(\frac{t-t_{i-1}}{h}\big)^3 - \big(\frac{t-t_{i-1}}{h}\big)\Big)\\
&+ M_{i-1}\Big(\big(\frac{t_i-t}{h}\big)^3 - \big(\frac{t_i-t}{h}\big)\Big)\Big] + \big(\frac{h}{w}\big)^4\Big[\frac{w^2}{3!}\Big(\big(\frac{t-t_{i-1}}{h}\big)^3 - \big(\frac{t-t_{i-1}}{h}\big)\Big)\\
&- \Big(\big(\frac{t-t_{i-1}}{h}\big) - \frac{1}{\sin w}\big(\sin w\big(\frac{t-t_{i-1}}{h}\big)\big)\Big)\Big]F_i + \big(\frac{h}{w}\big)^4\Big[\frac{w^2}{3!}\Big(\big(\frac{t_i-t}{h}\big)^3 - \big(\frac{t_i-t}{h}\big)\Big)\\
&- \Big(\big(\frac{t_i-t}{h}\big) - \frac{1}{\sin w}\big(\sin w\big(\frac{t_i-t}{h}\big)\big)\Big)\Big]F_{i-1},
\end{aligned} \tag{5}$$

where

$$S_\Delta(t_i) = u(t_i) = u_i, S''_\Delta(t_i) = M_i, S_\Delta^{(4)}(t_i) = F_i, \quad w = \tau h, \tau > 0. \tag{6}$$

We use the continuity of first and third derivatives of spline function (5) at t_i, and obtain the following result:

$$M_{i+1} + 4M_i + M_{i-1} = \frac{6}{h^2}\big(u_{i+1} - 2u_i + u_{i-1}\big) - 6h^2\big(\alpha_1 F_{i+1} + 2\beta_1 F_i + \alpha_1 F_{i-1}\big), \tag{7}$$

$$M_{i+1} - 2M_i + M_{i-1} = h^2\big(\alpha F_{i+1} + 2\beta F_i + \alpha F_{i-1}\big), \tag{8}$$

where

$$\alpha = \frac{1}{w^2}\big(w\csc w - 1\big), \qquad \beta = \frac{1}{w^2}\big(1 - w\cot w\big), \tag{9}$$

$$\alpha_1 = \frac{1}{w^2}\big(\frac{1}{6} - \alpha\big), \qquad \beta_1 = \frac{1}{w^2}\big(\frac{1}{3} - \beta\big). \tag{10}$$

Considering Eqs. (7), (8) and also some simple calculations, we can obtain the value of F_i as follows:

$$\begin{aligned}
F_i = &\frac{1}{12h^2(\alpha_1\beta - \alpha\beta_1)}\Big[(\alpha + 6\alpha_1)\big(M_{i+1} + M_{i-1}\big) + (4\alpha - 12\alpha_1)M_i\\
&- \frac{6\alpha}{h^2}\big(u_{i+1} - 2u_i + u_{i-1}\big)\Big].
\end{aligned} \tag{11}$$

Having used Eq. (11) and replaced $F_{i-1},\ F_i$ and F_{i+1} in Eq. (8), the following result is obtained:

$$ph^2\big(M_{i+2}+M_{i-2}\big)+h^2sM_i+h^2q\big(M_{i+1}+M_{i-1}\big)=\alpha\big(u_{i+2}+u_{i-2}\big)$$
$$+2(\beta-\alpha)\big(u_{i+1}+u_{i-1}\big)+\big(2\alpha-4\beta\big)u_i,\ \ i=2,3,\dots,n-2, \tag{12}$$

where

$$p=\frac{\alpha}{6}+\alpha_1,\ \ q=2\Big[\frac{1}{6}\big(2\alpha+\beta\big)-\big(\alpha_1-\beta_1\big)\Big],\ \ s=2\Big[\frac{1}{6}\big(\alpha+4\beta\big)+\big(\alpha_1-2\beta_1\big)\Big]. \tag{13}$$

