Synchronization of Noise Perturbed Hyperchaotic Lorenz Time-delay System via a Single Controller with One Variable

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Abstract: This work is devoted to investigating the synchronization of noise-perturbed hyperchaotic Lorenz time-delay system. Based on the Lyapunov stability theory, the sufficient conditions for achieving synchronization of hyperchaotic Lorenz time-delay systems with stochastic noise are derived, and a simple synchronization scheme only with a single variable controller is proposed. By numerical method, the synchronization condition of the control parameter \( k \) is also obtained. Numerical simulation results are presented to demonstrate the effectiveness of the proposed chaos synchronization scheme.

Keywords: Hyperchaotic; time-delay; linear feedback; control; single variable

1 Introduction

Chaos is very interesting nonlinear phenomenon and has applications in many areas such as biology, economics, secure communication, many other engineering systems, and so on [1]. During the last two decades, synchronization in chaotic dynamic systems has received a great deal of interest among scientists from various research fields since Pecora and Carroll [2] introduced a method to synchronize two identical chaotic systems with different initial conditions. In the literature, various synchronization schemes, such as OGY method [3], observer-based control [4] [5], active control [6], time-delay feedback approach[7], adaptive control [8] [9], backstepping design technique [10] [11], and so on, have been successfully applied to the chaos synchronization of chaotic or hyper chaotic systems.

Even though the notion of chaos synchronization is well studied in chaotic or hyper chaotic systems, there exists very little in-depth study in higher dimensional systems, such as time-delay systems, which are essentially infinite dimensional in nature and often exhibit high-dimensional, highly non-phase-coherent hyperchaotic attractors with complex topological structure [3]. Consequently, estimating phase explicitly to identify chaos synchronization in such systems is quite difficult. On the other hand, in the formulation of the control problem, the proposed controllers in previous works are, in most cases, too complex both in design and in implementation. In a practical way, smaller number of controllers and simpler form of controllers are practical greatly when synchronization of chaotic systems is considered via controllers [12]. Therefore, it is invited to investigate the chaotic synchronization problems of time delay chaotic systems by using single control input both for theoretical research and for practical applications. However, up to our knowledge, there have been few (if any) results of an investigation for the delay chaotic systems via a single controller with one variable in the literature, especially the time-delay hyperchaotic systems.

Motivated by above discussion, the synchronization of noise-perturbed hyperchaotic Lorenz time-delay system could be achieved only by a single variable controller. Based on the Lyapunov stability theory, a simple synchronization scheme is proposed, and the sufficient conditions for achieving synchronization are derived. By numerical method, the synchronization condition of the control parameter \( k \) is also obtained. Numerical simulation results are presented to demonstrate the effectiveness of the proposed synchronization scheme.

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2 Hyper chaotic Lorenz system with time delay

Wang et al. constructed a new hyper chaotic system by introducing state feedback control to the familiar Lorenz system [13], which is described as

\[
\begin{align*}
\dot{x} &= a(y - x) + rw, \\
\dot{y} &= cx - y - xz, \\
\dot{z} &= xy - bz, \\
\dot{w} &= -yz - dw,
\end{align*}
\]

(1)

where \(x, y, z, w\) are state variables, \(a, b, c, d\) and \(r\) are real constants. When \(a = 10, b = 8/3, c = 28, d = 1,\) and \(r = 1,\) system (1) has two positive Lyapunov exponents, i.e., \(\lambda_1 = 0.3381, \lambda_2 = 0.1586.\) The hyper chaotic attractors of system (1) are shown in Figure.1.

In this paper we consider the hyper chaotic Lorenz system with a time delay described as

\[
\begin{align*}
\dot{x} &= a(y - x) + rw(t - \tau), \\
\dot{y} &= cx - y - xz, \\
\dot{z} &= xy - bz, \\
\dot{w} &= -yz - dw,
\end{align*}
\]

(2)

where \(\tau > 0\) is the time delay. Time delay is ubiquitous in most physical, chemical, biological, neural and other natural systems. As the dynamical systems given by DDEs have an infinite dimensional state space, usually the attractors of the solutions are high dimensional. The hyper chaotic Lorenz time-delay system may exhibit more complicated complex behaviors. By the Galerkin approximation technique, an algorithm is considered by Ghosh for calculating Lyapunov exponents for system with time delay [14], when \(a = 10, b = 8/3, c = 28, d = 1,\) and \(r = 1,\) system (2) has two positive Lyapunov exponents, i.e., \(\lambda_1 = 0.6513, \lambda_2 = 0.1394\), which exhibits hyperchaotic behavior, the hyperchaotic attractors of system (2) are shown in Figure.2 (3D overview). Because it possesses at least two positive Lyapunov exponents and has more complex behavior than hyperchaotic system, the hyperchaotic Lorenz time-delay system is more suitable for some special engineering application such as secure communication. In the next section, the synchronization of noise-perturbed hyperchaotic Lorenz time-delay system will be investigated via a single feedback control only with one variable.

3 Synchronization of noise-perturbed hyperchaotic Lorenz time-delay system via a single variable control

Suppose the drive and response systems are given as the following forms:

\[
\begin{align*}
\dot{x}_1 &= a(y_1 - x_1) + rw_1(t - \tau), \\
\dot{y}_1 &= cx_1 - y_1 - x_1z_1, \\
\dot{z}_1 &= x_1y_1 - bz_1, \\
\dot{w}_1 &= -y_1z_1 - dw_1,
\end{align*}
\]

(3)

and the noise-perturbed response system is given as

\[
\begin{align*}
\dot{x}_2 &= [a(y_2 - x_2) + rw_2(t - \tau) + u_1]dt, \\
\dot{y}_2 &= [cx_2 - y_2 - x_2z_2 + u_2]dt, \\
\dot{z}_2 &= [x_2y_2 - bz_2 + u_3]dt, \\
\dot{w}_2 &= [-y_2z_2 - dw_2 + u_4]dt + \theta(w_2 - w_1)d\delta(t),
\end{align*}
\]

(4)

where \(u_1, u_2, u_3, u_4\) are controllers to be constructed, and \(\delta(t)\) is an one-dimensional Brownian motion defined on a complete probability space, \(\theta\) is the noise intensity. Subtracting the drive system (3) from the response system (4) yields
the following error dynamical system:

\[
\begin{align*}
    de_1 &= [a(e_2 - e_1) + re_4(t - \tau) + u_1]dt, \\
    de_2 &= [e_1 - e_2 - z_2e_1 - x_1e_3 + u_2]dt, \\
    de_3 &= [e_1y_2 + e_2x_1 - be_3 + u_3]dt, \\
    de_4 &= [-z_2e_2 - y_1e_3 - de_4 + u_4]dt + \theta e_4 dt(t),
\end{align*}
\]  

where \( e_1 = x_2 - x_1 \), \( e_2 = y_2 - y_1 \), \( e_3 = z_2 - z_1 \), \( e_4 = w_2 - w_1 \), \( u_i = -k_ie_i \) and \( k_i (i = 1, 2, 3, 4) \) is the positive feedback gains. Construct a positive definite Lyapunov function as following:

\[
V = 0.5[e_1^2 + e_2^2 + e_3^2 + e_4^2] + \alpha \int_{t-\tau}^{t} e_4^2 dt.
\]

where \( \alpha > 0 \). By the Itô-differential rule [15], calculating the time derivative of the Lyapunov function (6) along the trajectory of system (5) arrives at:

\[
\mathcal{L}V = -(a + k_1)e_1^2 - (1 + k_2)e_2^2 - (b + k_3)e_3^2 - (d + k_4 - \alpha - 0.5\theta^2)e_4^2 + (a + c - z_2)e_1e_2 + y_2e_1e_3 \\
- z_2e_2e_4 - y_1e_3e_4 + re_1e_4(t - \tau) - \alpha e_4^2(t - \tau).
\]

Since a chaotic system has bounded trajectories, there exists a positive constant \( M \), such that \( |x_i|, |y_i|, |z_i| \) and \( |w_i| \leq M(i = 1, 2) \), thus

\[
\mathcal{L}V \leq -(a + k_1 - 0.5r(\lambda)^{-1})e_1^2 - (1 + k_2)e_2^2 - (b + k_3)e_3^2 - (d + k_4 - \alpha - 0.5\theta^2)e_4^2 \\
+ (a + c + M)|e_1||e_2| + M|e_1||e_3| + M|e_2||e_4| + M|e_4||e_3| - (0.5r\lambda + \alpha)e_4^2(t - \tau) \\
= -([e_1, e_2, e_3, e_4]P[e_1, e_2, e_3, e_4]^T - (0.5r\lambda + \alpha)e_4^2(t - \tau),
\]

where

\[
P = \begin{pmatrix}
    k_1 + a_{11} & a_{12} & a_{13} & 0 \\
    a_{12} & k_2 + a_{22} & 0 & a_{13} \\
    a_{13} & 0 & k_3 + a_{33} & a_{13} \\
    0 & a_{13} & a_{13} & k_4 + a_{44}
\end{pmatrix}.
\]

Case 1 \( u_1 = -k_1e_1, u_i = 0, (i = 2, 3, 4) \), we define the following conditions

\[
A_{11} : k_1 > -a_{11}, \\
A_{12} : k_1 > a_{12}/a_{22} - a_{11}, \\
A_{13} : k_1 > a_{12}/a_{22} + a_{13}/a_{33} - a_{11}, \\
A_{14} : k_1 > [a_{13}^4 - a_{13}a_{13}^2 - a_{13}a_{44} - a_{12}a_{22} - a_{12}a_{33}a_{44}]/[a_{13}a_{33} + a_{22} - a_{22}a_{44}] - a_{11},
\]