For a numerical solution of the Bratu's problem (2) and (3), the interval $[0,1]$ is divided into a set of grid points with step size h. Setting $t=t_i=t_0+ih$, in Eq. (2), we obtain:

$$u''(t_i)=-\lambda e^{u(t_i)}, \tag{14}$$

by using the assumption $S''_{\Delta}(t_i)=M\_\{i\}$ we get:

$$M_i=-\lambda e^{u(t_i)}. \tag{15}$$

The following equation is obtained by replacing $M_{i+j},\ j=-2,-1,0,1,2$ as Eq. (15) in Eq.(12),

$$h^2\Big[p\big(-\lambda e^{u_{i+2}}-\lambda e^{u_{i-2}}\big)+q\big(-\lambda e^{u_{i+1}}-\lambda e^{u_{i-1}}\big)-s\lambda e^{u_i}\Big]=$$
$$\alpha\big(u_{i+2}+u_{i-2}\big)+2(\beta-\alpha)\big(u_{i+1}+u_{i-1}\big)+\big(2\alpha-4\beta\big)u_i, \tag{16}$$

after simplification, we have:

$$\alpha u_{i-2}+p\lambda h^2e^{u_{i-2}}+2(\beta-\alpha)u_{i-1}+q\lambda h^2e^{u_{i-1}}+\big(2\alpha-4\beta\big)u_i+s\lambda h^2e^{u_i}+$$
$$2(\beta-\alpha)u_{i+1}+q\lambda h^2e^{u_{i+1}}+\alpha u_{i+2}+p\lambda h^2e^{u_{i+2}}=0,\quad i=2,3,\dots,n-2, \tag{17}$$

where $u_0=0,\ u_n=0$. Using Taylor's series for Eq. (17), we can obtain local truncation error as follows:

$$t_i=h^4\Big[\frac{1}{6}\big(7\alpha+\beta\big)-\big(4p+q\big)\Big]u_i^{(4)}+h^6\Big[\frac{1}{180}\big(31\alpha+\beta\big)-\frac{1}{12}\big(16p+q\big)\Big]u_i^{(6)}$$
$$+h^8\Big[\frac{1}{10080}\big(127\alpha+\beta\big)-\frac{1}{360}\big(64p+q\big)\Big]u_i^{(8)}+O(h^9). \tag{18}$$

In Eq. (18), if $\alpha=\frac{1}{12}$ and $\beta=\frac{5}{12}$, then

$$p=\frac{1}{360},\qquad q=\frac{56}{360},\qquad t_i\cong O(h^8),\quad i=2,3,\dots,n-2. \tag{19}$$

The nonlinear system (17) consists of $(n-3)$ equation with $(n-1)$ unknowns $u_i,\ i=1,2,\dots,n-1$. To obtain unique solution, we need two more equations which are supplied by using method of undetermined coefficients.

$$4u_0-7u_1+2u_2+u_3=h^2\Big[\frac{71}{240}u''_0+\frac{43}{12}u''_1+\frac{7}{8}u''_2+\frac{1}{3}u''_3-\frac{5}{48}u''_4+\frac{1}{60}u''_5\Big], \tag{20}$$

$$4u_n-7u_{n-1}+2u_{n-2}+u_{n-3}=h^2\Big[\frac{71}{240}u''_n+\frac{43}{12}u''_{n-1}+\frac{7}{8}u''_{n-2}+\frac{1}{3}u''_{n-3}-\frac{5}{48}u''_{n-4}+\frac{1}{60}u''_{n-5}\Big]. \tag{21}$$