Case 2 \( u_2 = -k_2e_2, u_i = 0, (i = 1, 3, 4) \), in this case, the conditions are chosen as

\[
A_{21} : a_{11} > 0, \\
A_{22} : k_2 > a_{12}/a_{11} - a_{22}, \\
A_{23} : k_2 > a_{12}a_{33}/(a_{11}a_{33} - a_{13}^2) - a_{22}, \\
A_{24} : k_2 > [a_{13}^4 - a_{13}a_{13}^2 - a_{13}a_{44} - a_{12}a_{22} - a_{12}a_{33}a_{44}]/[a_{13}a_{33} + a_{22} - a_{22}a_{44}] - a_{22}.
\]

It is obvious that, for suitable values of \( \lambda, \alpha \) and \( k_i, (i = 1, 2, 3, 4) \), the conditions in case 1 and case 2 can be obtained, the matrix \( P \) is positive definite, and \( V \) is negative semi-definite. So one obtains \( e \to 0 \) and \( e_\tau \to \infty \) as \( t \to 0 \). It follows that the states of response system (4) and the states of drive system (3) are ultimately synchronized asymptotically.
4 Illustrative numerical simulation examples

In this section, some numerical examples are presented to illustrate the theoretical analysis. In the following numerical simulations, the fourth-order Runge-Kutta method is employed with time step size 0.001. The system parameters are selected as \( a = 10, \ b = 8/3, \ c = 28, \ d = 1, \ r = 1, \ \theta = 0.1 \) and the time delay \( \tau \) is chosen as 1. The initial values of drive system and the response system are chosen as \((x_1(0), y_1(0), z_1(0), w_1(0)) = (0.82, 0.29, 0.48, 0.1)\) and \((x_2(0), y_2(0), z_2(0), w_2(0)) = (0.78, 0.3, 0.25, 0.3)\), respectively.

When the controller is chosen as in case 1, the synchronization errors between the drive system and the noise-perturbed response system with the linear feedback controller \( u_1 = -k_1e_1, u_i = 0, (i = 2, 3, 4) \) (where \( k_1 = 6.0 \)) are displayed in Figure 3. From the numerical simulation results, it is obvious to obtain that the synchronization errors converge asymptotically to zero and the two hyperchaotic Lorenz time-delay systems are indeed achieved with synchronization. From the theoretical analysis, it is known that conditions for the synchronization obtained analytically are only the sufficient conditions. But in a practical way, \( k_1 \) can be obtained by numerical methods. In order to explore the synchronization behavior of the scheme, \( k_1 \) is taken as bifurcation parameter to see how the solution of evolves with the variation of the parameter. From the bifurcation diagram plotted in Figure 4, it can be observed that \( e_2 \) converges to zero at about \( k_1 > 5.57 \), which means that the synchronization can be achieved when \( k_1 > 5.57 \).

When the controller is chosen as in case 2, Figure 5 shows the synchronization errors between the drive system and the noise-perturbed response system with the linear feedback controller \( u_1 = -k_2e_2, u_i = 0, (i = 1, 3, 4) \) (where \( k_2 = 2.5 \)). It is obvious to obtain that the synchronization errors converge asymptotically to zero and the synchronization of the two systems are indeed achieved. When \( k_2 \) is chosen as the bifurcation parameter, the bifurcation diagram is plotted in Figure 6, it can be observed that \( e_2 \) converges to zero at about \( k_2 > 1.57 \), which means that the synchronization can be achieved when \( k_2 > 1.57 \).

5 Conclusions

In this paper the synchronization of the drive hyperchaotic time-delay system and the noise-perturbed response system with a single controller is investigated. The proposed controller is based on feedback control method which needs only...
Figure 2: 3D overview hyperchaotic attractor of Eq. (2) (a) $x, y, z$, (b) $x, y, w$, (c) $y, z, w$.

Figure 3: Dynamics of synchronization error states $e_1, e_2, e_3, e_4$, where $u_1 = -k_1 e_1, u_i = 0, (i = 2, 3, 4), k_1 = 6.0$.

Figure 4: Bifurcation diagram of $e_2$ versus the parameter $k_1$.
Figure 5: Dynamics of synchronization error states $e_1, e_2, e_3, e_4$, where $u_2 = -k_2 e_2, u_i = 0, (i = 1, 3, 4), k_2 = 2.5$.

Figure 6: Bifurcation diagram of $e_2$ versus the parameter $k_2$

A linear feedback controller, thus, synchronization can be realized by the physical configuration of the electronic set up, which is of important significance on using chaos synchronization for applications. The correctness of the proposed methods is verified by theoretical analysis. Furthermore, numerical simulations are provided to show the effectiveness of the developed methods.

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