Considering that $u_0=0$, $u_n=0$ and $u''_i=-\lambda e^{u_i}$, therefore we can rewrite the Eqs. (2) and (21) as follows:

$$-7u_1+\frac{43}{12}\lambda h^2e^{u_1}+2u_2+\frac{7}{8}\lambda h^2e^{u_2}+u_3+\frac{1}{3}\lambda h^2e^{u_3}-\frac{5}{48}\lambda h^2e^{u_4}+\frac{1}{60}\lambda h^2e^{u_5}+\lambda h^2\frac{71}{240}=0, \tag{22}$$

$$-7u_{n-1}+\frac{43}{12}\lambda h^2e^{u_{n-1}}+2u_{n-2}+\frac{7}{8}\lambda h^2e^{u_{n-2}}+u_{n-3}+\frac{1}{3}\lambda h^2e^{u_{n-3}}-\frac{5}{48}\lambda h^2e^{u_{n-4}}+\frac{1}{60}\lambda h^2e^{u_{n-5}}+\lambda h^2\frac{71}{240}=0. \tag{23}$$

The above nonlinear system consists of $(n-1)$ equations with $(n-1)$ unknowns $u_i,\ i=1,\dots,n-1$. Solving this nonlinear system by $Newton's$ method, we can obtain an approximation to the solution of (2).

3 Convergence analysis

Now we discuss the convergence of the parametric spline method for the Bratu's problem (2). We consider the Eqs. (17), (2) and (23) and then rewrite these equations in the matrix form which is the nonlinear system:

$$A_0'U + \lambda h^2 BF(U) = 0, \tag{24}$$

where $U = \big(u_1, u_2, \ldots, u_{n-1}\big)^T$. Also $A_0' = \big[a_{ij}\big]$, $B = \big[b_{ij}\big]$ are $(n-1)\times(n-1)$-dimensional and define as follows:

$$A_0' = \begin{pmatrix} 7 & -2 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-2}{3} & \frac{3}{2} & \frac{-2}{3} & \frac{-1}{12} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-1}{12} & \frac{-2}{3} & \frac{3}{2} & \frac{-2}{3} & \frac{-1}{12} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{-1}{12} & \frac{-2}{3} & \frac{3}{2} & \frac{-2}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -1 & -2 & 7 \end{pmatrix}, \tag{25}$$

$$B = \begin{pmatrix} \frac{-43}{12} & \frac{-7}{8} & \frac{-1}{3} & \frac{5}{48} & \frac{-1}{60} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-56}{360} & \frac{-246}{360} & \frac{-56}{360} & \frac{-1}{360} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-1}{360} & \frac{-56}{360} & \frac{-246}{360} & \frac{-56}{360} & \frac{-1}{360} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{-1}{360} & \frac{-56}{360} & \frac{-246}{360} & \frac{-56}{360} \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{-1}{60} & \frac{5}{48} & \frac{-1}{3} & \frac{-7}{8} & \frac{-43}{12} \end{pmatrix}. \tag{26}$$

and $F(U) = diag\big(e^{u_i}\big),\ i = 1, 2, \ldots, n-1$.

Theorem 1 *[17,Theorem 1.7.7]. Let M be a matrix such that $||M|| < 1$, and let I denote the unit matrix. Then $(I+M)^{-1}$ exists, and*

$$||(I+M)^{-1}|| < \frac{1}{1-||M||}.$$

Consider the matrix A_0' defined by (25), then we can write:

$$12A_0' = A_0A_1 + M, \tag{27}$$

where

$$A_0 = \begin{cases} 2, & i = j, \\ -1, & |i-j| = 1, \\ 0, & o.w, \end{cases} \qquad A_1 = \begin{cases} 4, & i = j, \\ 1, & |i-j| = 1, \\ 0, & o.w, \end{cases} \tag{28}$$

and

$$M = \begin{pmatrix} 77 & -22 & -11 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -6 & 12 & -6 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -6 & 12 & -6 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -6 & 12 & -6 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -11 & -22 & 77 \end{pmatrix}. \tag{29}$$

We know that the inverse of A_0 exists and is bounded as follows [18]:

$$||A_0^{-1}|| \le \frac{(b-a)^2}{8h^2}, \tag{30}$$

and also for A_1 we have [14]:

$$||A_1^{-1}|| \le \frac{1}{2}. \tag{31}$$

Considering the Eq. (29), we get $\|M\| = 110$. From Eq. (27) we can write

$$A_0' = \frac{1}{12} A_0 A_1 \big(I + A_1^{-1} A_0^{-1} M\big). \tag{32}$$

Now by using the following theorems, we show the inverse of A_0', defined by Eq. (32), exists and is bounded.

Theorem 2 *[18,Theorem 7.2]. A five-diagonal matrix $D = [d_{ij}]$ is irreducible, if and only if*

$$\begin{aligned} d_{i,i-1} &\neq 0 \ \ (i = 2, 3, \dots n), \quad d_{i,i+1} \neq 0 \ \ (i = 1, 2, \dots n-1), \\ d_{i,i-2} &\neq 0 \ \ (i = 3, 4, \dots n), \quad d_{i,i+2} \neq 0 \ \ (i = 1, 2, \dots n-2). \end{aligned}$$

Theorem 3 *[18,Theorem 7.4]. Let the matrix $M = [m_{ij}]$ be irreducible and satisfy the conditions*

$$(i) \ \ m_{ij} \leq 0, \ i \neq j, \ \ i, j = 1, 2, \dots, n,$$

$$(ii) \ \sum_{j=1}^{n} m_{ij} \begin{cases} \geq 0, & i = 1, 2, \dots n, \\ > 0, & \text{for at least one } i, \end{cases}$$

then M is monotone.

Theorem 4 *[18,section 7-2.2]. A monoton matrix is nonsingular.*

Since A_0' satisfies the conditions of Theorems 2 and 3, then according Theorem 4, A_0' is nonsingular. Considering the Eq. (32), we can write;

$$A_0'^{-1} = 12\big(I + A_1^{-1} A_0^{-1} M\big)^{-1} A_1^{-1} A_0^{-1}. \tag{33}$$

Hence, by applying Theorem 1 and Eqs. (30), (31) and $\|M\| = 110$, we conclude:

$$\|\big(I + A_1^{-1} A_0^{-1} M\big)^{-1}\| < \frac{8h^2}{8h^2 - 55(b-a)^2}. \tag{34}$$

Having used the Eqs. (33) and (34), we have:

$$\|A_0'^{-1}\| < \frac{6(b-a)^2}{\big(8h^2 - 55(b-a)^2\big)}. \tag{35}$$

In the following theorem we show that the inverse of

$$A = A_0' + \lambda h^2 BF, \tag{36}$$

exists.

Theorem 5 *If $Y = ||F|| < \frac{40\left(8h^2 - 55(b-a)^2\right)}{1179\left(\lambda h^2 (b-a)^2\right)}$ then the inverse of A, defined by Eq. (36), exists.*

Proof. Consider the Eq. (36), then

$$A = A_0' \big(I + \lambda h^2 A_0'^{-1} BF\big). \tag{37}$$

From Theorem 4 we know that the $A_0'^{-1}$ exists. Now, we need the existence of $\big(I + \lambda h^2 A_0'^{-1} BF\big)^{-1}$. According to Theorem 1 is sufficient, we show that $||\lambda h^2 A_0'^{-1} BF|| < 1$. Having used Eq. (35) and also $||B|| = \frac{1179}{240}$, we obtain:

$$\|\lambda h^2 A_0'^{-1} BF\| \leq \lambda h^2 ||A_0'^{-1}|| ||B|| ||F|| < \frac{1179}{40} . \frac{\lambda h^2 (b-a)^2}{\big(8h^2 - 55(b-a)^2\big)} Y. \tag{38}$$

Considering assumption $Y < \frac{40\left(8h^2 - 55(b-a)^2\right)}{1179\left(\lambda h^2 (b-a)^2\right)}$, we have:

$$\|\lambda h^2 A_0'^{-1} BF\| < 1. \tag{39}$$

Therefore, by using Theorem 1 and Eqs. (37) and (39) we conclude the existence of A^{-1}. ■

We can also obtain a bound on the errors $E = U - U_n$ in the maximum norm, where $U = \big(u(t_1), u(t_2), \dots, u(t_{n-1})\big)$ is the exact solution and $U_n = \big(u_1, u_2, \dots, u_{n-1}\big)$ is the approximate solution of Bratu's problem (2). From Theorem 5, we can derive a bound on $||E||$.

Theorem 6 *Let T be the vector of local truncation error and $AE = T$, then*

$$||E|| \cong O(h^6), \ \big(when\ \alpha = \frac{1}{12}, \ \ \beta = \frac{5}{12}\big). \tag{40}$$

Proof. By using Theorem 5 and $AE = T$, we can write:

$$E = A^{-1}T = \big(A_0' + \lambda h^2 BF\big)^{-1}T = \big(I + \lambda h^2 {A_0'}^{-1} BF\big)^{-1} {A_0'}^{-1} T, \tag{41}$$

therefore, we get

$$||E|| \leq ||\big(I + \lambda h^2 {A_0'}^{-1} BF\big)^{-1}||||{A_0'}^{-1}||||T||. \tag{42}$$

Having used Eq. (39) and Theorem 1 we obtain:

$$||\big(I + \lambda h^2 {A_0'}^{-1} BF\big)^{-1}|| \leq \frac{1}{1 - ||\lambda h^2 {A_0'}^{-1} BF||}. \tag{43}$$

Now, by applying Eqs. (35), (38) and (43), we have:

$$||\big(I + \lambda h^2 {A_0'}^{-1} BF\big)^{-1}|| \leq \frac{40K}{40K - 1179Y\lambda h^2 (b-a)^2}, \tag{44}$$

where $K = 8h^2 - 55(b-a)^2$. Considering the Eq. (18) and $\alpha = \frac{1}{12}$, $\beta = \frac{5}{12}$, we obtain $||T|| \leq \frac{h^8}{6048} M_8$, where $M_8 = \max\limits_{a \leq t \leq b} |u^{(8)}(t)|$. Therefore from Eq. (41), we conclude that:

$$||E|| \leq \frac{240(b-a)^2 h^8 M_8}{6048(40K - 1179Y\lambda h^2 (b-a)^2)} \cong O(h^6). \tag{45}$$

■

4 Numerical illustrations

In order to illustrate the performance of the parametric spline method for the Bratu equation (2) and justify the accuracy and efficiency of the method, we consider the following examples. The example have been solved by presented method with different values of λ. We take $\alpha = \frac{1}{12}$, $\beta = \frac{5}{12}$ and $n = 10$. The errors are reported on the set of uniform grid points

$$S = \{a = t_0, \ldots, t_i, \ldots, t_n = b\},$$

$$t_i = t_0 + ih, \quad i = 0, 1, 2, \ldots, n, \quad h = \frac{b-a}{n}. \tag{46}$$

The absolute error on the uniform grid points S is

$$|u(t_j) - u_j|, \quad 0 \leq j \leq n, \tag{47}$$

where $u(t_j)$ is the exact solution of the given example, and u_j is the computed solution by the parametric spline method. The exact solution of the equation (2) is given in [1-3] as:

$$u(t) = -2\ln\Big[\frac{\cosh\big((t-\frac{1}{2})\frac{\theta}{2}\big)}{\cosh\big(\frac{\theta}{4}\big)}\Big], \tag{48}$$

where θ satisfies

$$\theta = \sqrt{2\lambda}\cosh\big(\frac{\theta}{4}\big). \tag{49}$$

The Bratu problem has zero, one or two solutions when $\lambda > \lambda_c$, $\lambda = \lambda_c$ and $\lambda < \lambda_c$ respectively, where the critical value λ_c satisfies the equation:

Table 1: $\lambda = 1$, $\alpha = \frac{1}{12}$, $\beta = \frac{5}{12}$, $n = 10$.

x	$Present\ method$	$Laplace$[20]	$Decomposition$[21]	$B-spline$[19]
0.1	5.87×10^{-10}	1.98×10^{-6}	2.68×10^{-3}	2.98×10^{-6}
0.2	2.58×10^{-10}	3.94×10^{-6}	2.02×10^{-3}	5.46×10^{-6}
0.3	5.59×10^{-11}	5.85×10^{-6}	1.52×10^{-4}	7.33×10^{-6}
0.4	8.77×10^{-11}	7.70×10^{-6}	2.20×10^{-3}	8.50×10^{-6}
0.5	1.38×10^{-10}	9.47×10^{-6}	3.01×10^{-3}	8.89×10^{-6}
0.6	8.77×10^{-11}	1.11×10^{-5}	2.20×10^{-3}	8.50×10^{-6}
0.7	5.59×10^{-11}	1.26×10^{-5}	1.52×10^{-4}	7.33×10^{-6}
0.8	2.58×10^{-10}	1.35×10^{-5}	2.02×10^{-3}	5.46×10^{-6}
0.9	5.87×10^{-10}	1.20×10^{-5}	2.68×10^{-3}	2.98×10^{-6}

Table 2: $\lambda = 2$, $\alpha = \frac{1}{12}$, $\beta = \frac{5}{12}$, $n = 10$.

x	$Present\ method$	$Laplace$[20]	$Decomposition$[21]	$B-spline$[19]
0.1	1.25×10^{-8}	2.13×10^{-3}	1.52×10^{-2}	1.72×10^{-5}
0.2	1.95×10^{-8}	4.21×10^{-3}	1.47×10^{-2}	3.26×10^{-5}
0.3	2.73×10^{-8}	6.19×10^{-3}	5.89×10^{-3}	4.49×10^{-5}
0.4	3.31×10^{-8}	8.00×10^{-3}	3.25×10^{-3}	5.28×10^{-5}
0.5	3.53×10^{-8}	9.60×10^{-3}	6.98×10^{-3}	5.56×10^{-5}
0.6	3.31×10^{-8}	1.09×10^{-3}	3.25×10^{-3}	5.28×10^{-5}
0.7	2.73×10^{-8}	1.19×10^{-2}	5.89×10^{-4}	4.49×10^{-5}
0.8	1.95×10^{-8}	1.24×10^{-2}	1.47×10^{-2}	3.26×10^{-5}
0.9	1.25×10^{-8}	1.09×10^{-2}	1.52×10^{-2}	1.72×10^{-5}

$$1 = \frac{1}{4}\sqrt{2\lambda_c}\sinh\left(\frac{\theta_c}{4}\right). \tag{50}$$

It was evaluated in $[2-4]$ that the critical value λ_c is given by $\lambda_c = 3.513830719$. The maximum absolute errors in solutions of Bratu problems are compared with methods in [19-21] for n=10 and tabulated in Tables 1-2. The tables show that our results are more accurate.

Example 1 *Consider the following Bratu-type model*

$$u''(t) + e^{u(t)} = 0, \quad 0 \le t \le 1, \tag{51}$$

with boundary conditions

$$u(0) = 0, \quad u(1) = 0.$$

In Eq. (51), we have $\lambda = 1$. Applying the parametric spline method, following approximations are obtained and the numerical results are tabulated in Table 1.

Example 2 *Consider the following boundary value problem of the Bratu-type when $\lambda = 2$:*

$$\begin{aligned} &u''(t) + 2e^{u(t)} = 0, \quad 0 \le t \le 1,\\ &u(0) = u(1) = 0. \end{aligned} \tag{52}$$

In Table 2, parametric spline solutions for the case $\lambda = 2$ are compared with the numerical methods given in [19-21].

5 Conclusion

In this paper, parametric quintic spline method is applied for solving the Bratu equation. The parametric spline method reduce the computation of the Bratu equation to some nonlinear algebraic equations. The analytical results are illustrated with two numerical examples. The proposed scheme is simple and computationally attractive.

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